Gulf Journal of Mathematics
Vol 5, Issue 2 (2017) 56-61

LOWER BOUNDS FOR BLOW-UP TIME OF COUPLED NONLINEAR KLEIN-GORDON EQUATIONS

ERHAN PIŞKIN[1]

Abstract. This paper deals with the blow up of coupled nonlinear Klein-Gordon equations. Lower bounds for the time of blow up is derived if the solutions blow up.

1. Introduction

In this paper, we consider the following coupled nonlinear Klein-Gordon equations with nonlinear damping terms

$$\begin{cases} u_{tt}-\triangle u+m_1^2u+|u_t|^{p-1}u_t=f_1(u,v), & (x,t)\in\Omega\times(0,T), \\ v_{tt}-\triangle v+m_2^2v+|v_t|^{q-1}v_t=f_2(u,v), & (x,t)\in\Omega\times(0,T), \\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x), & x\in\Omega, \\ v(x,0)=v_0(x),\ v_t(x,0)=v_1(x), & x\in\Omega, \\ u(x,t)=v(x,t)=0, & x\in\partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$ in R^n $(n=1,2,3)$, $m_1,\ m_2>0$ and $p,q\geq 1$ are constants.

The coupled nonlinear Klein-Gordon equation which models the motion of charged mesons in an electromagnetic field is investigated [15].

In the absence of the m_1^2u and m_2^2v terms $(m_1=\ m_2=0)$ the problem (1.1) reduces to the following form

$$\begin{cases} u_{tt}-\triangle u+|u_t|^{p-1}u_t=f_1(u,v), \\ v_{tt}-\triangle v+|v_t|^{q-1}v_t=f_2(u,v). \end{cases} \tag{1.2}$$

Agre and Rammaha [2] studied the global existence and the blow up of solutions of the system (1.2) by using the same techniques as in [4]. For more related results, the reader is referred to [3],[5],[6].

Ye [16] proved the global existence and asymptotic stability of solutions of the system (1.1) for $p=q$. For $p=q=1$, the system was considered by many authors [7, 11, 17]. Wu [18] studied blow up of solutions of the system (1.1) for $n=3$. Recently, in [13, 14] we studied blow up and decay of solutions of the system (1.1).

Date: Received: Jan 10, 2017; Accepted: Mar 17, 2017.

[1] Corresponding author.

2010 *Mathematics Subject Classification.* 34B40; 35B44.

Key words and phrases. Lower bounds, Klein-Gordon equations, nonlinear damping and source terms.

In this work our aim is to find lower bound of the blow up time T^* for solutions of (1.1).

This paper is organized as follows: In the next section, we present some lemmas and the local existence theorem. In section 3, we establish lower bounds for the blow up time when the blow up occurs.

2. Preliminaries

In this section, we give some assumptions and lemmas which will be used throughout this work. Let $\|.\|$ and $\|.\|_p$ denote the usual $L^2(\Omega)$ norm and $L^p(\Omega)$ norm, respectively. We denote by C various positive constants which may be different at different occurrences.

Concerning the functions $f_1(u,v)$ and $f_2(u,v)$, we take

$$f_1(u,v) = (r+1)\left[a|u+v|^{r-1}(u+v) + b|u|^{\frac{r-3}{2}}u|v|^{\frac{r+1}{2}}\right],$$

$$f_2(u,v) = (r+1)\left[a|u+v|^{r-1}(u+v) + b|u|^{\frac{r+1}{2}}|v|^{\frac{r-3}{2}}v\right],$$

where $a, b > 0$ are constants and r satisfies

$$\begin{cases} 1 < r & \text{if } n \le 2, \\ 1 < r \le \frac{4-n}{n-2} & \text{if } n > 2. \end{cases} \tag{2.1}$$

We can easily verify that

$$u\ f_1(u,v) + v f_2(u,v) = (r+1)F(u,v),\ \forall (u,v) \in R^2, \tag{2.2}$$

where

$$F(u,v) = \left[a|u+v|^{r+1} + 2b|uv|^{\frac{r+1}{2}}\right]. \tag{2.3}$$

We define

$$J(t) = \frac{1}{2}\left(\|\nabla u\|^2 + \|\nabla v\|^2 + m_1^2\|u\|^2 + m_2^2\|v\|^2\right) - \int_\Omega F(u,v)\,dx, \tag{2.4}$$

and

$$I(t) = \|\nabla u\|^2 + \|\nabla v\|^2 + m_1^2\|u\|^2 + m_2^2\|v\|^2 - (r+1)\int_\Omega F(u,v)\,dx. \tag{2.5}$$

We also define the energy functional as follows

$$\begin{aligned} E(t) &= \frac{1}{2}\left(\|u_t\|^2 + \|v_t\|^2\right) + \frac{1}{2}\left(\|\nabla u\|^2 + \|\nabla v\|^2\right. \\ &\quad \left. + m_1^2\|u\|^2 + m_2^2\|v\|^2\right) - \int_\Omega F(u,v)\,dx. \end{aligned} \tag{2.6}$$

We also define

$$W_- = \left\{(u,v) : (u,v) \in H_0^1(\Omega) \times H_0^1(\Omega),\ I(u,v) < 0\right\}. \tag{2.7}$$

The next lemma shows that our energy functional (2.6) is a nonincreasing function along the solution of (1.1).

Lemma 2.1. *$E(t)$ is a nonincreasing function for $t \ge 0$ and*

$$E'(t) = -\left(\|u_t\|_{p+1}^{p+1} + \|v_t\|_{q+1}^{q+1}\right) \le 0. \tag{2.8}$$

Proof. Multiplying the first equation of (1.1) by u_t and the second equation by v_t, integrating over Ω, using integrating by parts and summing up the product results, we get

$$E(t) - E(0) = -\int_0^t \left(\|u_\tau\|_{p+1}^{p+1} + \|v_\tau\|_{q+1}^{q+1} \right) d\tau \text{ for } t \geq 0. \tag{2.9}$$

□

Lemma 2.2. *(Sobolev-Poincare inequality) [1]. Let p be a number with $2 \leq p < \infty$ $(n = 1, 2)$ or $2 \leq p \leq 2n/(n-2)$ $(n \geq 3)$, then there is a constant $C_* = C_*(\Omega,\ p)$ such that*

$$\|u\|_p \leq C_* \|\nabla u\|, \ \forall u \in H_0^1(\Omega).$$

Lemma 2.3. *There exist two positive constants c_1 and c_2 such that*

$$\int_\Omega |f_1(u,v)|^2 dx \leq c_1 \left(\|\nabla u\|^2 + \|\nabla v\|^2 \right)^r$$

and

$$\int_\Omega |f_2(u,v)|^2 dx \leq c_2 \left(\|\nabla u\|^2 + \|\nabla v\|^2 \right)^r$$

are satisfied.

Proof. Our techniques of proof follows carefully the steps in [10, 12], with necessary modifications imposed by the nature of our problem.

$$\begin{aligned} \int_\Omega |f_1(u,v)|^2 dx &\leq (r+1)^2 \int_\Omega \left(|u+v|^r + |u|^{\frac{r-1}{2}} |v|^{\frac{r+1}{2}} \right)^2 dx \\ &\leq (r+1)^2 \int_\Omega \left(|u|^r + |v|^r + |u|^{\frac{r-1}{2}} |v|^{\frac{r+1}{2}} \right)^2 dx. \end{aligned}$$

Now, we consider the following two ceses.

Case 1. For $|u| \leq |v|$, we get

$$\begin{aligned} \int_\Omega |f_1(u,v)|^2 dx &\leq (r+1)^2 \int_\Omega \left(|v|^r + |v|^r + |v|^{\frac{r-1}{2}} |v|^{\frac{r+1}{2}} \right)^2 dx \\ &\leq (r+1)^2 \int_\Omega (3|v|^r)^2 dx \\ &\leq 9(r+1)^2 \|v\|_{2r}^{2r}, \end{aligned}$$

Thus, by the Sobolev-Poincare inequality, we have

$$\begin{aligned} \int_\Omega |f_1(u,v)|^2 dx &\leq c_1 (r+1)^2 \|\nabla v\|^{2r} \\ &\leq c_1 (r+1)^2 \left(\|\nabla u\|^2 + \|\nabla v\|^2 \right)^r. \end{aligned}$$

Case 2. If $|u| > |v|$, we get

$$\begin{aligned}\int_\Omega |f_1(u,v)|^2 dx &\leq (r+1)^2 \int_\Omega \left(|u|^r + |u|^r + |u|^{\frac{r-1}{2}} |u|^{\frac{r+1}{2}}\right)^2 dx \\ &\leq (r+1)^2 \int_\Omega (3|u|^r)^2 dx \\ &\leq 9(r+1)^2 \|u\|_{2r}^{2r},\end{aligned}$$

Thus, by the Sobolev-Poincare inequality, we have

$$\begin{aligned}\int_\Omega |f_1(u,v)|^2 dx &\leq c_1 (r+1)^2 \|\nabla u\|^{2r} \\ &\leq c_1 (r+1)^2 \left(\|\nabla u\|^2 + \|\nabla v\|^2\right)^r.\end{aligned}$$

□

Next, we state the local existence theorem that can be established by combining arguments of [4, 16].

Theorem 2.4. *(Local existence). Suppose that (2.1) holds. Then there exist p, q satisfying*

$$\begin{cases} 1 \leq p, q & \text{if } n \leq 2, \\ 1 \leq p, q \leq \frac{n+2}{n-2} & \text{if } n > 2 \end{cases}$$

and further $(u_0, v_0) \in H_0^1(\Omega) \times H_0^1(\Omega)$, $(u_1, v_1) \in L^2(\Omega) \times L^2(\Omega)$. Then problem (1.1) has a unique local solution

$$u, v \in C\left([0,T); H_0^1(\Omega)\right),$$

$u_t \in C\left([0,T); L^2(\Omega)\right) \cap L^{p+1}(\Omega \times [0,T))$ *and* $v_t \in C\left([0,T); L^2(\Omega)\right) \cap L^{q+1}(\Omega \times [0,T))$. *Moreover, at least one of the following statements holds true:*

i) $T = \infty$,
ii) $\|u_t\|^2 + \|v_t\|^2 + \|\nabla u\|^2 + \|\nabla v\|^2 + m_1^2 \|u\|^2 + m_2^2 \|v\|^2 \longrightarrow \infty$ as $t \longrightarrow T^*$.

Theorem 2.5. *(see [13])Assume $r > \max\{p, q\}$, the initial energy $E(0) < 0$. Then the solution of this system blows up in finite time T^*.*

3. Lower bounds for the blow up time

In this section, we prove lower bounds for the blow up time T^*.

Theorem 3.1. *Assume that (2.1) holds. Assume further that $(u_0, v_0) \in W_-$, $(u_0, v_0) \in H_0^1(\Omega) \times H_0^1(\Omega)$, $(u_1, v_1) \in L^2(\Omega) \times L^2(\Omega)$ and $1 < p, q < r$. Let u be the solution of the problem (1.1), which blows up at a finite time T^*. Then*

$$\int_{\psi(0)}^{\infty} \frac{d\tau}{\psi(\tau) + E(0) + \frac{c_2+c_3}{2}(r+1)^r \psi^r(\tau)} \leq T^*.$$

Proof. We define

$$\psi(t)=\int_{\Omega}F(u,v)\,dx \tag{3.1}$$

Therefore

$$\psi'(t)=\int_{\Omega}\left(u_tF_u+v_tF_v\right)dx. \tag{3.2}$$

Using Young's inequality, we have

$$\psi'(t)\leq\frac{1}{2}\int_{\Omega}\left(u_t^2+v_t^2\right)dx+\frac{1}{2}\int_{\Omega}\left(F_u^2+F_v^2\right)dx.$$

By the Lemma 2.3, we obtain

$$\psi'(t)\leq\frac{1}{2}\int_{\Omega}\left(u_t^2+v_t^2\right)dx+\frac{c_2+c_3}{2}\left(\|\nabla u\|^2+\|\nabla v\|^2\right)^r. \tag{3.3}$$

Since $I(t)<0$, we get

$$\|\nabla u\|^2+\|\nabla v\|^2+m_1^2\|u\|^2+m_2^2\|v\|^2<(r+1)\int_{\Omega}F(u,v)\,dx. \tag{3.4}$$

Inserting (3.4) into (3.3), to get

$$\begin{aligned}\psi'(t) &\leq \frac{1}{2}\int_{\Omega}\left(u_t^2+v_t^2\right)dx+\frac{c_2+c_3}{2}\left((r+1)\int_{\Omega}F(u,v)\,dx\right)^r\\ &= \frac{1}{2}\int_{\Omega}\left(u_t^2+v_t^2\right)dx+\frac{c_2+c_3}{2}(r+1)^r\left(\int_{\Omega}F(u,v)\,dx\right)^r\\ &= \frac{1}{2}\int_{\Omega}\left(u_t^2+v_t^2\right)dx+\frac{c_2+c_3}{2}(r+1)^r\psi^r(t).\end{aligned} \tag{3.5}$$

By the definition $E(t)$, we have

$$\begin{aligned}&\left(\|u_t\|^2+\|v_t\|^2\right)+\|\nabla u\|^2+\|\nabla v\|^2+m_1^2\|u\|^2+m_2^2\|v\|^2\\ =\;& 2\int_{\Omega}F(u,v)\,dx+2E(t)\\ \leq\;& 2\psi(t)+2E(0).\end{aligned} \tag{3.6}$$

Combining (3.5)-(3.6), we get

$$\psi'(t)\leq\psi(t)+E(0)+\frac{c_2+c_3}{2}(r+1)^r\psi^r(t). \tag{3.7}$$

Applying Theorem 2.5, we have

$$\lim_{t\to T^*}\int_{\Omega}F(u,v)\,dx=\infty. \tag{3.8}$$

According to (3.7) and (3.8), we obtain

$$\int_{\psi(0)}^{\infty}\frac{d\tau}{\psi(\tau)+E(0)+\frac{c_2+c_3}{2}(r+1)^r\psi^r(\tau)}\leq T^*.$$

This completes the proof. □

References

1. R.A. Adams and J.J.F. Fournier, Sobolev Spaces, Academic Press, 2003.
2. K. Agre and M.A. Rammaha, Systems of nonlinear wave equations with damping and source terms, Diff. Integral Eqns. 19(11) (2006), 1235-1270.
3. C.O. Alves, M.M. Cavalcanti, V.N. Domingos Cavalcanti, M.A. Rammaha and D. Toundykov, On the existence, uniform decay rates and blow up of solutions to systems of nonlinear wave equations with damping and source terms, Discrete and Continuous Dynamical Systems-Series S. 2(3) (2009), 583-608.
4. V. Georgiev and G. Todorova, Existence of a solution of the wave equation with nonlinear damping and source term, J. Diff. Eqns. 109 (1994), 295-308.
5. B.S. Houari, Global nonexistence of positive initial-energy solutions of a system of nonlinear wave equations with damping and source terms, Diff. Integral Eqns. 23(1-2) (2010), 79-92.
6. B.S. Houari, Global existence and decay of solutions of a nonlinear system of wave equations, Appl. Anal. 91(3) (2012), 475-489.
7. M.O. Korpusov, Blow up the solution of a nonlinear system of equations with positive energy, Theoretical and Mathematical Physics. 171(3) (2012), 725-738.
8. S.A. Messaoudi and B.S. Houari, Global nonexistence of positive initial-energy solutions of a system of nonlinear viscoelastic wave equations with damping and source terms, J. Math. Anal. Appl. 365 (2010), 277-287.
9. S.A. Messaoudi, Blow up in a nonlinearly damped wave equation, Math. Nachr. 231 (2001), 105-111.
10. S.A. Messaoudi, Blow up of solutions with positive initial energy in a nonlinear viscoelastic wave equations, J. Math. Anal. Appl. 320 (2006), 902-915.
11. M.M. Miranda and L.A. Medeiros, On the existence of global solutions of a coupled nonlinear Klein-Gordon equations, Funkcial Ekvac. 30 (1987), 147-161.
12. A. Peyravi, Lower bounds of blow up time for a system of semilinear hyperbolic Petrovsky equations, Acta Math. Sci. 36B(3) (2016), 683-688 .
13. E. Pişkin, Uniform decay and blow-up of solutions for coupled nonlinear Klein-Gordon equations with nonlinear damping terms, Math. Meth. Appl. Sci. 37(18) (2014), 3036-3047.
14. E. Pişkin, Blow-up of solutions for coupled nonlinear Klein-Gordon equations with weak damping terms, Math. Sci. Letters, 3(3) (2014) 189-191.
15. I. Segal, Nonlinear partial differential equations in quantum field theory. Proc Symp Appl Math AMS. 17 (1965), 210-226.
16. Y. Ye, Global existence and asymptotic stability for coupled nonlinear Klein-Gordon equations with nonlinear damping terms. Dynamical Syst. 28(2) (2013), 287-298.
17. S.T. Wu, Blow-up results for system of nonlinear Klein-Gordon equations with arbitrary positive initial energ, Electron J Diff Equations. 2012 (2012), 1-13.
18. S.T. Wu, Blow-up of solutions for a system of nonlinear wave equations with nonlinear damping, Electron J Diff Equations. 2009 (2009), 1-11.

[1] Dicle University, Department of Mathematics, 21280 Diyarbakir, Turkey.
E-mail address: episkin@dicle.edu.tr