

SOME OBSERVATIONS ON NUMBERS IN SHORT INTERVALS

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ABSTRACT. We obtain some results about the primes and the prime factorizations of the positive integers in the interval $(x, x + x^\theta]$, where $0 < \theta < 1$. We also obtain some results about the primes and the prime factorizations of the positive integers in the interval $(x, x + f(x)]$, where $f(x)$ is a function of slow increase. For instance, $f(x) = \log^\alpha x$, where $\alpha > 0$.

1. INTRODUCTION AND MAIN RESULTS

There exist in the literature many works on the distribution of composite numbers in the interval $[1, x]$ with their prime factors restricted in some form. Numbers with exactly k prime factors in their prime factorization (see, for example, [4]), square-free numbers and in general k -free numbers (see, for example, [7]), square-full numbers and in general k -full numbers (see, for example, [5]), smooth numbers (see, for example, [11]), etc. Also, there exist works on special prime factors. Least prime factor (see, for example, [8]), greatest prime factor (see, for example, either [1] or [9]).

In this article we obtain some results about the primes and the prime factorizations of the positive integers in the short interval $(x, x + x^\theta]$, where $0 < \theta < 1$. We also obtain some results about the primes and the prime factorizations of the positive integers in the short interval $(x, x + f(x)]$, where $f(x)$ is a function of slow increase. For instance, either $f(x) = \log^\alpha x$ ($\alpha > 0$) or $f(x) = \log \log x$.

We need the following lemmas.

Lemma 1.1. (*Legendre's rule*) *Let p be a positive prime not exceeding x . Then the exponent of p in the prime factorization of $[x]!$ is*

$$\sum_{i=1}^{\infty} \left\lfloor \frac{x}{p^i} \right\rfloor.$$

Proof. See [2] (Chapter 3). □

Lemma 1.2. *The following asymptotic formula holds*

$$\log([x])! = x \log x - x + O(\log x).$$

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Proof. See [2] (Chapter 3). □

Lemma 1.3. *If $x \geq 0$ and $y \geq 0$ then the following inequality holds*

$$\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor \leq \lfloor x \rfloor + \lfloor y \rfloor + 1.$$

Proof. See [3] (Chapter 6). □

Lemma 1.4. *(Prime number theorem) We have*

$$\pi(x) \sim \frac{x}{\log x},$$

where $\pi(x)$ is the prime counting function.

We have

$$\sum_{p \leq x} \log p \sim x.$$

Proof. See [4]. □

Let $0 < \theta < 1$ be a positive real number. The prime factorization of $\lfloor x^\theta \rfloor!$ is

$$\lfloor x^\theta \rfloor! = \prod_{p \leq x^\theta} p^{A(p)} \quad (1.1)$$

and the prime factorization of $\prod_{x < n \leq x+x^\theta} n$ is

$$\prod_{x < n \leq x+x^\theta} n = \frac{\lfloor x+x^\theta \rfloor!}{\lfloor x \rfloor!} = \prod_{p \leq x^\theta} p^{E(p)} \prod_{x^\theta < p \leq x+x^\theta} p^{F(p)}, \quad (1.2)$$

where n denotes a positive integer and the exponents $F(p)$ are not necessarily greater than zero.

If n satisfies the inequality $x < n \leq x+x^\theta$ we can write $n = g(n)a(n)$, where either $g(n) = 1$ or $g(n)$ is a product of primes p such that $x^\theta < p \leq x+x^\theta$ and where either $a(n) = 1$ or $a(n)$ is a product of primes p such that $p \leq x^\theta$. In particular, if there exists a prime p such that $x < p \leq x+x^\theta$ then $g(p) = p$ and $a(p) = 1$. Therefore we have (see (1.2))

$$\prod_{\substack{x < n \leq x+x^\theta \\ g(n) \neq 1}} g(n) = \prod_{x^\theta < p \leq x+x^\theta} p^{F(p)} \quad (1.3)$$

and

$$\prod_{\substack{x < n \leq x+x^\theta \\ a(n) \neq 1}} a(n) = \prod_{p \leq x^\theta} p^{E(p)}. \quad (1.4)$$

The number of factors in the product $\prod_{\substack{x < n \leq x+x^\theta \\ g(n) \neq 1}} g(n)$ (see equation (1.3)) will be denoted $N(x)$. That is, $N(x)$ is the number of $g(n)$ such that $g(n) \neq 1$, where $x < n \leq x+x^\theta$. The number of distinct primes p such that $F(p) > 0$ in the right hand of equation (1.3) will be denoted $M(x)$.

Theorem 1.5. *The following asymptotic formulas hold*

$$\log ([x^\theta]!) = \log \left(\prod_{p \leq x^\theta} p^{A(p)} \right) = \theta x^\theta \log x - x^\theta + o(x^\theta), \quad (1.5)$$

$$\log \left(\prod_{x < n \leq x+x^\theta} n \right) = \log \left(\prod_{p \leq x^\theta} p^{E(p)} \prod_{x^\theta < p \leq x+x^\theta} p^{F(p)} \right) = x^\theta \log x + o(x^\theta), \quad (1.6)$$

$$\log \left(\prod_{\substack{x < n \leq x+x^\theta \\ a(n) \neq 1}} a(n) \right) = \log \left(\prod_{p \leq x^\theta} p^{E(p)} \right) = \theta x^\theta \log x + O(x^\theta), \quad (1.7)$$

$$\log \left(\prod_{\substack{x < n \leq x+x^\theta \\ g(n) \neq 1}} g(n) \right) = \log \left(\prod_{x^\theta < p \leq x+x^\theta} p^{F(p)} \right) = (1-\theta) x^\theta \log x + O(x^\theta). \quad (1.8)$$

If $n_1 \neq n_2$, where $x < n_1 \leq x+x^\theta$ and $x < n_2 \leq x+x^\theta$, then

$$\gcd(g(n_1), g(n_2)) = 1, \quad (1.9)$$

where $\gcd$ denotes the greatest common divisor.

The following inequalities hold for all $\epsilon > 0$, arbitrarily small, from a certain $x = x_\epsilon$

$$N(x) \geq (1-\theta-\epsilon) x^\theta, \quad (1.10)$$

$$M(x) \geq N(x) \geq (1-\theta-\epsilon) x^\theta. \quad (1.11)$$

Proof. Lemma 1.2 gives

$$\log ([x^\theta]!) = \theta x^\theta \log x - x^\theta + o(x^\theta). \quad (1.12)$$

Therefore equations (1.1) and (1.12) give equation (1.5).

Lemma 1.2 also gives

$$\begin{aligned} \log \left(\prod_{x < n \leq x+x^\theta} n \right) &= \log \left(\frac{[x+x^\theta]!}{[x]!} \right) = (x+x^\theta) \left(\log x + \log \left(1 + \frac{x^\theta}{x} \right) \right) \\ &\quad - x^\theta - x \log x + O(\log x) = x^\theta \log x + o(x^\theta), \end{aligned} \quad (1.13)$$

where we have used the formula $\log(1+y) \sim y$ as $y \rightarrow 0$. Therefore equations (1.2) and (1.13) give equation (1.6).

Let p be a positive prime such that $p \leq x^\theta$. If i is a positive integer then Lemma 1.3 gives

$$\left\lfloor \frac{x^\theta}{p^i} \right\rfloor \leq \left\lfloor \frac{x+x^\theta}{p^i} \right\rfloor - \left\lfloor \frac{x}{p^i} \right\rfloor \leq \left\lfloor \frac{x^\theta}{p^i} \right\rfloor + 1. \quad (1.14)$$

Let us consider the equality $p^j = x + x^\theta$. We have $j = \frac{\log(x+x^\theta)}{\log p}$. Therefore if we put $a_p = \left\lfloor \frac{\log(x+x^\theta)}{\log p} \right\rfloor + 1$ then we obtain (see (1.14))

$$\sum_{i=1}^{a_p} \left\lfloor \frac{x^\theta}{p^i} \right\rfloor \leq \sum_{i=1}^{a_p} \left\lfloor \frac{x + x^\theta}{p^i} \right\rfloor - \sum_{i=1}^{a_p} \left\lfloor \frac{x}{p^i} \right\rfloor \leq \sum_{i=1}^{a_p} \left\lfloor \frac{x^\theta}{p^i} \right\rfloor + a_p, \quad (1.15)$$

where by Lemma 1.1 and equations (1.1) and (1.2) we have

$$A(p) = \sum_{i=1}^{a_p} \left\lfloor \frac{x^\theta}{p^i} \right\rfloor, \quad (1.16)$$

$$E(p) = \sum_{i=1}^{a_p} \left\lfloor \frac{x + x^\theta}{p^i} \right\rfloor - \sum_{i=1}^{a_p} \left\lfloor \frac{x}{p^i} \right\rfloor. \quad (1.17)$$

Equations (1.15), (1.16) and (1.17) give

$$A(p) \leq E(p) \leq A(p) + a_p. \quad (1.18)$$

Equations (1.18) and (1.1) give

$$\lfloor x^\theta \rfloor! = \prod_{p \leq x^\theta} p^{A(p)} \leq \prod_{p \leq x^\theta} p^{E(p)} \leq \prod_{p \leq x^\theta} p^{A(p)+a_p} = \lfloor x^\theta \rfloor! \prod_{p \leq x^\theta} p^{a_p}. \quad (1.19)$$

Equation (1.19) gives

$$\log(\lfloor x^\theta \rfloor!) \leq \log \left(\prod_{p \leq x^\theta} p^{E(p)} \right) \leq \log(\lfloor x^\theta \rfloor!) + \log \left(\prod_{p \leq x^\theta} p^{a_p} \right). \quad (1.20)$$

Now, by Lemma 1.4 we have

$$\begin{aligned} 0 &\leq \log \left(\prod_{p \leq x^\theta} p^{a_p} \right) = \sum_{p \leq x^\theta} a_p \log p = \sum_{p \leq x^\theta} \left(\left\lfloor \frac{\log(x+x^\theta)}{\log p} \right\rfloor + 1 \right) \log p \\ &\leq \log(x+x^\theta) \pi(x^\theta) + \sum_{p \leq x^\theta} \log p \leq cx^\theta, \end{aligned} \quad (1.21)$$

where c is a positive constant.

Equations (1.4), (1.5), (1.20) and (1.21) give equation (1.7).

Equations (1.3), (1.6) and (1.7) give equation (1.8).

Suppose $x < n_1 < n_2 \leq x + x^\theta$. If either $g(n_1) = 1$ or $g(n_2) = 1$ then clearly (1.9) holds. Suppose that $g(n_1) \neq 1$, $g(n_2) \neq 1$ and suppose that there exists a prime p such that $g(n_1)$ and $g(n_2)$ are multiple of p . Then n_1 and n_2 are multiple of p and we obtain

$$x^\theta = (x + x^\theta) - x \geq n_2 - n_1 = pk > x^\theta. \quad (1.22)$$

That is, an evident contradiction. Therefore (1.9) holds. Note that in (1.22) $p > x^\theta$ (by definition of $g(n)$) and $k \geq 1$.

Suppose that $x < n \leq x + x^\theta$. We have $\log g(n) \leq \log n \leq \log(x + x^\theta) = \log x + o(1)$. Therefore we have (see (1.8))

$$(1 - \theta)x^\theta \log x + O(x^\theta) \leq N(x) (\log x + o(1)).$$

From here, by use of the equation $\frac{1}{1+o(\frac{1}{\log x})} = 1 + o(\frac{1}{\log x})$, we find inequality (1.10).

Inequality (1.11) is an immediate consequence of inequalities (1.3), (1.9) and (1.10). Since if $g(n) \neq 1$ then it has at least one prime factor. The theorem is proved. $\square$

Let $\Omega(g(n))$ ($g(n) \neq 1$) be the total number of prime factors in the prime factorization of $g(n)$. Clearly, as $g(n) \neq 1$ then $\Omega(g(n)) \geq 1$.

Theorem 1.5 holds in the particular case $\theta = \frac{1}{k}$ ($k \geq 2$) and $x = s^k$, where s is a positive integer. In this case we can add the following results. The interval $[1, x^\theta = s]$ contains the s positive integers $1, 2, \dots, s$ and the interval $(x = s^k, x + x^\theta = s^k + s]$ contains the s positive integers $s^k + 1, s^k + 2, \dots, s^k + s$. In this case the inequality $p > x^\theta$ becomes $p \geq s + 1$. Therefore $1 \leq \Omega(g(n)) \leq k - 1$ ($g(n) \neq 1$). Since $(s + 1)^k > s^k + s = x + x^\theta$.

Therefore, in the case $\theta = \frac{1}{2}$, we have $\Omega(g(n)) = 1$ ($g(n) \neq 1$) and the corresponding $a(n)$ satisfies $1 \leq a(n) \leq \frac{s^2 + s}{s + 1} = s$.

Definition 1.6. Let $f(x)$ be a function defined on the interval $[a, \infty)$ such that $f(x) > 0$, $\lim_{x \rightarrow \infty} f(x) = \infty$ and with continuous derivative $f'(x) > 0$. The function $f(x)$ is of slow increase if and only if the following limit holds

$$\lim_{x \rightarrow \infty} \frac{xf'(x)}{f(x)} = 0. \quad (1.23)$$

Typical functions of slow increase are $\log^\alpha x$ ($\alpha > 0$), $\log \log x$, $\frac{\log x}{\log \log x}$, etc. The functions of slow increase are studied in [6]. If $f(x)$ is a function of slow increase then $\log(f(x))$ is also a function of slow increase (see [6]).

The functions $f(x)$ of slow increase have the following property

$$\lim_{x \rightarrow \infty} \frac{f(x)}{x^\alpha} = 0, \quad (1.24)$$

for all $\alpha > 0$. See [6].

On the other hand, we have the following limit (use L'Hospital's rule and (1.23))

$$\lim_{x \rightarrow \infty} \frac{\log(f(x))}{\log x} = 0. \quad (1.25)$$

Let $f(x)$ be a function of slow increase. The prime factorization of $\lfloor f(x) \rfloor!$ is

$$\lfloor f(x) \rfloor! = \prod_{p \leq f(x)} p^{A(p)} \quad (1.26)$$

and the prime factorization of $\prod_{x < n \leq x+f(x)} n$ is

$$\prod_{x < n \leq x+f(x)} n = \frac{\lfloor x+f(x) \rfloor!}{\lfloor x \rfloor!} = \prod_{p \leq f(x)} p^{E(p)} \prod_{f(x) < p \leq x+f(x)} p^{F(p)}, \quad (1.27)$$

where n denotes a positive integer and the exponents $F(p)$ are not necessarily greater than zero.

If n satisfies the inequality $x < n \leq x+f(x)$ we can write $n = g(n)a(n)$, where either $g(n) = 1$ or $g(n)$ is a product of primes p such that $f(x) < p \leq x+f(x)$ and where either $a(n) = 1$ or $a(n)$ is a product of primes p such that $p \leq f(x)$. In particular, if there exists a prime p such that $x < p \leq x+f(x)$ then $g(p) = p$ and $a(p) = 1$. Therefore we have

$$\prod_{\substack{x < n \leq x+f(x) \\ g(n) \neq 1}} g(n) = \prod_{f(x) < p \leq x+f(x)} p^{F(p)} \quad (1.28)$$

and

$$\prod_{\substack{x < n \leq x+f(x) \\ a(n) \neq 1}} a(n) = \prod_{p \leq f(x)} p^{E(p)}. \quad (1.29)$$

The number of factors in the product (see (1.28)) $\prod_{\substack{x < n \leq x+f(x) \\ g(n) \neq 1}} g(n)$ will be denoted $N(x)$. That is, $N(x)$ is the number of $g(n)$ such that $g(n) \neq 1$, where $x < n \leq x+f(x)$. The number of distinct primes p such that $F(p) > 0$ in the right hand of equation (1.28) will be denoted $M(x)$.

Theorem 1.7. *The following asymptotic formulas hold*

$$\log(\lfloor f(x) \rfloor!) = \log \left(\prod_{p \leq f(x)} p^{A(p)} \right) \sim f(x) \log f(x), \quad (1.30)$$

$$\log \left(\prod_{x < n \leq x+f(x)} n \right) = \log \left(\prod_{p \leq f(x)} p^{E(p)} \prod_{f(x) < p \leq x+f(x)} p^{F(p)} \right) \sim f(x) \log x, \quad (1.31)$$

$$\log \left(\prod_{\substack{x < n \leq x+f(x) \\ a(n) \neq 1}} a(n) \right) = \log \left(\prod_{p \leq f(x)} p^{E(p)} \right) = o(f(x) \log x), \quad (1.32)$$

$$\log \left(\prod_{\substack{x < n \leq x+f(x) \\ g(n) \neq 1}} g(n) \right) = \log \left(\prod_{f(x) < p \leq x+f(x)} p^{F(p)} \right) \sim f(x) \log x. \quad (1.33)$$

If $n_1 \neq n_2$, where $x < n_1 \leq x+f(x)$ and $x < n_2 \leq x+f(x)$, then

$$\gcd(g(n_1), g(n_2)) = 1, \quad (1.34)$$

where $\gcd$ denotes the greatest common divisor.

$$N(x) \sim f(x). \quad (1.35)$$

$$M(x) \geq N(x) \sim f(x). \quad (1.36)$$

Proof. Equation (1.30) is an immediate consequence of Lemma 1.2.

Note that the number of positive integers n such that $x < n \leq x + f(x)$ is $f(x) + O(1)$ and besides taking logarithm in both sides we obtain $\log x < \log n \leq \log(x + f(x)) = \log x + o(1)$ (see (1.24)). Now

$$\log \left(\prod_{x < n \leq x + f(x)} n \right) = \sum_{x < n \leq x + f(x)} \log n$$

and consequently

$$\begin{aligned} f(x) \log x &\sim (f(x) + O(1)) \log x \leq \sum_{x < n \leq x + f(x)} \log n \\ &\leq (\log x + o(1)) (f(x) + O(1)) \sim f(x) \log x. \end{aligned} \quad (1.37)$$

Equations (1.27) and (1.37) give equation (1.31).

As in the proof of Theorem 1.5 we find that (note that now, $a_p = \left\lfloor \frac{\log(x+f(x))}{\log p} \right\rfloor + 1$) (see (1.30), Lemma 1.4 and (1.25))

$$\begin{aligned} \log \left(\prod_{p \leq f(x)} p^{E(p)} \right) &\leq \log \left(\prod_{p \leq f(x)} p^{A(p)} \right) + \log \left(\prod_{p \leq f(x)} p^{a_p} \right) \\ &\leq \log([f(x)]!) + \log(x + f(x)) \sum_{p \leq f(x)} 1 + \sum_{p \leq f(x)} \log p \\ &= h_1(x) f(x) \log f(x) + h_2(x) (\log x + o(1)) \frac{f(x)}{\log f(x)} + h_3(x) f(x) \\ &= o(f(x) \log x), \end{aligned}$$

where $h_i(x) \rightarrow 1$ as $x \rightarrow \infty$ ($i = 1, 2, 3$). Therefore equation (1.32) is proved.

Equation (1.33) is an immediate consequence of equations (1.31) and (1.32).

The proof of equation (1.34) is the same as the proof of equation (1.9) in Theorem 1.5.

We have (see above in the proof) $\log g(n) \leq \log n \leq \log(x + f(x)) = \log x + o(1)$. Therefore (see equation (1.33)) we obtain

$$h_4(x) f(x) \log x \leq N(x) (\log x + o(1)), \quad (1.38)$$

where $h_4(x) \rightarrow 1$ as $x \rightarrow \infty$. Equation (1.38) gives

$$N(x) \geq h_5(x) f(x), \quad (1.39)$$

where $h_5(x) \rightarrow 1$ as $x \rightarrow \infty$.

On the other hand $N(x)$ does not exceed the number of positive integers n such that $x < n \leq x + f(x)$ and this number is $f(x) + O(1)$ (see above in the proof). Therefore

$$N(x) \leq f(x) + O(1). \quad (1.40)$$

Inequalities (1.39) and (1.40) give (1.35).

Inequality (1.36) is an immediate consequence of the definition of $M(x)$, equation (1.34) and equation (1.35). The theorem is proved. $\square$

Let $\Omega(n)$ be (as usual, see [4]) the total number of prime factors in the prime factorization of n . The following formula holds (see [4])

$$\sum_{n \leq x} \Omega(n) = x \log \log x + Cx + o(x), \quad (1.41)$$

where C is a constant.

We have the following theorem, where we use the definitions of $a(n)$ and $g(n)$ above Theorem 1.5.

Theorem 1.8. *The following asymptotic formulas hold*

$$\sum_{n \leq x^\theta} \Omega(n) = \sum_{p \leq x^\theta} A(p) = x^\theta \log \log x + O(x^\theta), \quad (1.42)$$

$$\sum_{\substack{x < n \leq x + x^\theta \\ a(n) \neq 1}} \Omega(a(n)) = \sum_{p \leq x^\theta} E(p) = x^\theta \log \log x + O(x^\theta), \quad (1.43)$$

$$\sum_{\substack{x < n \leq x + x^\theta \\ g(n) \neq 1}} \Omega(g(n)) = \sum_{x^\theta < p \leq x + x^\theta} F(p) = O(x^\theta), \quad (1.44)$$

$$\sum_{x < n \leq x + x^\theta} \Omega(n) = x^\theta \log \log x + O(x^\theta), \quad (1.45)$$

$$\sum_{x < n \leq x + x^\theta} \Omega(n) \sim \sum_{n \leq x^\theta} \Omega(n) \sim x^\theta \log \log x. \quad (1.46)$$

Proof. Equation (1.42) is an immediate consequence of equation (1.41) and equation (1.1).

Equation (1.18) gives

$$\sum_{p \leq x^\theta} A(p) \leq \sum_{p \leq x^\theta} E(p) \leq \sum_{p \leq x^\theta} A(p) + \sum_{p \leq x^\theta} a(p). \quad (1.47)$$

Now,

$$\begin{aligned} \sum_{p \leq x^\theta} a(p) &= \sum_{p \leq x^\theta} \left(\left\lfloor \frac{\log(x + x^\theta)}{\log p} \right\rfloor + 1 \right) \leq \frac{\log(x + x^\theta)}{\log 2} h_1(x) \frac{x^\theta}{\theta \log x} \\ &+ h_1(x) \frac{x^\theta}{\theta \log x} \leq cx^\theta, \end{aligned} \quad (1.48)$$

where $h_1(x) \rightarrow 1$ as $x \rightarrow \infty$ and c is a positive constant.

Therefore equations (1.4), (1.42), (1.47) and (1.48) give equation (1.43).

The number of positive integers n such that $x < n \leq x + x^\theta$ is $x^\theta + O(1)$. On the other hand, there exists a positive integer k such that $(x^\theta)^k > x + x^\theta$ and consequently $\Omega(g(n)) \leq k$. Hence (see (1.3))

$$\sum_{\substack{x < n \leq x + x^\theta \\ g(n) \neq 1}} \Omega(g(n)) = \sum_{x^\theta < p \leq x + x^\theta} F(p) \leq k(x^\theta + O(1)).$$

That is, equation (1.44).

Equation (1.45) is an immediate consequence of equations (1.43) and (1.44).

Equation (1.46) is an immediate consequence of equations (1.42) and (1.45).

The theorem is proved. $\square$

Now, we examine the large interval $[1, x]$ and we compare with the short interval $(x, x + x^\theta]$.

Theorem 1.9. *We have the following asymptotic formulas*

$$\sum_{n \leq x} \log n = \sum_{n \leq x} \log(a(n)g(n)) = x \log x + O(x), \quad (1.49)$$

$$\sum_{\substack{n \leq x \\ a(n) \neq 1}} \log a(n) = \theta x \log x + O(x), \quad (1.50)$$

$$\sum_{\substack{n \leq x \\ g(n) \neq 1}} \log g(n) = (1 - \theta) x \log x + O(x), \quad (1.51)$$

$$\sum_{n \leq x} \Omega(n) = \sum_{n \leq x} (\Omega(a(n)) + \Omega(g(n))) = x \log \log x + Cx + o(x), \quad (1.52)$$

where C is a positive constant.

$$\sum_{\substack{n \leq x \\ a(n) \neq 1}} \Omega(a(n)) = x \log \log x + (C + \log \theta)x + o(x), \quad (1.53)$$

$$\sum_{\substack{n \leq x \\ g(n) \neq 1}} \Omega(g(n)) = (-\log \theta)x + o(x). \quad (1.54)$$

Proof. Equation (1.49) is Lemma 1.2. We use the definitions of $a(n)$ and $g(n)$ above Theorem 1.5.

We have the well-known formula (see, for example, [4])

$$\sum_{p \leq x} \frac{\log p}{p} = \log x + O(1).$$

On the other hand, there exists a positive integer k such that $(x^\theta)^k > x$. Therefore Lemma 1.1 and Lemma 1.4 give

$$\begin{aligned}
\sum_{\substack{n \leq x \\ g(n) \neq 1}} \log g(n) &= \sum_{x^\theta < p \leq x} \log p \left(\left\lfloor \frac{x}{p} \right\rfloor + \left\lfloor \frac{x}{p^2} \right\rfloor + \cdots + \left\lfloor \frac{x}{p^k} \right\rfloor \right) \\
&= x \sum_{x^\theta < p \leq x} \left(\frac{\log p}{p} + \frac{\log p}{p^2} + \cdots + \frac{\log p}{p^k} \right) + O(x) = x \sum_{x^\theta < p \leq x} \frac{\log p}{p} + O(x) \\
&= (1 - \theta) x \log x + O(x).
\end{aligned}$$

That is, equation (1.51).

Equations (1.49) and (1.51) give equation (1.50).

Equation (1.52) is equation (1.41). We have the well-known formula (see, for example, [4])

$$\sum_{p \leq x} \frac{1}{p} = \log \log x + M + o(1),$$

where M is Mertens's constant.

On the other hand, there exists a positive integer k such that $(x^\theta)^k > x$. Therefore Lemma 1.1 and Lemma 1.4 give

$$\begin{aligned}
\sum_{\substack{n \leq x \\ g(n) \neq 1}} \Omega(g(n)) &= \sum_{x^\theta < p \leq x} \left(\left\lfloor \frac{x}{p} \right\rfloor + \left\lfloor \frac{x}{p^2} \right\rfloor + \cdots + \left\lfloor \frac{x}{p^k} \right\rfloor \right) \\
&= x \sum_{x^\theta < p \leq x} \left(\frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^k} \right) + o(x) = x \sum_{x^\theta < p \leq x} \frac{1}{p} + o(x) \\
&= (-\log \theta) x + o(x).
\end{aligned}$$

That is, equation (1.54).

Equations (1.52) and (1.54) give equation (1.53). The theorem is proved. $\square$

Compare Theorem 1.5 and Theorem 1.8 with Theorem 1.9.

Now, we examine the Alladi-Erdős function $S(n)$ in the large interval $[1, x]$ and we compare with the short interval $(x, x + x^\theta]$. The Alladi-Erdős function $S(n)$ is the sum of all primes in the prime factorization of n . For example, if $n = 3^4 7^2 19$ then $S(n) = 3 + 3 + 3 + 3 + 7 + 7 + 19$. Alladi and Erdős proved that (see [1])

$$\sum_{n \leq x} S(n) = \frac{\pi^2}{6} \frac{1}{2} \frac{x^2}{\log x} + o\left(\frac{x^2}{\log x}\right). \quad (1.55)$$

Theorem 1.10. *The following asymptotic formulas and limits hold*

$$\sum_{n \leq x} S(n) = \sum_{n \leq x} (S(a(n)) + S(g(n))) = \frac{\pi^2}{6} \frac{1}{2} \frac{x^2}{\log x} + o\left(\frac{x^2}{\log x}\right), \quad (1.56)$$

$$\sum_{n \leq x} S(a(n)) = \frac{x^{1+\theta}}{\theta \log x} + O\left(\frac{x^{1+\theta}}{\log^2 x}\right), \quad (1.57)$$

$$\sum_{n \leq x} S(g(n)) = \frac{\pi^2}{6} \frac{1}{2} \frac{x^2}{\log x} + o\left(\frac{x^2}{\log x}\right), \quad (1.58)$$

$$\lim_{x \rightarrow \infty} \frac{\sum_{n \leq x} S(g(n))}{\sum_{n \leq x} S(a(n))} = \infty. \quad (1.59)$$

Proof. Equation (1.56) is equation (1.55). Limit (1.59) is an immediate consequence of equations (1.57) and (1.58). Equation (1.58) is an immediate consequence of equations (1.56) and (1.57). Therefore we have to prove equation (1.57). Note that it is well-known (see [6]) the formulas

$$\sum_{p \leq x} p = \frac{1}{2} \frac{x^2}{\log x} + o\left(\frac{x^2}{\log x}\right), \quad (1.60)$$

$$\sum_{p \leq x} \frac{1}{p} \sim \log \log x, \quad (1.61)$$

and the following stronger form of the prime number theorem [10]

$$\pi(x) = \sum_{p \leq x} 1 = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right).$$

Let us consider the equation $2^i = x$, then $i = \frac{\log x}{\log 2}$. Consequently if we put $s = \left\lfloor \frac{\log x}{\log 2} \right\rfloor + 1$ then

$$\begin{aligned} & \sum_{n \leq x} S(a(n)) \\ &= \sum_{p \leq x^\theta} p \left(\left\lfloor \frac{x}{p} \right\rfloor + \left\lfloor \frac{x}{p^2} \right\rfloor + \cdots \right) = \sum_{p \leq x^\theta} p \left(\sum_{k=1}^s \left\lfloor \frac{x}{p^k} \right\rfloor \right) = x \sum_{p \leq x^\theta} \left(\sum_{k=1}^s \frac{1}{p^{k-1}} \right) \\ &- \sum_{p \leq x^\theta} p \left(\sum_{k=1}^s \left\{ \frac{x}{p^k} \right\} \right) = x \sum_{p \leq x^\theta} \frac{1 - \frac{1}{p^s}}{1 - \frac{1}{p}} - \sum_{p \leq x^\theta} p \left(\sum_{k=1}^s \left\{ \frac{x}{p^k} \right\} \right) \\ &= x \sum_{p \leq x^\theta} 1 + x \sum_{p \leq x^\theta} \frac{1}{p-1} - x \sum_{p \leq x^\theta} \frac{1}{p^s} \frac{1}{1 - \frac{1}{p}} - \sum_{p \leq x^\theta} p \left(\sum_{k=1}^s \left\{ \frac{x}{p^k} \right\} \right) \\ &= \frac{x^{1+\theta}}{\theta \log x} + O\left(\frac{x^{1+\theta}}{\log^2 x}\right). \end{aligned}$$

That is, equation (1.57). Note that (see (1.60))

$$0 \leq \sum_{p \leq x^\theta} p \left(\sum_{k=1}^s \left\{ \frac{x}{p^k} \right\} \right) \leq s \sum_{p \leq x^\theta} p = \left(\left\lfloor \frac{\log x}{\log 2} \right\rfloor + 1 \right) h_1(x) \frac{x^{2\theta}}{2\theta \log x} < c_1 x^{2\theta},$$

where $h_1(x) \rightarrow 1$ as $x \rightarrow \infty$ and c_1 is a positive constant.

Note also that (see (1.61))

$$x \sum_{p \leq x^\theta} \frac{1}{p-1} \leq 2x \sum_{p \leq x^\theta} \frac{1}{p} = h_2(x)x \log \log x,$$

where $h_2(x) \rightarrow 1$ as $x \rightarrow \infty$.

Finally, note that

$$0 \leq \sum_{p \leq x^\theta} \frac{1}{p^s} \frac{1}{1 - \frac{1}{p}} \leq 2 \sum_{p \leq x^\theta} \frac{1}{p^s} \leq 2 \sum_p \frac{1}{p^s} = o(1),$$

since (as it is well-known) $(\zeta(s) - 1) \rightarrow 0$ as $s \rightarrow \infty$, where $\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \dots$ (as usual) denotes the zeta function.

Note that if we desired a more precise formula than (1.57) (an asymptotic expansion) we can use the well-known asymptotic expansion for $\pi(x)$ (see [10]) namely

$$\pi(x) = \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2!x}{\log^3 x} + \dots + o\left(\frac{x}{\log^m x}\right). \quad (1.62)$$

The theorem is proved. $\square$

Theorem 1.11. *The following formulas hold*

$$\sum_{n \leq x^\theta} S(n) = \sum_{p \leq x^\theta} pA(p) \sim \frac{1}{\theta} \frac{1}{2} \frac{\pi^2}{6} \frac{x^{2\theta}}{\log x}, \quad (1.63)$$

$$c_2 \frac{x^{2\theta}}{\log x} < \sum_{\substack{x < n \leq x+x^\theta \\ a(n) \neq 1}} S(a(n)) = \sum_{p \leq x^\theta} pE(p) < c_1 \frac{x^{2\theta}}{\log x}, \quad (1.64)$$

where c_1 and c_2 are positive constants.

$$c_3 x^{2\theta} < \sum_{\substack{x < n \leq x+x^\theta \\ g(n) \neq 1}} S(g(n)) \leq \sum_{x < n \leq x+x^\theta} S(n) \leq \sum_{x < n \leq x+x^\theta} n = h_1(x)x^{1+\theta}, \quad (1.65)$$

where c_3 is a positive constant and $h_1(x) \rightarrow 1$ as $x \rightarrow \infty$

The following limit holds.

$$\lim_{x \rightarrow \infty} \frac{\sum_{\substack{x < n \leq x+x^\theta \\ g(n) \neq 1}} S(g(n))}{\sum_{\substack{x < n \leq x+x^\theta \\ a(n) \neq 1}} S(a(n))} = \infty. \quad (1.66)$$

Proof. Equation (1.63) is an immediate consequence of equation (1.55).

Equation (1.60) and Abel summation give

$$\sum_{p \leq x} \frac{p}{\log p} = \frac{x^2}{2 \log^2 x} + o\left(\frac{x^2}{\log^2 x}\right). \quad (1.67)$$

Equation (1.18) gives

$$pA(p) \leq pE(p) \leq pA(p) + pa_p. \quad (1.68)$$

Now, equation (1.60) and equation (1.65) give

$$\begin{aligned} \sum_{p \leq x^\theta} p a_p &= \sum_{p \leq x^\theta} p \left(\left\lfloor \frac{\log(x + x^\theta)}{\log p} \right\rfloor + 1 \right) \leq \log(x + x^\theta) \sum_{p \leq x^\theta} \frac{p}{\log p} + \sum_{p \leq x^\theta} p \\ &< c_4 \frac{x^{2\theta}}{\log x}, \end{aligned} \quad (1.69)$$

where c_4 is a positive constant.

Equations (1.63), (1.66) and (1.67) give equation (1.64).

On the other hand, we have (see (1.10))

$$\sum_{\substack{x < n \leq x + x^\theta \\ g(n) \neq 1}} S(g(n)) \geq x^\theta M(x) > c_3 x^{2\theta}.$$

The number of n such that $x < n \leq x + x^\theta$ is $x^\theta + O(1)$. Consequently

$$x^{1+\theta} \sim x(x^\theta + O(1)) \leq \sum_{x < n \leq x + x^\theta} n \leq (x + x^\theta)(x^\theta + O(1)) \sim x^{1+\theta}.$$

Therefore equation (1.65) is proved.

Limit (1.66) is an immediate consequence of (1.64) and (1.65). The theorem is proved. $\square$

Let us consider an interval $(a, b]$, where $0 < a < b$ are real numbers. Let N_1 be the number of positive integers n such that $a < n \leq b$. Then it is easy to prove that $|N_1 - (b - a)| \leq 3$. Therefore if c is a positive integer then the number N_c of multiple of c in the interval satisfies $|N_c - \frac{b-a}{c}| \leq 3$, since the inequality $a < cn \leq b$ is equivalent to the inequality $\frac{a}{c} < n \leq \frac{b}{c}$. Consequently we can write $N_c = \frac{b-a}{c} + O(1)$, where $|O(1)| \leq 3$.

Theorem 1.12. *Let $0 < \theta < 1$ be. Let us consider the interval $I_x = (x, x + x^\theta]$. The number of primes p in the interval (if there is some) will be denoted $\pi(I_x)$. Then $\pi(I_x) = \sum_{x < p \leq x + x^\theta} 1$. The following formulas hold*

$$\pi(I_x) = o(x^\theta) \quad (x \rightarrow \infty), \quad (1.70)$$

$$\sum_{x < p \leq x + x^\theta} p = o(x^{1+\theta}) \quad (x \rightarrow \infty).$$

Proof. Let p_n be the n -th prime number. It is well-known the limit (see [4])

$$\lim_{k \rightarrow \infty} \prod_{i=1}^k \left(1 - \frac{1}{p_i} \right) = 0.$$

Therefore given $\epsilon > 0$ there exists k_ϵ such that

$$\prod_{i=1}^{k_\epsilon} \left(1 - \frac{1}{p_i} \right) < \frac{\epsilon}{2}.$$

If we eliminate the multiples of p_i ($i = 1, 2, \dots, k$) in the interval $I_x = (x, x + x^\theta]$ then the prime numbers are among the rest of the numbers. Therefore the inclusion-exclusion principle gives

$$\begin{aligned} \pi(I_x) &\leq (x^\theta + O(1)) - \sum_{1 \leq i \leq k} \left(\frac{x^\theta}{p_i} + O(1) \right) + \sum_{1 \leq i < j \leq k} \left(\frac{x^\theta}{p_i p_j} + O(1) \right) - \dots \\ &= x^\theta \prod_{i=1}^{k_\epsilon} \left(1 - \frac{1}{p_i} \right) + O_f(1) < \epsilon x^\theta \quad (x \geq x_\epsilon), \end{aligned}$$

where $|O_f(1)| \leq 3(2^k)$. Therefore, limit (1.70) is proved. Now, we have

$$o(x^{1+\theta}) = x o(x^\theta) \leq \sum_{x < p \leq x+x^\theta} p \leq (x + x^\theta) o(x^\theta) = o(x^{1+\theta}).$$

The theorem is proved. $\square$

In the next theorem we obtain a more precise result than equation (1.65).

Theorem 1.13. *The following formulas hold*

$$\sum_{\substack{x < n \leq x+x^\theta \\ g(n) \neq 1}} S(g(n)) = o(x^{1+\theta}) \quad (x \rightarrow \infty), \quad (1.71)$$

$$\sum_{x < n \leq x+x^\theta} S(n) = o(x^{1+\theta}) \quad (x \rightarrow \infty). \quad (1.72)$$

Proof. The following formula is well-known

$$\sum_{t=1}^s \frac{1}{t} = h_2(s) \log s, \quad (1.73)$$

where $h_2(s) \rightarrow 1$ as $s \rightarrow \infty$.

There exists a positive integer k such that $(x^\theta)^k > x + x^\theta$.

Let $\epsilon > 0$ be. There exists h depending of ϵ such that

$$\sum_{t=h}^{\infty} \frac{1}{t^2} < \frac{\epsilon}{2k}. \quad (1.74)$$

Let us consider the interval $I_{x,t} = \left(\frac{x}{t}, \frac{x+x^\theta}{t} \right]$, where t is a positive integer. Let $N(I_{x,t})$ be the number of positive integers in the interval, then $N(I_{x,t}) = \frac{x^\theta}{t} + O(1)$, where $|O(1)| \leq 3$. Let $\pi(I_{x,t})$ be the number of prime numbers in the interval. If $t = 1, 2, \dots, h-1$ we have $\pi(I_{x,t}) = o\left(\frac{x^\theta}{t}\right)$ (the proof is the same as the proof of Theorem 1.12).

We recall that in the interval $I(x, 1) = (x, x + x^\theta]$ there are a number $N(x)$ of n_i ($i = 1, 2, \dots, N(x)$) such that $g(n_i) \neq 1$ and $\gcd(g(n_i), g(n_j)) = 1$ if $i \neq j$. The number of prime factors in the prime factorization of $g(n_i)$ does not exceed k , since $(x^\theta)^k > x + x^\theta$ (see above), that is $1 \leq \Omega(g(n_i)) \leq k$ ($i = 1, 2, \dots, N(x)$). Let P_i be the greatest prime factor in the prime factorization of $g(n_i)$. We have

$x < n_i = g(n_i)a(n_i) = P_i t \leq x + x^\theta$ and therefore $\frac{x}{t} < P_i \leq \frac{x+x^\theta}{t}$. Note that the inequality $\frac{x+x^\theta}{t} < x^\theta$ holds if $t \geq s = \lfloor x^{1-\theta} + 1 \rfloor + 1$. Consequently (see (1.74))

$$\begin{aligned}
\sum_{\substack{x < n \leq x+x^\theta \\ g(n) \neq 1}} S(g(n)) &\leq k \sum_{t=1}^s \frac{x+x^\theta}{t} \pi(I_{x,t}) \leq k \sum_{t=1}^{h-1} \frac{x+x^\theta}{t} \pi(I_{x,t}) \\
&+ k \sum_{t=h}^s \frac{x+x^\theta}{t} N(I_{x,t}) = k \sum_{t=1}^{h-1} \frac{x+x^\theta}{t} o\left(\frac{x^\theta}{t}\right) + k \sum_{t=h}^s \frac{x+x^\theta}{t} \left(\frac{x^\theta}{t} + O(1)\right) \\
&= k \sum_{t=1}^{h-1} o(x^{1+\theta}) + k(x+x^\theta)x^\theta \sum_{t=h}^\infty \frac{1}{t^2} + k(x+x^\theta) \sum_{t=h}^s \frac{O(1)}{t} \\
&\leq ko(x^{1+\theta}) + kh_1(x)x^{1+\theta} \frac{\epsilon}{2k} + kh_4(x)xO(\log x) \leq k2\frac{\epsilon}{2k}x^{1+\theta} \\
&= \epsilon x^{1+\theta} \quad (x \geq x_\epsilon),
\end{aligned}$$

where $h_1(x) \rightarrow 1$ and $h_4(x) \rightarrow 1$ as $x \rightarrow \infty$. Therefore equation (1.71) is proved, since $\epsilon > 0$ can be arbitrarily small.

Note that (see equation (1.73))

$$\begin{aligned}
\left| \sum_{t=h}^s \frac{O(1)}{t} \right| &\leq \sum_{t=h}^s \frac{|O(1)|}{t} \leq 3 \sum_{t=h}^s \frac{1}{t} \leq 3 \sum_{t=1}^s \frac{1}{t} = 3h_2(s) \log s \\
&= 3h_3(x)(1-\theta) \log x \leq 6(1-\theta) \log x,
\end{aligned} \tag{1.75}$$

where $h_3(x) \rightarrow 1$ as $x \rightarrow \infty$.

Now, $S(n) = S(a(n)g(n)) = S(a(n)) + S(g(n))$, and consequently equation (1.72) is an immediate consequence of equations (1.71) and (1.64). The theorem is proved. $\square$

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