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ON THE X-RANKS ASSOCIATED TO A LINEAR SUBSPACE

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Abstract. Let $X \subset \mathbb{P}^N$ be an integral non-degenerate variety. For any $q \in \mathbb{P}^N$ the X-rank $r_X(q)$ of q is the minimal cardinality of a set $S \subset X$ whose linear span contains q. For any r-dimensional linear subspace $V \subset \mathbb{P}^N$ the X-rank $r_X(V)$ of V is the minimal cardinality of a set $S \subset X$ whose linear span contains V. We define several invariants related to the X-rank, e.g. the open rank of V, $r+1$ integers $r_X(q_i)$, $0 \le i \le r$, with $r_X(q_i) \le r_X(q_j)$ for all $i < j$ and $q_0, \dots, q_r$ a basis of V and minimal $\eta_X(V) := r_X(q_0) + \cdots + r_X(q_r)$. We always have $\eta_X(V) \ge r_X(V)$ and we give an upper bound for $\eta_X(V)$, which ensures that we may compute $r_X(V)$ with knowledge of $q_0, \dots, q_r$ and in particular check if $\eta_X(V) = r_X(V)$ (the general case, but not always). The integer $r_X(q_r)$ is the minimal integer t such that V is spanned by points of X-rank $\le t$. Many of these invariants of V give lower bounds for the invariants of any linear subspace of V.

1. The main definitions and the main results

Let $X \subset \mathbb{P}^N$ be an integral and non-degenerate variety over $\mathbb{C}$ (but we also consider the real field $\mathbb{R}$ in section 3, while Proposition 1.3 and the definitions of the integers $\eta_X(V)$, $r_{X,i}(V)$, $0 \le i \le \dim(V)$ work over an arbitrary field). For each $q \in \mathbb{P}^N$ the X-rank $r_X(q)$ of q is the minimal cardinality of a finite set $S \subset X$ such that $q \in \langle S \rangle$, where $\langle\ \rangle$ denotes the linear span. When $N+1 = \prod_{i=1}^{k}(n_i+1)$, $k \ge 2$, and X is the Segre embedding of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ the X-rank $r_X(q)$ of q is the tensor rank of the tensor of format $(n_1+1, \dots, n_k+1)$ associated to q. When $k = 2$ and X is the Segre embedding of $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ the tensor rank $r_X(q)$ is just the rank of the $(n_1+1) \times (n_2+1)$ matrix associated to q, but even in this classical case this notion leads to many research topics if (as in this paper) instead of a single point q we consider a linear space $V \subset \mathbb{P}^N$. Rank metric codes can be seen as a vector space of $n_1 \times n_2$ matrices where the entries are from a finite field and the rank distance is the rank of the difference between two matrices ([17], [21], [26], [30], [30], [32], [34]). The motivation of many of these works came from network codings ([27]). For linear spaces of matrices over a field with an upper/lower bound of their rank or of constant rank, see [4], [10], [17], [19], [28], [29]. When $N+1 = \binom{n+d}{n}$ with n a positive integer, $d \ge 2$, and X is the order d Veronese embedding of $\mathbb{P}^n$, then $r_X(q)$ is the minimal number of addendum of the

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degree d homogeneous polynomial $f(x_0,\dots,x_n) \in \mathbb{C}[x_0,\dots,x_n]$ associated to q in a decomposition $f(x_0,\dots,x_n) = \sum_i \ell_i(x_0,\dots,x_n)^d$, where each $\ell_i(x_0,\dots,x_n)$ is a linear form; for this set-up the papers [15], [18], [23], [33, 1.2] look at the rank of a general r-dimensional linear subspace. Abo and Wan considered the case of several anti-symmetric forms ([1]).

For any integer $t > 0$ the t-secant variety $\sigma_t(X)$ of X is the closure in $\mathbb{P}^N$ of the set of all $q \in \mathbb{P}^N$ with $r_X(q) = t$; here over $\mathbb{C}$ we may take the closure either in the euclidean topology or the Zariski topology and get the same set $\sigma_t(X)$. The set $\sigma_t(X)$ is an irreducible projective variety, which contains all $q \in \mathbb{P}^N$ with $r_X(q) \le t$ (but often it contains points with higher X-rank). We have $\sigma_1(X) = X$ and $\sigma_t(X) \subseteq \sigma_{t+1}(X)$ for all $t > 0$. Let $r_{X,\mathrm{gen}}$ be the minimal integer x such that $\sigma_x(X) = \mathbb{P}^N$, i.e. the only integer x such that $r_X(q) = x$ for all q in a non-empty open subset of $\mathbb{P}^N$ (either for the euclidean or the Zariski topology of $\mathbb{P}^N$).

In this paper we look at properties related to the X-rank, but for linear subspaces $V \subset \mathbb{P}^N$. For each linear space $V \subseteq \mathbb{P}^N$ the X-rank $r_X(V)$ of X is the minimal cardinality of a finite set $S \subset X$ such that $V \subseteq \langle S\rangle$. Obviously $r_X(\{q\}) = r_X(q)$ for all $q \in \mathbb{P}^N$.

Adapting the proofs in [12] we get the following statement.

Proposition 1.1. *Let $V \subset \mathbb{P}^N$ be an r-dimensional linear subspace. Let ρ be the minimal value of all $r_X(o)$, $o \in V$. Then:*

(i) $r_X(V) \le \min\{(r+2)r_{X,\mathrm{gen}}, (r+1)r_{X,\mathrm{gen}} + \rho\}$;

(ii) If $u_1,\dots,u_k \in V$, $k \ge 2$, are linearly independent, then $r_X(V) \le (r + 2 - k)r_{X,\mathrm{reg}} + r_X(u_1) + \cdots + r_X(u_k)$.

There are two notions related to the X-rank and for which the proof of Proposition 1.1 may be extended with some effort. These two notions are related. They capture what happens if a part of X cannot be used for the interpolation, i.e. the finite set S used in the definition of X-rank must avoid a proper closed subset of X. We fix a proper closed subset $T \subset X$ for the Zariski topology. For any $q \in \mathbb{P}^N$ (resp. linear subspace $V \subseteq \mathbb{P}^N$) let $r_{X\setminus T}$ (resp. $r_{X\setminus T}(V)$) be the minimal cardinality of a finite set $S \subset X \setminus T$ such that $q \in \langle S\rangle$ (resp. $V \subseteq \langle S\rangle$). Obviously $r_{X\setminus T}(q) \ge r_X(q)$ and $r_{X\setminus T}(V) \ge r_X(V)$. The proof of Proposition 1.1 and of Proposition 3.1 below works verbatim using $r_{X\setminus T}$ and $r_{X\setminus T,\mathbb{R}}$ instead of r_X and $r_{X,\mathbb{R}}$, respectively. The notion of open rank is more subtle (it was introduced by J. Jelisiejew in [25], inspired by a related notion for affine varieties introduced in [8] and [9]). The *open rank* $\mathrm{ora}_X(q)$ of $q \in \mathbb{P}^N$ is the minimal integer $t > 0$ such that for every Zariski closed proper subset $T \subsetneq X$ there is a set $S_T \subset X \setminus T$ with $\sharp(S_T) \le t$ and $q \in \langle S_T\rangle$. Obviously $\mathrm{ora}_X(q) \ge r_X(q)$ and $\mathrm{ora}_X(q) \ge r_{X\setminus T}(q)$ for any closed $T \subsetneq X$, but often $\mathrm{ora}_X(q) > r_X(q)$ (always if $r_X(q) = 1$, i.e. if $q \in X$). There is a case for the symmetric tensor rank in which the maximal value of the rank is strictly less than the maximal value of the open rank ([6], i.e. trivariate homogeneous polynomials of degree 4). There are cases in which the generic rank is not the generic open rank, i.e. there is no non-empty open subset $U \subset \mathbb{P}^N$ (for the Zariski or the euclidean topology) such that $\mathrm{ora}_X(q) = r_{X,\mathrm{gen}}$ for all $q \in U$. Take any non-defective $X \subset \mathbb{P}^N$ such that $a := (N+1)/(\dim(X)+1)$ is an integer; when $a := (N+1)/(\dim(X)+1) \in \mathbb{N}$, X is non-defective if and only

if $\sigma_a(X) = \mathbb{P}^N$, i.e. if and only if $a = r_{X,\mathrm{gen}}$. By assumption and a dimensional count there is a non-empty open subset $U \subset \mathbb{P}^N$ for the Zariski topology such that $r_X(q) = a$ for all $q \in U$ and for each $q \in U$ there are only finitely many $S \subset X$ with $\sharp(S) = a$ and $q \in \langle S\rangle$. We claim that $\mathrm{ora}_X(q) > a$ for every $q \in U$. Fix $q \in U$ and take the finitely many sets $S_1, \dots, S_k$ such that $\sharp(S_i) = a$ and $q \in \langle S_i\rangle$. Set $T := S_1 \cup \dots \cup S_k$. By the definition of T there is no $S \subset X \setminus T$ with $\sharp(S) = a$ and $q \in \langle S\rangle$; if $\dim(X) > 1$ we may even take instead of $S_1 \cup \dots \cup S_k$ an irreducible $T_1 \subsetneq X$ with $T_1 \supset S_1 \cup \dots \cup S_k$. Hence the generic open rank is at least $a+1$. We may find many examples $X \subset \mathbb{P}^N$ as above among the Veronese embeddings of a projective space $\mathbb{P}^n$ with respect to the complete linear system $|\mathcal{O}_{\mathbb{P}^n}(d)|$, $d > 2$, i.e. in the set up of the rank 1 decomposition of homogeneous polynomials; in this case $N+1 = \binom{n+d}{n}$ and it is sufficient to assume $\binom{n+d}{n} \equiv 0 \pmod{n+1}$ and that (n,d) is not in the Alexander-Hischowitz list of defective Veronese embeddings of degree > 2 ([2], [3], [13]), i.e., with the assumption $\binom{n+d}{n} \equiv 0 \pmod{n+1}$ and $d > 2$, if and only if $(n,d) \notin \{(2,4),(4,3)\}$. For instance if $n = 2$ it is sufficient to take any integer $d \geq 5$ with $d \equiv 1, 2 \pmod 3$.

For any linear subspace $V \subset \mathbb{P}^N$ the *open rank* of V is the minimal integer $t > 0$ such that for every Zariski closed proper subset $T \subsetneq X$ there is a set $S_T \subset X \setminus T$ with $\sharp(S_T) \leq t$ and $q \in \langle S_T\rangle$.

In section 2 we adapt the proof of Proposition 1.1 to the open rank and get the following result.

Theorem 1.2. *Let $V \subset \mathbb{P}^N$ be an r-dimensional linear subspace. Let ρ be the minimal value of all $\mathrm{ora}_X(o)$, $o \in V$. Then*

(i) $\mathrm{ora}_X(V) \leq \min\{(r+2)r_{X,\mathrm{gen}}, (r+1)r_{X,\mathrm{gen}} + \rho\}$.

(ii) If $u_1, \dots, u_k \in V$, $k \geq 2$, are linearly independent, then $r_X(V) \leq (r + 2 - k)r_{X,\mathrm{reg}} + \mathrm{ora}_X(u_1) + \dots + \mathrm{ora}_X(u_k)$.

As in [12] we may extend this type of results over $\mathbb{R}$ with very mild assumptions (see section 3).

For a linear subspace V with $r := \dim(V) > 0$ many other invariants may be defined in the following way. For any integer r such that $0 \leq r \leq N$ let $G(r,N)$ denote the Grassmannian of all r-dimensional linear subspaces of $\mathbb{P}^N$. Let $V \subset \mathbb{P}^N$ be a linear subspace. Set $r_{X,\max}(V) := \max_{q\in V} r_X(q)$ and $r_{X,\min}(V) := \min_{q\in V} r_X(q)$. Let $r_{X,\mathrm{gen}}(V)$ denote the only integer t such that there is a non-empty open subset $U \subseteq V$ with $r_X(q) = t$ for all $q \in U$ (it is called the *generic* X-rank of V). If X is defined over $\mathbb{R}$ with $X_{\mathrm{reg}}(\mathbb{R}) \neq \emptyset$, then $r_{X,\mathbb{R},\max}(V) := \max_{q\in V} r_{X,\mathbb{R}}(q)$ and $r_{X,\min}(V) := \min_{q\in V} r_{X,\mathbb{R}}(q)$. In this case we may also define the X-typical ranks of V, i.e. the integers t for which there is a non-empty euclidean open set $E \subset V$ with $r_{X,\mathbb{R}}(q) = t$ (see section 3). In certain situations related to coding theory small $r_{X,\min}(V)$ is bad, while in other situations small $r_{X,\min}(V)$ is good. For instance, software programs which compute the tensor rank of tensors and the rank of homogeneous polynomials are more effective for low rank tensors. By Proposition 1.1 small $r_{X,\min}(V)$ helps to get an upper bound for $r_X(V)$. For a general $V \in G(r,N)$ the invariant $r_{X,\min}(V)$ is computed in terms of the dimensions of the secant varieties $\sigma_t(V)$, $t > 0$.

For any $V \in G(r,N)$ let $\eta_X(V)$ be the minimal integer x such that there is $S \subset V$ with S spanning V and $x = \sum_{q\in S} r_X(q)$. We obviously have

$$r_X(V) \le \eta_X(V) \le (r+1)r_{X,\min}(V)$$

Define the flag $U_0 \subset \cdots \subset U_h$, $h \ge 0$, and the integers $\ell_{X,0}(V), \dots, \ell_{X,h}(V)$, $r_{X,0}(V) \le \cdots \le r_{X,r}(V)$, in the following way. Let U_0 be the linear span of all $v \in V$ with $r_X(v) = r_{X,\min}(V)$. Set $\ell_{X,0}(V) := \dim(U_0)$ and $r_{X,i}(V) = r_{X,\min}(V)$ for all $i = 0, \dots, \ell_{X,0}(V)$. If $U_0 = V$, i.e. if $\ell_{X,0}(V) = r$, then we take $h = 0$ and stop. Now assume $\ell_{X,0}(V) < r$. Let $r_{X,\ell_{X,0}(V)+1}(V)$ be the minimal integer $r_X(v)$ for some $v \in V \setminus U_0$. Let $U_1 \subseteq V$ be the linear span of the union of U_0 and all $v \in V \setminus U_0$ with $r_X(v) = r_{X,\ell_{X,0}(V)+1}(V)$. Set $\ell_{X,1}(V) = \dim(U_1)$ and $r_{X,i}(V) = r_{X,\ell_{X,0}(V)+1}(V)$ for all $x \in \{\ell_{X,0}(V)+1, \dots, \ell_{X,1}(V)\}$. If $U_1 = V$, i.e. if $\ell_{X,1}(V) = r$, then we stop. Now assume $\ell_{X,1}(V) < r$. Let $r_{X,\ell_{X,1}(V)+1}(V)$ be the minimal integer $r_X(v)$ for some $v \in V \setminus U_1$. Let $U_2 \subseteq V$ be the linear span of the union of U_1 and all $v \in V \setminus U_1$ with $r_X(v) = r_{X,\ell_{X,1}(V)+1}(V)$. Set $\ell_{X,2}(V) = \dim(U_2)$ and $r_{X,i}(V) = r_{X,\ell_{X,1}(V)+1}(V)$ for all $x \in \{\ell_{X,1}(V)+1, \dots, \ell_{X,2}(V)\}$. And so on.

Proposition 1.3. *Fix $s \in \{0, \dots, r\}$ and let $W \subseteq V$ be any s-dimensional linear subspace. Then $\eta_X(W) \le \sum_{i=0}^{s} r_{X,i}(V)$.*

Using Proposition 1.3 we get the following recipe to compute all integers $\eta_X(V)$, $\ell_{X,j}(V)$ and $r_{X,i}(V)$. Take $q_0 \in V$ such that $r_X(q_0) = r_{X,\min}(V)$. If $r = 0$, then stop. If $r > 0$, then take $q_1 \in V \setminus \{q_0\}$ with minimal rank. If $r = 1$, then set $\eta_X(V) = r_X(q_0) + r_X(q_1)$ and then stop. Now assume $r > 1$. Assume to have defined linearly independent points $q_0, \dots, q_s \in V$, $s < r$, such that for $h = 1, \dots, s-1$, $r_X(q_h)$ is the minimal X-rank of a point of $V \setminus \langle\{q_0, \dots, q_{h-1}\}\rangle$. Since $s < r$, we have $V \supsetneq \langle\{q_0, \dots, q_s\}\rangle$. Let q_{s+1} be any point of $V \setminus \langle\{q_0, \dots, q_s\}\rangle$ with minimal X-rank. And so on. We obtain in this way $r+1$ points $q_0, \dots, q_r$ with non-decreasing X-rank and spanning V.

Claim: We have $r_{X,i}(V) = r_X(q_i)$ for all i, $q_i \in U_e$ if and only if $i \le \ell_{X,e}(V)$ and each U_j is spanned by $U_j \cap \{q_0, \dots, q_r\}$.

Proof of the Claim: Since $q_0, \dots, q_r$ are linearly independent, the second and the third statement of the Claim are equivalent. The case $i = 0$ of the first statement is true, because $r_X(q_0) = r_{X,\min}(V)$. Moreover $q_0 \in U_0$ by the definitions of U_0 and of q_0. Now assume $i > 0$ and that $r_{X,j}(V) = r_X(q_j)$ for all $j < i$. Set $U_{-1} := \emptyset$. Let e be the minimal non-negative integer such that $i \le \ell_{X,e}(V)$. If $i - 1 \le \ell_{X,e-1}$, then set $f := e - 1$, otherwise set $f := e$. By the inductive assumption we have $q_0, \dots, q_{i-1} \in U_f$. First assume $f < e$, i.e. $f = e - 1$, i.e $r_{X,i}(V) > r_{X,i-1}(V)$. In this case we have $U_f = \langle\{q_0, \dots, q_{i-1}\}\rangle$. Since $q_i \notin U_f$, $U_f = \langle\{q_0, \dots, q_{i-1}\}\rangle$ and $r_{X,i}(V)$ is the minimal X-rank of an element of $V \setminus U_f$, we have $r_X(q_i) \ge r_{X,i}(V)$. Since $\langle\{q_0, \dots, q_{i-1}\rangle = U_f$, our choice of q_i gives $r_X(q_i) \le r_{X,i}(V)$. Now assume $f = e$. In this case we have $r_{X,i}(V) = r_{X,i-1}(V)$. Set $t := \ell_{X,e-1}(V)$. We have $U_{e-1} = \langle\{q_0, \dots, q_t\}\rangle$. There is an $\ell_{X,e}(V)$-dimensional linear subspace U_e of V spanned by U_{e-1} and $\ell_{X,e}(V) - \ell_{X,e-1}(V)$ linearly independents element with X-rank $r_{X,i}(V) = r_{X,i-1}(V)$, while $U_e \setminus U_{e-1}$ has no element with X-rank $< r_{X,i}(V)$. By the inductive assumption we also

have $q_j \in U_e$ for all $j < i$ and $U_{e-1} = \langle\{q_0, \dots, q_t\}\rangle$. The definition of q_i gives $r_X(q_i) = r_{X,i}(V)$.

This procedure shows that the integer $\delta_X(V) := r_X(q_r)$ is an invariant of X and the linear space V. The integer $\delta_X(V)$ is the minimal integer t such that V is spanned by the points of V with X-rank at most t. If W is an s-dimensional linear subspace of V, then the Claim implies $\delta_X(W) \geq r_{X,s}(V)$. Obviously $\langle\{q_0, \dots, q_s\}\rangle$ is an s-dimensional subspace of V with $\delta_X(\langle\{q_0, \dots, q_s\}\rangle) = r_{X,s}(V)$.

Take any $V \in G(r, N)$ and assume that the set of all $q \in V$ with $r_X(q) = r_{X,\min}(V)$ spans V. In this case we obviously have $r_X(V) \leq (r+1)r_{X,\min}(V)$. Can we improve this upper bound for $r_X(V)$? If $r_{X,\min}(V) = 1$, i.e. if $V \cap X \neq \emptyset$, then this bound cannot be improved, because $r_X(V) \geq r+1$ for all $V \in G(r, N)$. Now assume that V is general in $G(r, N)$ and set $z := r_{X,\min}$ and $e := \deg(\sigma_z(X))$ and $g := N - \dim(\sigma_z(X))$.

Proposition 1.4. *Set $n := \dim(X)$, $a := r_{X,\mathrm{gen}}$. For any $i \in \{1, \dots, a\}$ set $\sigma(i) := \dim(\sigma_i(X))$. Set $\sigma(0) = -1$ and $\sigma_0(X) = \emptyset$. Fix a general $V \in G(r, N)$.*

(i) We have $r_{X,\mathrm{gen}}(V) = a$.

(ii) If $r < N - \sigma(a-1)$, then $r_{X,\min}(V) = a$.

(iii) For any $i < a$ there is $q \in V$ with $r_X(q) = i$ if and only if $r \geq N - \sigma(i)$.

(iv) Assume $r \geq N - \sigma(i)$. The integer $z := r_{X,\min}(V)$ is the positive integer $\leq a-1$ such that $N - \sigma(z) \leq r < N - \sigma(z-1)$.

(v) We have $r_{X,i}(V) = r_{X,\min}(V)$ for all $i = 0, \dots, r$, and hence $r_X(V) \leq \eta_X(V) = (r+1)r_{X,\min}(V)$.

In general the inequality $r_X(V) \leq \eta_X(V)$ is very rough. If we are only interested in the case of a general $V \in G(r, N)$ in some important cases (e.g. if $X \subset \mathbb{P}^N$ is a Veronese embedding of a projective space as in [15], [18], [23]), this integer is the minimal integer t such that $\sigma_t(\widetilde{X}) = \mathbb{P}^{\widetilde{N}}$, where $\widetilde{X} \subset \mathbb{P}^{\widetilde{N}}$ is a certain embedding of a variety $\widetilde{X}$ constructed from the embedding $X \subset \mathbb{P}^N$. For an arbitrary X we know a general result for the uniqueness of the set evincing the X-rank of a point $q \in \mathbb{P}^N$ ([14, Theorem 1.18] is stated for the symmetric tensor rank, but its proof works in the general case). For these type results it was introduced the following invariant of the embedding $X \subset \mathbb{P}^N$. Let $\rho(X)$ be the maximal positive integer t such that any set $S \subset X$ with $\sharp(S) \leq t$ is linearly independent. We have $2 \leq \rho(X) \leq N+1$. It is easy to check that $\rho(X) = N+1$ if and only if X is a rational normal curve of $\mathbb{P}^N$. We have $\rho(X) = d+1$ if $X \subset \mathbb{P}^N$, $N = -1 + \binom{n+d}{n}$, is the order d Veronese embedding of $\mathbb{P}^n$. In section 2 we prove the following result.

Theorem 1.5. *Take any $V \in G(r, N)$.*

(i) If $2r_X(V) \leq \rho(X) + r$, then there is a unique set $S \subset X$ such that $\sharp(S) \leq r_X(V)$ and $V \subseteq \langle S\rangle$.

(ii) Assume $2\eta_X(V) \leq \rho(X)+r$. The integer $r_V(V)$ is computed in the following way. Take $q_0, \dots, q_r \in V$ such that $r_X(q_i) = r_{X,i}(V)$ and $\langle\{q_0, \dots, q_r\}\rangle = V$. Take any $S_i \subset X$ such that $\sharp(S_i) = r_X(q_i)$ and $q_i \in \langle S_i\rangle$. Set $S := \cup_{i=0}^r S_i$. Then $r_X(V) = \sharp(S)$ and S is the only subset $A \subset X$ with $\sharp(A) = r_X(V)$ and $V \subseteq \langle A\rangle$.

Now we shows why part (ii) of Theorem 1.5 shows (for any $r > 0$) how to construct examples $V, M \in G(r, N)$ with $r_X(V) = \eta_X(V)$, $r_X(M) < \eta_X(M)$ and $\eta_X(V) = \eta_X(M) \le (\rho(X) + r)/2$. Fix any $r + 1$ linearly independents points $q_0, \dots, q_r$ with $r_X(q_i) \le r_X(q_j)$ for all $i \le j$ and $\sum_{i=0}^{r} r_X(q_i) \le 2\rho(X) + r$. Set $V := \langle \{q_0, \dots, q_r\} \rangle$. Since $2r_X(q_i) \le \rho(X)$, there is a unique $S_i \subset X$ such that $\sharp(S_i) = r_X(q_i)$ and $q_i \in \langle S_i \rangle$. Set $S := \cup_{i=1}^{r} S_i$. By part (ii) of Theorem 1.5 we have $r_X(V) = \sharp(S)$. By the Claim and Proposition 1.3 we have $\eta_X(V) = \sum_{i=0}^{r} r_X(q_i) = \sum_{i=0}^{r} \sharp(S_i)$. Hence $r_X(V) = \ell_X(V)$ if and only if $S_i \cap S_j$ for all $i \neq j$. We may also reverse the construction. We start with sets $S_i \subset X$, $i = 0, \dots, r$, with $S_0 \neq \emptyset$, $\sharp(S_i) \le \sharp(S_j)$ for all $i \le j$ and $\sum_{i=0}^{r} \sharp(S_i) \le 2\rho(X) + r$. This inequality implies that each S_i is linearly independent and that if $p_i \in \langle S_i \rangle$, then $r_X(p_i) = \sharp(S_i)$ if and only if there is no $S' \subsetneq S_i$ with $p_i \in \langle S' \rangle$. We also assume that for all $i = 1, \dots, r$, then $S_i \nsubseteq \langle \cup_{h=0}^{i-1} S_h \rangle$. Take a general $q_i \in \langle S_i \rangle$. We saw that $r_X(q_i) = \sharp(S_i)$. Since $S_i \nsubseteq \langle \cup_{h=0}^{i-1} S_h \rangle$ for $i = 1, \dots, r$ we have $q_i \notin \langle \{q_0, \dots, q_{i-1}\} \rangle$. Hence we may easily construct V, M with $r_X(V) = \eta_X(V)$ and $r_X(M) < \eta_X(M)$ (note that we may take arbitrary set $S_i \neq \emptyset$ with $\sum_{i=0}^{r} 2\sharp(S_i) \le \rho(X) + r$).

In all cases with uniqueness (e.g. in Theorem 1.5) we obviously have $\mathrm{ora}_X(V) > r_X(V)$ and $\mathrm{ora}_X(q_i) > r_X(q_i)$ for all i.

2. The proofs

Proof of Proposition 1.1: Fix $o \in V$ with $r_X(o) = \rho$ and set $a := r_{X,\mathrm{gen}}$. Take a non-empty open subset $U \subset \mathbb{P}^N$ (either for the euclidean or the Zariski topology of $\mathbb{P}^N$) such that $r_X(q) = a$ for all $q \in U$. Fix $q \in U$ and set $W := \{q\} \cup V$. We have $\dim(W) \le r+1$ and $W \supseteq V$. Since $q \in W \cap U$, the set $U \cap W$ is a non-empty open subset of W (for the euclidean or the Zariski topology). Hence there are $e_1, \dots, e_{r+2} \in U \cap W$ spanning W. Since $r_X(e_i) = q$, there is $S_i \subset X$ such that $\sharp(S_i) = a$ and $e_i \in \langle S_i \rangle$. Set $S := S_1 \cup \dots \cup S_{r+2}$. We have $\sharp(S) \le (r + 2)a$ and $W \subseteq \langle S \rangle$. Since $W \supseteq V$, we get $r_X(V) \le (r + 2)a$. Since $o \in W$ and $U \cap W$ contains a non-empty open subset of W, there are $f_1, \dots, f_{r+1} \in U \cap W$ such that $\{o, f_1, \dots, f_{r+1}\}$ spans W. As above we get $r_X(V) \le r_X(o) + (r + 1)a$. In the same way we get part (ii). □

Proof of Theorem 1.2: Fix $o \in V$ with $r_X(o) = \rho$ and set $a := r_{X,\mathrm{gen}}$. Fix a closed subset $T \subsetneq X$. By the definition of open rank there are $A, A_i \subset X \setminus T$, $1 \le i \le k$, such that $\sharp(A) = \rho$, $\sharp(A_i) = \mathrm{ora}_X(u_i)$ for all i, $o \in \langle A \rangle$ and $u_i \in \langle A_i \rangle$ for all i.

Claim 1: There is a non-empty open subset $U' \subset \mathbb{P}^N$ such that for each $q \in U'$ there is $S_q \subset X \setminus T$ with $\sharp(S_q) = a$ and $q \in \langle S_q \rangle$.

Proof of Claim 1: By assumption we have $\sigma_a(X) = \mathbb{P}^N$, i.e. $\mathbb{P}^N$ is the Zariski closure of the union of all linear spaces $\langle S \rangle$ with $S \subset X$, S linearly independent and $\sharp(S) = a$. Any point of X is a limit of a family of points of $X \setminus T$. Hence $\mathbb{P}^N$ is the Zariski closure of the union of all linear spaces $\langle S \rangle$ with $S \subset X \setminus T$, S linearly independent and $\sharp(S) = a$. Let $Z \subset (X \setminus T)^a$ be the set of all $(b_1, \dots, b_a) \in (X \setminus T)^a$ which are linearly independent. Z is a non-empty Zariski open subset of $(X \setminus T)^a$. Let $\mathbb{I} \subset Z \times \mathbb{P}^N$ be the set of all pairs (b, q) with $b = (b_1, \dots, b_a)$ and $q \in \langle \{b_1, \dots, b_a\} \rangle$. $\mathbb{I}$ is an irreducible variety of dimension $a \dim(X) - 1$. Let $\pi : \mathbb{I} \to \mathbb{P}^N$ the projection onto the last factor. By assumption

$\mathbb{P}^N$ is the Zariski closure of the image $\mathrm{Im}(\pi)$ of π. By Chevalley's theorem $\mathrm{Im}(\pi)$ is a constructible subset of $\mathbb{P}^N$ for the Zariski topology. Hence the image of π contains a non-empty open subset U' for the Zariski topology. The set U' satisfies the thesis of Claim 1.

With Claim 1 the proof of Proposition 1.1 goes verbatim for the open rank, using U' instead of U. □

Proof of Proposition 1.3: We use induction on the integer r (applied to all linear subspaces of $\mathbb{P}^N$). For any fixed integer r we use induction on the integer s. For any r the case $s = 0$ is trivially true, because for any linear space M $r_{X,0}(M) = r_{X,\min}(M)$ and $r_{X,\min}(N) \le r_{X,\min}(M)$ if $N \supseteq M$. Hence we may assume that $r \ge s > 0$. Let $W(0) \subset \cdots \subset W(h') = W$, $h' \ge 0$, be the flag of linear subspaces of W obtained as in the definition of the integer $h \ge 0$ and the flag $U_0 \subset \cdots \subset U_h = V$ of V using W instead of V. Take a basis $\{q_0, \dots, q_r\}$ of V with $r_X(q_i) = r_{X,i}(V)$ for all i and such that each U_i is spanned by $U_i \cap \{q_0, \dots, q_r\}$. Take a basis $\{p_0, \dots, p_s\}$ of W with $r_X(p_i) = r_{X,i}(W)$, $i = 0, \dots, s$ and such that each $W(i)$ is spanned by $W(i) \cap \{p_0, \dots, p_s\}$. Let W' be the linear span of $\{p_0, \dots, p_{s-1}\}$. By the inductive assumption for each integer $j < s$ we have $\sum_{i=0}^{j} r_X(p_i) \ge \sum_{i=0}^{j} q_i$. If $r_X(p_s) \ge r_X(q_s)$, then we are done. Now assume $r_X(p_s) < r_X(q_s)$. Since $r_X(p_s) \ge r_X(p_i)$ for all $i \le s$, we get that the linear span T of the set of all $v \in V$ with $r_X(v) < r_X(q_s)$ has dimension at least s. Hence $r_{X,s}(V) < r_X(q_s)$, a contradiction. □

Remark 2.1. The case $s = r$ of Proposition 1.3 implies that $\eta_X(V) = \sum_{i=0}^{r} r_{X,i}(V)$.

Proof of Proposition 1.4: Since V is general, it contains a general point of $\mathbb{P}^N$ and hence part (i) is trivial. For any integer $i \in \{1, \dots, a-1\}$ set $\sigma_i^o(X) := \{q \in \sigma_i(X) \mid r_X(q) = i\}$. The set $\sigma_i^o(X)$ contains a non-empty open subset of $\sigma_i(X)$ and in particular the closure of $\sigma_i(X) \setminus \sigma_i^o(X)$ is a finite union of algebraic sets of dimension $< \sigma(i)$. Since V is general, for any $i > 0$ we have $V \cap \sigma_i(X) \neq \emptyset$ if and only if $r \ge N - \sigma(i)$. Hence $r_{X,\min}(V) > i$ if $r < N - \sigma(i)$. Now assume $r \ge N - \sigma(i)$ and hence $V \cap \sigma_i(X) \neq \emptyset$. We get $\dim(V \cap \sigma_i(X)) = \sigma(i) + r - N$, $\dim(V \cap (\sigma_i(X) \setminus \sigma_i^o(X))) < \sigma(i) + r - N$ and so $V \cap (\sigma_i(X) \setminus \sigma_i^o(X)) \neq \emptyset$.

Now we prove part (v). Set $z := r_{X,\min}(V)$. It is sufficient to prove that V is spanned by $r+1$ points with z as their X-rank. If $z = a$, then we are done, because a non-empty Zariski open subset of V is formed by points q with $r_X(q) = z$. Now assume $z < a$. By assumption $N - \sigma(z) \le r < N - \sigma(z-1)$. Let W be a general linear subspace of V with $\dim(W) = N - \sigma(z)$. Since V is general, W is a general element of $G(N - \sigma(z), N)$. Hence $W \cap \sigma_z(X)$ is finite and contained in $\sigma_i^o(X)$. Since X is irreducible, $\sigma_z(X)$ is irreducible. Hence $\sigma_z(X) \cap M$ is an irreducible curve for a general $M \in G(N - \sigma(z) + 1, N)$. Since W is general, we may take as M a subspace containing W. Take any hyperplane $H \subset \mathbb{P}^N$. Since $\sigma_z(W)$ is reduced and connected, we have $h^1(\mathbb{P}^N, \mathcal{I}_{\sigma_z(X)}) = 0$. Thus the exact sequence

$$0 \to \mathcal{I}_{\sigma_z(X)} \to \mathcal{I}_{\sigma_z(X)}(1) \to \mathcal{I}_{\sigma_z \cap H, H}(1) \to 0$$

gives that $H \cap \sigma_z(X)$ spans H. Taking a hyperplane of H and iterating the trick several times we first obtain that the integral curve $\sigma_z(X) \cap M$ spans M and then

that the finite set $\sigma_z(X) \cap W$ spans W. Since $\sigma_z(X) \cap W = \sigma_i^o(X) \cap W$, W is spanned by points with X-rank z. Varying W in V we get that V is spanned by points with X-rank z. □

Proof of Theorem 1.5: We first prove part (i). We use induction on r. Since the case $r = 0$ is true by the proof of [14, Theorem 1.18], we may assume $r > 0$. Take $A, S \subset X$ with $\sharp(S) \le r_X(V)$, $\sharp(A) \le r_X(V)$, $V \subseteq \langle S \rangle \cap \langle A \rangle$ and $S \neq A$. The definition of the integer $r_X(V)$ gives $\sharp(A) = \sharp(S) = r_X(V)$.

First assume $S \cap A = \emptyset$. Take $A_2 \subset A$ with $\sharp(A_2) = A - r$ and $V \cap \langle A_2 \rangle$ a single point. Since $V \subseteq \langle S \rangle$ and $A_2 \cap S = \emptyset$, $A_2 \cup S$ is linearly dependent. Since $\sharp(S \cup A_2) \le \rho(X)$, we get a contradiction.

Now assume $S \cap A \neq \emptyset$ and set $A_1 := A \setminus A \cap S$, $S_1 := S \setminus S \cap A$, $e := \sharp(A \cap S)$ and $W := V \cap \langle A_1 \rangle \cap \langle S_1 \rangle$. Since $S \neq A$ and no proper subset of A spans V, we have $S_1 \neq A_1$, $S_1 \neq \emptyset$, $A_1 \neq \emptyset$, $A_1 \neq S_1$ and $\dim(W) = r - e$. Apply the first part to W, A_1 and S_1.

Now we prove part (ii). Since part (ii) follows from part (i) when $r = 0$, we may assume that $r > 0$ and use induction on r. We have $\sharp(S) \le \sum_{i=0}^{r} r_X(q_i) = \eta_X(V)$ and $V \subseteq \langle S \rangle$. Hence $r_X(V) \le \sharp(S)$. Let S' be the minimal subset of S such that $V \subseteq \langle S' \rangle$. Take any $A \subset X$ with $\sharp(A) = r_X(V)$ and $V \subseteq \langle A \rangle$ and assume $S \neq A$. Since $\sharp(A) + \sharp(S) \le 2\eta_X(V) \le \rho(X) + r$, the proof of part (i) gives $A = S'$. If $S' = S$, then the proof of part (ii) is over. Now assume $S' \nsubseteq S$. Set $M := \langle \{q_0, \dots, q_{r-1}\} \rangle$. We have $\dim(M) = r - 1$. The definition of $q_0, \dots, q_{r-1}$ gives $r_{X,i}(M) = r_{X,i}(V)$. Since $\dim(M) = r - 1$, the inductive assumption implies $S_i \cap S' = S$ for all $i < r$. Take $N := \langle \{q_1, \dots, q_r\} \rangle$. The Claim in the introduction gives $r_{X,i}(N) = r_{X,i+1}(V)$ for all $i < r$. The inductive assumption gives $S_i \cap S' = S_i$ for all $i > 0$ and in particular $S_r \subseteq S'$. Thus $S' = S$, a contradiction. □

3. Over $\mathbb{R}$

Let $X \subset \mathbb{P}^N$ be a geometrically integral variety defined over $\mathbb{R}$. We write $X(\mathbb{R}) \subset \mathbb{P}^N(\mathbb{R})$ for the real points of X and $X(\mathbb{C}) \subset \mathbb{P}^N(\mathbb{C})$ for the complex points of X (we called them X in the other sections). Recall that " geometrically integral " means that the complex variety $X_{\mathbb{C}}$ (the same real equations of X, but seen in $\mathbb{P}^N(\mathbb{C})$) is an integral complex variety. We require that $X(\mathbb{R})$ contains at least one smooth point of $X(\mathbb{C})$ (usually this is written down as $X_{\mathrm{reg}}(\mathbb{R}) \neq \emptyset$). This assumption implies that $X_{\mathrm{reg}}(\mathbb{R})$ spans the real projective space $\mathbb{P}^N(\mathbb{R})$ (linear span with real coefficients) and hence the complex projective space $\mathbb{P}^N(\mathbb{C})$ (linear span with complex coefficients). A *typical X-rank* is an integer $t > 0$ such that there is a non-empty open subset $E \subset \mathbb{P}^N$ for the euclidean topology with $r_{X,\mathbb{R}}(q) = t$ for all $q \in E$ ([7], [11], [12], [16]). The generic X-rank of $X(\mathbb{C}) \subset \mathbb{P}^N(\mathbb{C})$ is the minimum of all typical X-ranks and if x_1 and x_2 are typical X-rank, then every integer x with $x_1 \le x \le x_2$ is a typical X-rank ([7, Theorem 2.2]). Fix $V \in G(r, N)(\mathbb{R})$. For any $S \subset \mathbb{P}^N(\mathbb{R})$ let $\langle S \rangle_{\mathbb{R}}$ denote its $\mathbb{R}$-linear span in the real projective space $\mathbb{P}^N(\mathbb{R})$ and let $\langle S \rangle_{\mathbb{C}}$ denote its $\mathbb{C}$-linear span in the complex projective space $\mathbb{P}^N(\mathbb{C})$. We have $\dim_{\mathbb{R}}(\langle S \rangle_{\mathbb{R}}) = \dim_{\mathbb{C}}(\langle S \rangle_{\mathbb{C}})$ and

$\langle S\rangle_{\mathbb{R}} = \langle S\rangle_{\mathbb{C}} \cap \mathbb{P}^N(\mathbb{R})$. The real X-rank $r_{X,\mathbb{R}}(V)$ of V is the minimal cardinality of a finite set $S \subset X(\mathbb{R})$ such that $V \subseteq \langle S\rangle_{\mathbb{R}}$.

Proposition 3.1. *Let $V \subset \mathbb{P}^N(\mathbb{R})$ be an r-dimensional $\mathbb{R}$-linear subspace. Let ρ be the minimal value of all $r_{X,\mathbb{R}}(o)$, $o \in V$. Then*

(i) $r_X(V) \le \min\{(r+2)r_{X,\mathrm{gen}}, (r+1)r_{X,gen} + \rho\}$.

(ii) If $u_1, \dots, u_k \in V$ are linearly independent, then $r_{X,\mathbb{R}}(V) \le (r+2-k)r_{X,\mathrm{reg}} + r_{X,\mathbb{R}}(u_1) + \cdots + r_{X,\mathbb{R}}(u_k)$.

Proof. Fix $o \in V$ with $r_X(o) = \rho$ and set $a := r_{X,\mathrm{gen}}$. Take a non-empty open subset $U \subset \mathbb{P}^N(\mathbb{R})$ for the euclidean topology of $\mathbb{P}^N$ such that $r_{X,\mathbb{R}}(q) = a$ for all $q \in U$. Fix $q \in U$ and set $W := \{q\} \cup V$. W is an $\mathbb{R}$-linear subspace, $\dim(W) \le r+1$ and $W \supseteq V$. Since $q \in W \cap U$, the set $U \cap W$ is a non-empty open subset of W for the euclidean or the Zariski topology). Then we continue as in the proof of Proposition 1.1. □

Remark 3.2. Take the the assumptions of Proposition 3.1, but look at the proof of Claim 1 in the proof of Theorem 1.2. Set $n := \dim(X)$. We fix a proper Zariski closed subset $T \subsetneq X$. Take a non-empty open and connected subset $E \subset X_{\mathrm{reg}}(\mathbb{R})$ for the euclidean topology such that $E \cap T(\mathbb{C}) = \emptyset$. E has euclidean dimension n and in particular it is a connected topological manifold of dimension n. Let Z_E be the set of all $b = (b_1, \dots, b_a) \in E^a$ such that $b_1, \dots, b_a$ are linearly independent. Z_E is a topological manifold of dimension na, but we do not claim that it is connected. Let $\mathbb{I}_E$ be the set of all pairs (b, u) with $b = (b_1, \cdots, b_a) \in Z_E$ and $u \in \langle\{b_1, \dots, b_a\}\rangle_{\mathbb{R}}$. Let $\pi_E : \mathbb{I}_E \to \mathbb{P}^N(\mathbb{R})$ denote the projection onto the second factor. Since π is dominant, its complex differential has rank N for a Zariski dense open subset of $\mathbb{I}$. This open subset meets $\mathbb{I}_E$. Hence there is $c \in \mathbb{I}_E$ at which the complex differential of π has rank N. The differential of π and of π_E at c are given by the same real matrix. Hence the image of π_E contains a non-empty euclidean subset of $\mathbb{P}^N(\mathbb{R})$. With this observation Theorem 1.2 is extended to the open real rank.

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