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SOLUTIONS OF FRACTIONAL FOAM DRAINAGE AND ZAKHAROV-KUZNETSOV EQUATIONS USING A NEW ALGORITHM

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ABSTRACT. The Daftardar-Gejji Jafari Method (DGJM) has been used extensively in the recent one and a half decades to solve various non-linear equations such as algebraic equations, integral equations, partial differential equations, ordinary and fractional differential equations, and so on. In this paper, we present a new time-efficient algorithm for DGJM and solve the non-linear fractional foam drainage and Zakharov-Kuznetsov equations. We compare the DGJM solutions with those obtained by the Adomian decomposition method and the homotopy perturbation method. Moreover, the computational procedure of the new algorithm is more effective, and time-efficient and does not include any tedious calculations.

1. Introduction

Nonlinear partial differential equations (PDEs) describe various physical and artificial irregular phenomena and hence play a vital role in sciences and technology. Several methods such as transform methods, decomposition/iterative methods and numerical methods have been developed in the literature for solving linear/nonlinear ordinary and fractional PDEs[1, 2, 3]. Some of the well- established and studied decomposition methods are the Adomian decomposition method (ADM) [4], the homotopy perturbation method (HPM) [5] and the Daftardar-Gejji and Jafari method (DGJM) [6]. The DGJM has been used effectively in the literature to solve various linear and non-linear equations of integer and fractional orders. In addition, various hybrid analytic and numerical methods have been developed with the help of DGJM [7, 8, 9]. For more details about DGJM and its applications, we refer to the review article [10]. Further, in 2020, Kumar *et.al* [11] have developed a new algorithm for DGJM which is very simple and time-efficient as compared to the original one. In the present paper, our aim is to solve the following two equations using the new algorithm of DGJM.

(i) Foam Drainage Equation: Foaming occurs in many distillation and absorption processes. Foam drainage is a natural process that describes fluid flows

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out of a foam [12, 13]. Foams are very useful in many industrial and technological processes from the practical and scientific point of view [14]. There have been developed many applications for foams, such as cleansing, water purification, minerals extraction, etc. [12, 15]. In [16], Verbist and Weaire developed a model that idealizes the network of Plateau borders. We consider the following time and space-fractional foam drainage equation [17]

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{1}{2}u\frac{\partial^2 u}{\partial x^2} + 2u^2\frac{\partial^\beta u}{\partial x^\beta} - \left(\frac{\partial^\beta u}{\partial x^\beta}\right)^2 = 0,\ 0 < \alpha, \beta \leq 1,\ x > 0. \tag{1}$$

where α and β are the orders of the fractional time and space derivatives respectively.

(ii) Zakharov-Kuznetsov equations: Recently, the fractional Zakharov - Kuznetsov equations have been used for modeling various kinds of weakly nonlinear ion-acoustic waves in plasma. This has led to a significant interest in the study of these equations. We consider the fractional version of the Zakharov Kuznetsov equations (ZKE(m, n, k))[18, 19] of the form:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + a\frac{\partial}{\partial x}u^m + b\frac{\partial^3}{\partial x^3}u^n + c\frac{\partial^3}{\partial y \partial y \partial x}u^k = 0, \tag{2}$$

where $u = u(x, y, t)$, $0 < \alpha \leq 1, a, b$ and c are arbitrary constants and m, n and k are non-zero integers.

This paper is organized as follows: In section 2, we give some basic definitions of fractional calculus. In section 3, we present the new algorithm of DGJM for solving a general functional equation. In section 4, we solve the foam drainage equation (1) and fractional Zakharov-Kuznetsov equations (2) using the new algorithm of DGJM and compare the results with those derived by HPM and ADM. Finally, we draw the conclusions in section 5.

2. Preliminaries

In this section, we give some basic definitions and properties of the fractional operators [20].

Definition 2.1. *Riemann-Liouville time-fractional integral of order $\alpha > 0$, of a real valued function $u(x, t)$ is defined as*

$$I_t^\alpha u(x, t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t - s)^{\alpha - 1} u(x, s) ds.$$

Definition 2.2. *The Caputo time-fractional derivative operator of order $\alpha > 0$, of a real-valued function $u(x, t)$ is defined as*

$$\begin{aligned} \frac{\partial^\alpha u(x, t)}{\partial t^\alpha} &= I_t^{n-\alpha}\left[\frac{\partial^n u(x, t)}{\partial t^n}\right], \\ &= \begin{cases} \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-y)^{n-\alpha-1}\frac{\partial^n u(x,y)}{\partial y^n}dy, & n - 1 < \alpha < n, \\ \frac{\partial^n u(x,t)}{\partial t^n}, & \alpha = n \in \mathbb{N}. \end{cases} \end{aligned}$$

Similarly, the Caputo space-fractional derivative operator of order $\beta > 0$ of $u(x,t)$ is defined as

$$\frac{\partial^\beta u(x,t)}{\partial x^\beta} = I_x^{n-\beta}\left[\frac{\partial^n u(x,t)}{\partial x^n}\right],$$

$$= \begin{cases} \frac{1}{\Gamma(n-\beta)}\int_0^x (x-\tau)^{n-\beta-1}\frac{\partial^n u(x,\tau)}{\partial \tau^n}d\tau, & n-1<\beta<n, \\ \frac{\partial^n u(x,t)}{\partial x^n}, & \beta = n \in \mathbb{N}. \end{cases}$$

Theorem 2.1. *Let $u(x,t) \in C^n[0,T]$ and $n-1<\alpha<n, n \in N$ then*

$$I_t^\alpha\left(\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}\right) = u(x,t) - \sum_{k=0}^{n-1}\frac{u^k(x,0)}{k!}t^k, \ t>0.$$

3. A New Algorithm for DGJM

In 2006, Daftardar-Gejji and Jafari [6] proposed a decomposition method to solve the general function equation of the following form:

$$u = f + N(u), \tag{3}$$

where $u = u(x_1, x_2, \cdots, x_n)$ is a function of n-variables $x_1, x_2, \cdots, x_n$; f a known function and N a known nonlinear operator from a Banach space $B \to B$. Recently, Kumar et. al [11] proposed a new algorithm for DGJM, which reduces the computational procedure and time to a large extent as compared to the original one. The computational procedure of the new algorithm is discussed below: In DGJM, a solution of the equation (3) is assumed in terms of the following infinite series:

$$u = \sum_{i=0}^{\infty} u_i. \tag{4}$$

Let $S_n = u_0 + u_1 + \cdots u_n, \ n = 0,1,2\cdots$. Then the terms $u_i's$ of eqn (4) are calculated as follows:

$$u_0 = S_0 = f, \tag{5}$$
$$u_1 = N(S_0), \tag{6}$$
$$u_2 = N(S_1) - N(S_0), \tag{7}$$
$$u_3 = N(S_2) - N(S_1), \tag{8}$$
$$\vdots$$
$$u_n = N(S_{n-1}) - N(S_{n-2}) \tag{9}$$

On adding the equations (5-9), we get

$$u_0 + u_1 + \cdots + u_n = f + N(S_{n-1}),$$

which is equivalent to

$$S_n = f + N(S_{n-1}). \tag{10}$$

(Note that as $n \to \infty$ (10) converges to (3)) Thus we get the following recursive formula for calculating $S_n' s$:

$$\left.\begin{aligned} &S_0 = f, \\ &S_n = S_0 + N(S_{n-1}),\ n = 1, 2, \cdots, \end{aligned}\right\} \tag{11}$$

It is clear that as $n \to \infty$, S_n converges to u *i.e.* $\lim_{n\to\infty} S_n = u$, which is the solution of equation (3). The formula defined in (11) is referred to as a new algorithm for DGJM. The Convergence analysis of the new algorithm (11) is discussed in [11] and reiterated here.

Theorem 3.1. *Let $N : B \to B$ be continuous and Fréchet differentiable with bounded Fréchet derivative DN. If $\|DN\| = \max_{\|u\|=1} \|DN(u)\| \leq k < 1$, then the sequence of successive iterations $\{S_n\}$ given in (11) converge uniformly to $\lim_{n\to\infty} S_n = \tilde{u}$ (say), which is a solution of (3) i.e. $\tilde{u} = f + N(\tilde{u})$.*

Proof. Note that S_m can be written as

$$S_m = S_0 + \sum_{j=0}^{m-1} (S_{j+1} - S_j).$$

In view of the new algorithm (11) and mean value inequality for Banach spaces [21], we have

$$\begin{aligned} \|S_{j+1} - S_j\| &= \|N(S_j) - N(S_{j-1})\| \leq \|DN\| \|S_j - S_{j-1}\| \\ &\leq k\|N(S_{j-1}) - N(S_{j-2})\| \leq k^2 \|S_{j-1} - S_{j-2}\| \\ &\vdots \\ &\leq k^j \|S_1 - S_0\|. \end{aligned}$$

Denote $M_j = k^j \|S_1 - S_0\|$. In view of the Weierstrass M-test, $\sum_{j=0}^{\infty} M_j$ converges, hence $\{S_m\}$ converge uniformly to a continuous function $\tilde{u}$ (say), which is a solution of (3).□

4. Applications

In this section, we solve the equations (1) and (2) using a new algorithm of DGJM and also compare the obtained results with HPM and ADM.

4.1. Time-fractional Foam Drainage equation.

Consider the following form of the time-fractional foam drainage equation (take $\beta = 1$ in (1)):

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{1}{2} u \frac{\partial^2 u}{\partial x^2} - 2u^2 \frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2,\ 0 < \alpha \leq 1, \tag{12}$$

with the initial condition

$$u(x, 0) = -\sqrt{c} \tanh(\sqrt{c} x). \tag{13}$$

For $\alpha = 1$, the exact solution of (12-13) is given as [22]:

$$u(x,t) = \begin{cases} -\sqrt{c}\tanh(\sqrt{c}(x-ct)), & x \le ct, \\ 0, & x > ct, \end{cases} \tag{14}$$

where c is the velocity of the wavefront.

Applying the inverse operator I_t^α, in (12) and using the initial condition (13), we get

$$u(x,t) = u(x,0) + I_t^\alpha \left[\frac{1}{2} u \frac{\partial^2 u}{\partial x^2} - 2u^2 \frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x} \right)^2 \right], \tag{15}$$

where $N(u) = I_t^\alpha \left[\frac{1}{2} u \frac{\partial^2 u}{\partial x^2} - 2u^2 \frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x} \right)^2 \right]$. Now by using the recurrence relation (11), we get

$$\begin{aligned} S_0 &= u(x,0) = -\sqrt{c}\tanh(\sqrt{c}x), \\ S_1 &= S_0 + N(S_0) = -\sqrt{c}\tanh(\sqrt{c}x) + c^2 \mathrm{sech}^2\left(\sqrt{c}x\right) \frac{t^\alpha}{\Gamma(\alpha+1)}, \\ S_2 &= S_0 + N(S_1) = c^2 \mathrm{sech}^2\left(\sqrt{c}x\right) \frac{t^\alpha}{\Gamma(\alpha+1)} - \sqrt{c}\tanh\left(\sqrt{c}x\right) \\ &\quad + \frac{t^{2\alpha} B_1 \left(B_0 - B_3 \left(B_5\left(\sqrt{c}x\right) - B_6 \tanh\left(\sqrt{c}x\right)\right)\right)}{B_7}, \end{aligned} \tag{16}$$

where

$$\begin{aligned} &B_0 = B_2 \tanh\left(\sqrt{c}x\right) \mathrm{sech}^4\left(\sqrt{c}x\right), \ B_1 = 4^{-\alpha} c^{7/2} \mathrm{sech}^2\left(\sqrt{c}x\right), \\ &B_2 = \sqrt{\pi} 4^\alpha c^3 \Gamma\left(\alpha + \frac{1}{2}\right) \Gamma(3\alpha+1)^2 t^{2\alpha}, \ B_3 = \alpha^3 \Gamma(\alpha)^2 \Gamma(4\alpha), \\ &B_4 = 16^\alpha c^{3/2} \Gamma\left(\alpha + \frac{1}{2}\right)^2 t^\alpha, \ B_5 = \left(B_4(\cosh\left(2\sqrt{c}x\right) - 2\right) \mathrm{sech}^4\left(\sqrt{c}x\right), \\ &B_6 = 2\pi\Gamma(3\alpha+1), \ B_7 = \sqrt{\pi}\alpha^4 \Gamma(\alpha)^3 \Gamma(4\alpha) \Gamma\left(\alpha + \frac{1}{2}\right) \Gamma(3\alpha+1). \end{aligned}$$

The three-term solution of (12-13) obtained by the new algorithm of DGJM is given in (16). For $\alpha = 1$, the obtained results are compared numerically as well as graphically in tables 1 to 3 and figs. 1 to 3 respectively. Further, it is observed that the obtained results are in very good agreement with ADM.

x	$\lvert u_{Exact} - u_{HPM}\rvert$	$\lvert u_{Exact} - u_{ADM}\rvert$	$\lvert u_{Exact} - u_{New\ algo}\rvert$
-10	0	4.44089×10^{-16}	4.44089×10^{-16}
-8	2.4×10^{-18}	2.24265×10^{-13}	4.44089×10^{-16}
-6	2.1059×10^{-15}	2.29754×10^{-10}	1.77636×10^{-15}
-4	2.14918×10^{-12}	2.34498×10^{-7}	2.14939×10^{-12}
-2	2.17229×10^{-9}	0.000236656	2.18081×10^{-9}
-1	5.10302×10^{-8}	0.00523834	5.70554×10^{-8}
0	1.45797×10^{-9}	5.2479×10^{-8}	2.91598×10^{-9}

TABLE 1. Absolute errors ($t = 0.001, c = 3$) for (12-13).

x	$\lvert u_{Exact} - u_{HPM}\rvert$	$\lvert u_{Exact} - u_{ADM}\rvert$	$\lvert u_{Exact} - u_{New\ algo}\rvert$
-10	2.00000×10^{-18}	1.77636×10^{-15}	4.44089×10^{-16}
-8	2.01600×10^{-15}	1.86962×10^{-12}	2.66454×10^{-15}
-6	2.05744×10^{-12}	1.9087×10^{-9}	2.05791×10^{-12}
-4	2.09993×10^{-9}	1.94811×10^{-6}	2.09994×10^{-9}
-2	0.000002123	0.00197296	2.13177×10^{-6}
-1	0.000050433	0.0485679	0.0000565495
0	0.000014557	0.00051592	0.0000291443

TABLE 2. Absolute errors ($t = 0.01, c = 3$) for (12-13).

x	$\lvert u_{Exact} - u_{HPM}\rvert$	$\lvert u_{Exact} - u_{ADM}\rvert$	$\lvert u_{Exact} - u_{New\ algo}\rvert$
-10	1.59300×10^{-15}	1.42109×10^{-14}	1.77636×10^{-15}
-8	1.62593×10^{-12}	1.40941×10^{-11}	1.62625×10^{-12}
-6	1.65952×10^{-9}	1.43841×10^{-8}	1.65952×10^{-9}
-4	1.69379×10^{-6}	0.0000146727	1.6938×10^{-6}
-2	0.001716375	0.0064218	0.00172493
-1	0.043767689	0.094494	0.0506295
0	0.126259125	3.71367	0.27862

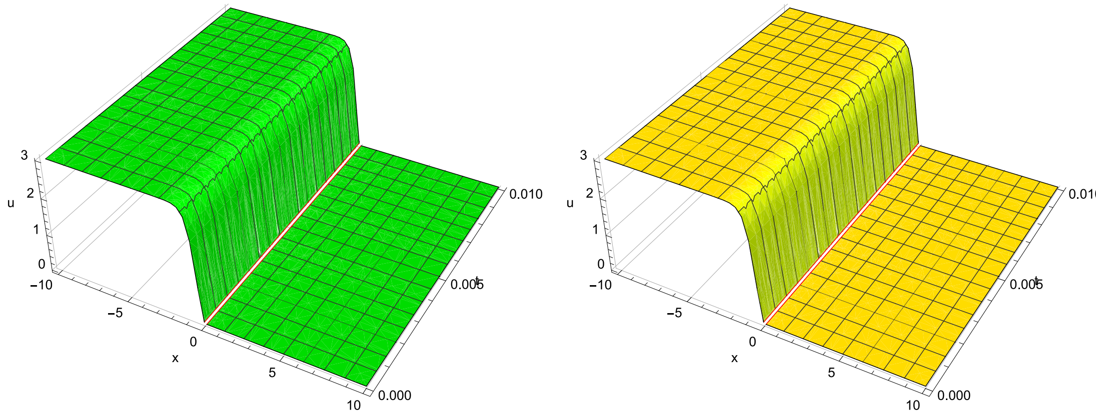
TABLE 3. Absolute errors ($t = 0.1, c = 3$) for (12-13).

4.2. **Space-Fractional Foam Drainage Equation.** Consider the following form of the space-fractional foam drainage equation (taking $\alpha = 1$ in (1))

$$\frac{\partial u}{\partial t} = \frac{1}{2}u\frac{\partial^2 u}{\partial x^2} - 2u^2\frac{\partial^\beta u}{\partial x^\beta} + \left(\frac{\partial^\beta u}{\partial x^\beta}\right)^2 \quad 0 < \beta \leq 1, \tag{17}$$

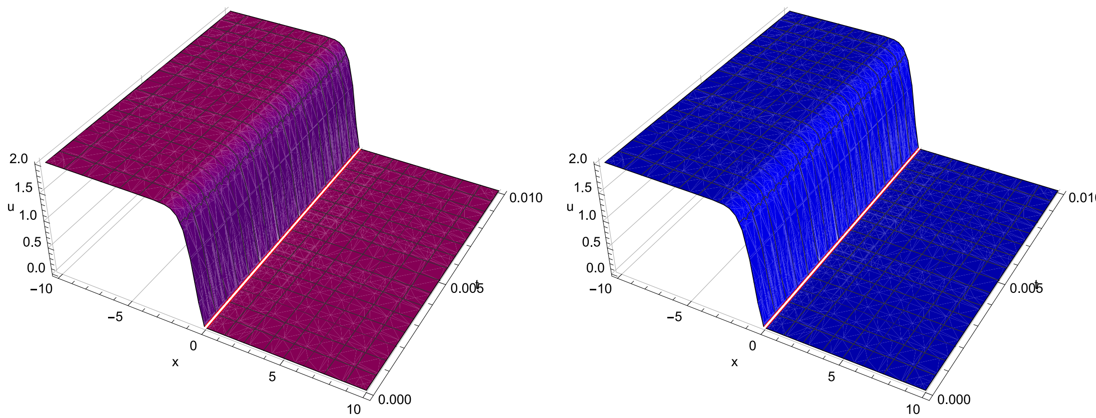
with the following initial condition

$$u(x,0) = x^2. \tag{18}$$



(A) (B)

FIGURE 1. For $c = 3$, (A) exact solution (B) DGJM solution of (12-13).



(A) (B)

FIGURE 2. For $c = 2$ (A) exact solution (B) DGJM solution of (12-13).

The initial value problem (17-18) is equivalent to the following integral equation

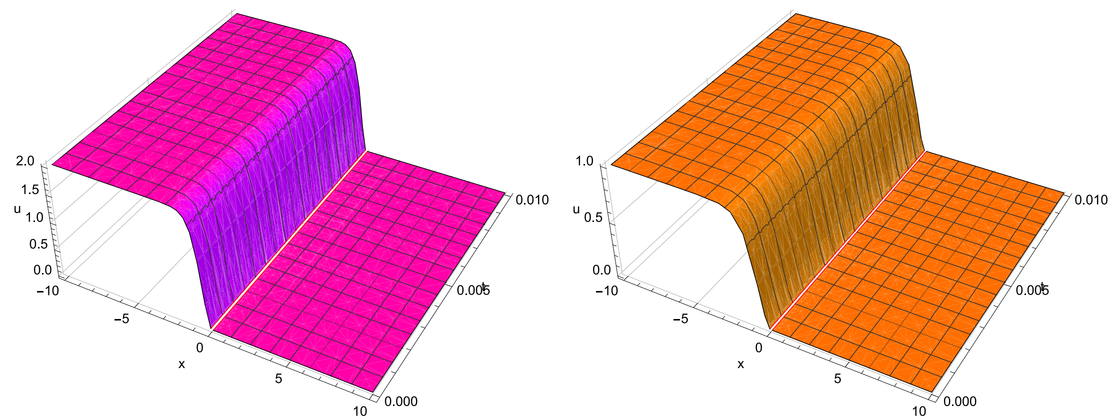
$$u(x,t) = u(x,0) + I_t\left[\frac{1}{2}u\frac{\partial^2 u}{\partial x^2} - 2u^2\frac{\partial^\beta u}{\partial x^\beta} + \left(\frac{\partial^\beta u}{\partial x^\beta}\right)^2\right], \tag{19}$$

where $I_t = \int_0^t ()dt$ and $N(u) = I_t\left[\frac{1}{2}u\frac{\partial^2 u}{\partial x^2} - 2u^2\frac{\partial^\beta u}{\partial x^\beta} + \left(\frac{\partial^\beta u}{\partial x^\beta}\right)^2\right]$.

Using the recurrence relation (11), we get

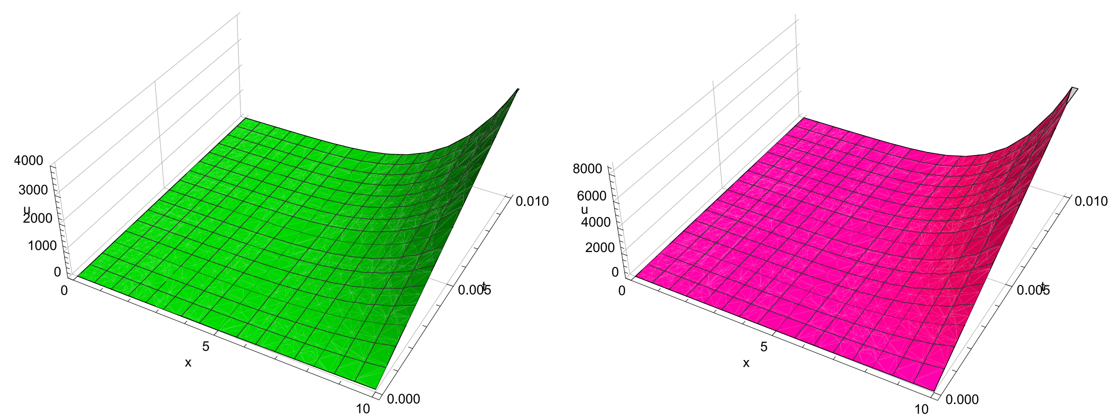
$$S_0 = u(x,0) = x^2, \tag{20}$$

$$\begin{aligned} S_1 &= S_0 + N(S_0) \\ &= x^2 - t\left(x^2 - \frac{4x^{6-\beta}}{\Gamma(3-\beta)} + \frac{4x^{4-2\beta}}{\Gamma(3-\beta)^2}\right). \end{aligned} \tag{21}$$



(A) (B)

FIGURE 3. For $c = 1$ (A) exact solution (B) DGJM solution of (12-13).



(A) $\beta = 1$ (B) $\beta = \frac{1}{2}$

FIGURE 4. Two-term DGJM solutions of (17-18).

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Figures 4(A) and 4(B) are the graphical representation of the two-term DGJM solution (21) of the space-fractional equation (17-18).

4.3. **Time Fractional Zakharov-Kuznetsov Equations.** In this subsection, we solve Zakharov-Kuznetsov equation (2) for $m = 2, n = 2, k = 2$ and for $m = 3, n = 3, k = 3$ using new algorithm. For $\alpha = 1$, we also compare the obtained results with HPM and exact solutions.

4.3.1. Consider the following time fractional ZKE(2,2,2) of the form:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial}{\partial x}u^2 + \frac{1}{8}\frac{\partial^3}{\partial x^3}u^2 + \frac{1}{8}\frac{\partial^3}{\partial y \partial y \partial x}u^2 = 0, \tag{22}$$

where $0 < \alpha \leq 1$. For $\alpha = 1$, the exact solution of (22) along with the following initial condition [23]

$$u(x, y, 0) = \frac{4}{3}\rho \sinh^2[x + y], \tag{23}$$

where ρ is an arbitrary constant, is given by

$$u(x, y, t) = \frac{4}{3}\rho \sinh^2[x + y - \rho t]. \tag{24}$$

Equation (22) is equivalent to the following integral equation:

$$u(x, y, t) = u(x, y, 0) + I_t^\alpha \left[-\frac{\partial}{\partial x}u^2 - \frac{1}{8}\frac{\partial^3}{\partial x^3}u^2 - \frac{1}{8}\frac{\partial^3}{\partial y \partial y \partial x}u^2 \right], \tag{25}$$

where I_t^α is an integral operator of order $\alpha > 0$ with respect to t. Let $S_0 = u_0 = u(x, y, 0) = \frac{4}{3}\rho \sinh^2[x + y]$ and $N(u) = I_t^\alpha \left[-\frac{\partial}{\partial x}u^2 - \frac{1}{8}\frac{\partial^3}{\partial x^3}u^2 - \frac{1}{8}\frac{\partial^3}{\partial y \partial y \partial x}u^2 \right]$.

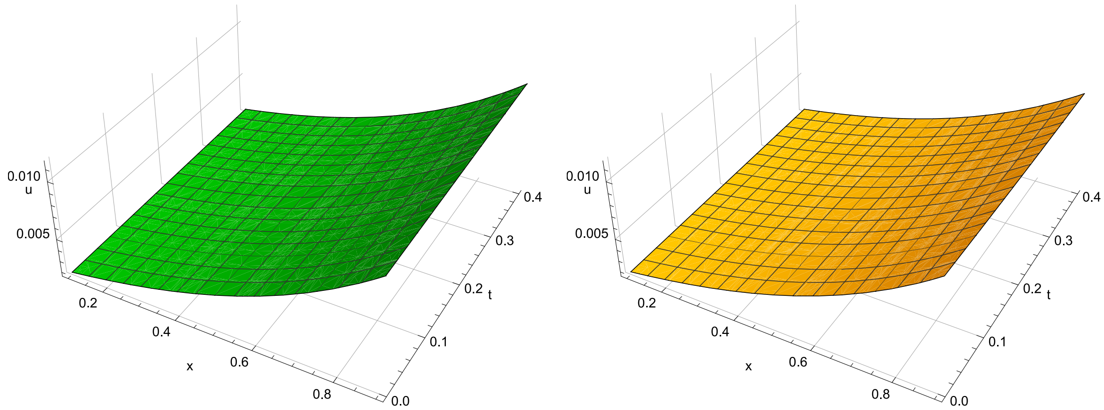
In view of (11), we get

$$\begin{aligned}
S_1 &= S_0 + N(S_0) = \frac{4}{3}\rho \sinh^2 z + 8p^2 \Big(4\sinh(2z) - 5\sinh[4z] \Big) \frac{t^\alpha}{9\Gamma(\alpha + 1)}. \\
S_2 &= S_0 + N(S_1) = \frac{4}{3}p \sinh^2 z + 8p^2 \Big[4\sinh(2z) - 5\sinh(4z) \Big] \frac{t^\alpha}{9\Gamma(\alpha + 1)} \\
&+ 64p^3 \Big[13\cosh(2z) - 70\cosh(4z) + 75\cosh(6z) \Big] B \frac{t^{2\alpha}}{27\Gamma(2\alpha + 1)} \\
&- 1280Cp^4 \Big[4\sinh(2z) + 8\sinh(4z) - 60\sinh(6(z)) \\
&+ 85\sinh(8(z)) \Big] \frac{t^{3\alpha}}{81B^2\Gamma(3\alpha + 1)},
\end{aligned} \tag{26}$$

where $B = \Gamma(\alpha + 1)$ and $C = \Gamma(2\alpha + 1), z = (x + y)$. In table 4, we compare the DGJM solution (26), the exact and HPM solutions [24, 25] for equation (22-23) numerically. Further, in table 5 three-term DGJM solutions are computed for $\alpha = 0.50, 0.60$ and 0.75. In fig. 5, three-term DGJM and exact solutions of (22-23) are depicted. Further, in fig. 6 the DGJM solutions are plotted for various values of the fractional derivative operator α.

x	y	t	*Exact solution*	*HPMsolution*	*NewAlgo*
0.1	0.1	0.2	$5.393877159 \times 10^{-5}$	$5.354824505 \times 10^{-5}$	$5.355357167 \times 10^{-5}$
		0.3	$5.388407669 \times 10^{-5}$	$5.329624424 \times 10^{-5}$	$5.330822346 \times 10^{-5}$
		0.4	$5.382941057 \times 10^{-5}$	$5.304291051 \times 10^{-5}$	$5.306419678 \times 10^{-5}$
0.6	0.6	0.2	$3.036507411 \times 10^{-3}$	$2.985667896 \times 10^{-3}$	$2.990009136 \times 10^{-3}$
		0.3	$3.035778955 \times 10^{-3}$	$2.957882201 \times 10^{-3}$	$2.967622123 \times 10^{-3}$
		0.4	$3.035050641 \times 10^{-3}$	$2.929005004 \times 10^{-3}$	$2.946270879 \times 10^{-3}$
0.9	0.9	0.2	$1.153697757 \times 10^{-2}$	$1.087461656 \times 10^{-2}$	$1.104133904 \times 10^{-2}$
		0.3	$1.153454074 \times 10^{-2}$	$1.047729580 \times 10^{-2}$	$1.084887611 \times 10^{-2}$
		0.4	$1.153210438 \times 10^{-2}$	$1.003750659 \times 10^{-2}$	$1.069179110 \times 10^{-2}$

TABLE 4. Numerical solutions of (22-23) when $\alpha = 1, \rho = 0.001$.



(A) (B)

FIGURE 5. For $\alpha = 1$ (A) exact solution; (B) three-term DGJM solution of (22-23).

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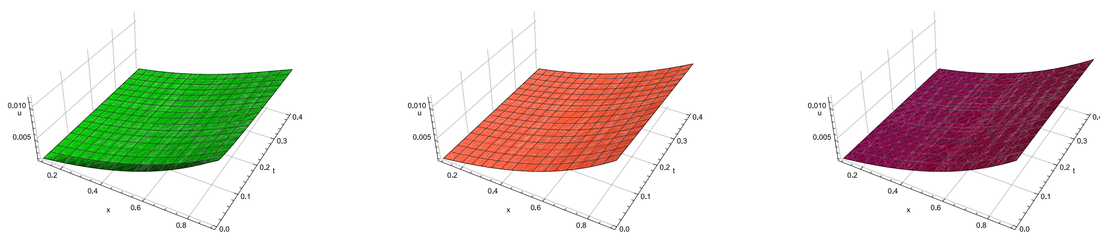

(A) $\alpha = 0.25$ (B) $\alpha = 0.50$ (C) $\alpha = 0.75$

FIGURE 6. Three-term DGJM solutions of (22-23), when $y = 0.9$.

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x	y	t	$\alpha = 0.50$	$\alpha = 0.60$	$\alpha = 0.75$
0.1	0.1	0.2	$5.281686358 \times 10^{-5}$	$5.300424327 \times 10^{-5}$	$5.324727333 \times 10^{-5}$
		0.3	$5.254651973 \times 10^{-5}$	$5.272213413 \times 10^{-5}$	$5.296653952 \times 10^{-5}$
		0.4	$5.232041702 \times 10^{-5}$	$5.247791132 \times 10^{-5}$	$5.271052936 \times 10^{-5}$
0.6	0.6	0.2	$2.930071427 \times 10^{-3}$	$2.943605015 \times 10^{-3}$	$2.963023855 \times 10^{-3}$
		0.3	$2.910660825 \times 10^{-3}$	$2.910660825 \times 10^{-3}$	$2.939466956 \times 10^{-3}$
		0.4	$2.895587979 \times 10^{-3}$	$2.904141940 \times 10^{-3}$	$2.919197294 \times 10^{-3}$
0.9	0.9	0.2	$1.072412964 \times 10^{-2}$	$1.074679319 \times 10^{-2}$	$1.083753036 \times 10^{-2}$
		0.3	$1.068554861 \times 10^{-2}$	$1.066027133 \times 10^{-2}$	$1.069157491 \times 10^{-2}$
		0.4	$1.068318572 \times 10^{-2}$	$1.062142862 \times 10^{-2}$	$1.059844176 \times 10^{-2}$

TABLE 5. Three-term new algorithm solution of (22-23) , when $\alpha = 0.50, 0.60$ and 0.75 and $\rho = 0.001$.

4.3.2. Consider the following time fractional FZKE(3,3,3)

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial}{\partial x}u^3 + 2\frac{\partial^3}{\partial x^3}u^3 + 2\frac{\partial^3}{\partial y \partial y \partial x}u^3 = 0, \tag{27}$$

where $0 < \alpha \leq 1$. For $\alpha = 1$, the exact solution of (27) with respect to the following initial condition [23]

$$u(x, y, 0) = \frac{3}{2} p \sinh\left(\frac{x+y}{6}\right), \tag{28}$$

where ρ is an arbitrary constant, is defined as

$$u(x, y, t) = \frac{3}{2} p \sinh\left(\frac{1}{6}(x + y - pt)\right). \tag{29}$$

Eq. (27) is equivalent to the following integral equation

$$u(x, y, t) = u(x, y, 0) + I_t^\alpha \left[-\frac{\partial}{\partial x} u^3 - 2\frac{\partial^3}{\partial x^3} u^3 - 2\frac{\partial^3}{\partial y \partial y \partial x} u^3\right]. \tag{30}$$

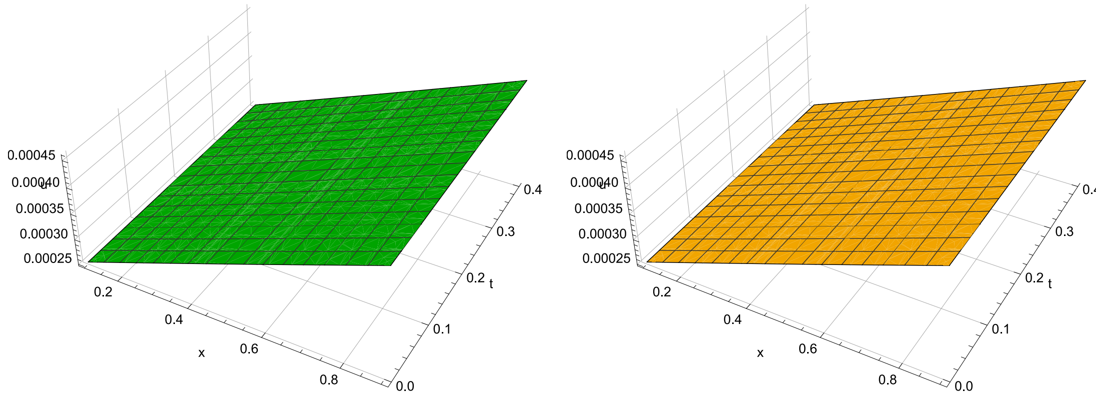
Let $S_0 = \frac{3}{2} p \sinh\left(\frac{x+y}{6}\right)$ and $N(u) = I_t^\alpha \left[-\frac{\partial}{\partial x} u^3 - 2\frac{\partial^3}{\partial x^3} u^3 - 2\frac{\partial^3}{\partial y \partial y \partial x} u^3\right]$.
In view of (11), we get

$$\begin{aligned} S_1 = S_0 + N(S_0) &= \frac{3}{2} p \sinh\left(\frac{z}{6}\right) \\ &+ 3p^3 \left(5\cosh\left(\frac{z}{6}\right) - 9\cosh\left(\frac{z}{2}\right)\right) \frac{t^\alpha}{32\Gamma(\alpha+1)}, \end{aligned} \tag{31}$$

$$\begin{aligned} S_2 = S_0 + N(S_1) &= \frac{3}{2} p \sinh\left(\frac{z}{6}\right) \\ &+ 3p^3 \left(5\cosh\left(\frac{z}{6}\right) - 9\cosh\left(\frac{z}{2}\right)\right) \frac{t^\alpha}{32\Gamma(\alpha+1)} \\ &+ 768p^5 \left[-621\sinh\left(\frac{z}{2}\right) + 70\sinh\left(\frac{z}{6}\right) + \right. \\ &\left. 765\sinh\left(\frac{5(z)}{6}\right)\right] \frac{t^{2\alpha}}{131072\Gamma[2\alpha+1]} - \\ &3p^7 C \left[1385\cosh\left(\frac{z}{6}\right) + 9\left(75\cosh\left(\frac{z}{2}\right) - \right.\right. \\ &\left.\left. 1615\cosh\left(\frac{5(z)}{6}\right) + 1827\cosh\left(\frac{7(z)}{6}\right)\right)\right] \frac{t^{3\alpha}}{8192B^2\Gamma(3\alpha+1)} \\ &- 3p^9 t^{4\alpha} \left[3550\sinh\left(\frac{z}{6}\right) - 9\right) - 3412\sinh\left(\frac{z}{2}\right) - \\ &10935\sinh\left(\frac{3(z)}{2}\right) + 1700\sinh\left(\frac{5(z)}{6}\right) + \\ &\left. 9135\sinh\left(\frac{7(z)}{6}\right)\right)\right] \frac{t^4\alpha}{131072B^3\Gamma(4\alpha+1)}, \end{aligned} \tag{32}$$

where $B = \Gamma(\alpha + 1), C = \Gamma(2\alpha + 1), z = (x + y)$.

In table 6, three-term DGJM solutions (32) are compared with the exact and HPM solutions for equation (27-28). In table 7, three-term DGJM solutions are calculated for $\alpha = 0.50, 0.60$ and 0.75. In fig. 7, for $\alpha = 1$ the exact and the DGJM solutions are plotted. Besides, in fig. 8 three-term DGJM solutions are represented graphically for different values of α.



(A) (B)

FIGURE 7. For $\alpha = 1$ (A) exact solution; (B) three-term DGJM solution of (27-28).

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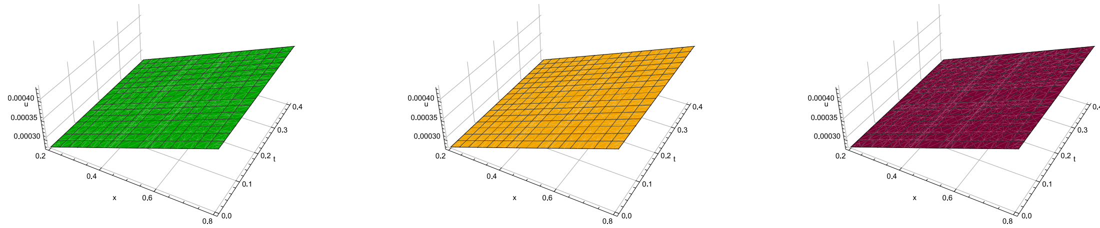
x	y	t	*Exact*	*New Algo*	*HPM*
0.1	0.1	0.2	$4.995923204 \times 10^{-5}$	$5.000918398 \times 10^{-5}$	$5.000895773 \times 10^{-5}$
		0.3	$4.993421817 \times 10^{-5}$	$5.000914609 \times 10^{-5}$	$5.000880670 \times 10^{-5}$
		0.4	$4.990920434 \times 10^{-5}$	$5.000910819 \times 10^{-5}$	$5.000865568 \times 10^{-5}$
0.6	0.6	0.2	$3.019530008 \times 10^{-4}$	$3.020038994 \times 10^{-4}$	$3.020036280 \times 10^{-4}$
		0.3	$3.019274992 \times 10^{-4}$	$3.020038472 \times 10^{-4}$	$3.020034401 \times 10^{-4}$
		0.4	$3.019019978 \times 10^{-4}$	$3.020037950 \times 10^{-4}$	$3.020032522 \times 10^{-4}$
0.9	0.9	0.2	$4.567281735 \times 10^{-4}$	$4.567802963 \times 10^{-4}$	$4.567799629 \times 10^{-4}$
		0.3	$4.567020404 \times 10^{-4}$	$4.567802244 \times 10^{-4}$	$4.567797243 \times 10^{-4}$
		0.4	$4.566759074 \times 10^{-4}$	$4.567801525 \times 10^{-4}$	$4.567794858 \times 10^{-4}$

TABLE 6. Numerical solutions of (27) when $\alpha = 1$ and $p = 0.001$.

x	y	t	$\alpha = .5$	$\alpha = 0.6$	$\alpha = 0.75$
0.1	0.1	0.2	$5.000906854 \times 10^{-5}$	$5.000909830 \times 10^{-5}$	$5.000913646 \times 10^{-5}$
		0.3	$5.000902556 \times 10^{-5}$	$5.000905382 \times 10^{-5}$	$5.000909263 \times 10^{-5}$
		0.4	$5.000898933 \times 10^{-5}$	$5.000901502 \times 10^{-5}$	$5.000905238 \times 10^{-5}$
0.6	0.6	0.2	$3.020037404 \times 10^{-4}$	$3.020037814 \times 10^{-4}$	$3.020038339 \times 10^{-4}$
		0.3	$3.020036811 \times 10^{-4}$	$3.020037201 \times 10^{-4}$	$3.020037735 \times 10^{-4}$
		0.4	$3.020036312 \times 10^{-4}$	$3.020036667 \times 10^{-4}$	$3.020037181 \times 10^{-4}$
0.9	0.9	0.2	$4.567800773 \times 10^{-4}$	$4.567801337 \times 10^{-4}$	$4.567802061 \times 10^{-4}$
		0.3	$4.567799957 \times 10^{-4}$	$4.567800493 \times 10^{-4}$	$4.567801230 \times 10^{-4}$
		0.4	$4.567799270 \times 10^{-4}$	$4.567799757 \times 10^{-4}$	$4.567800466 \times 10^{-4}$

TABLE 7. Three-term new algorithm solution to (27), when $\alpha = 0.5, 0.6$ and $\alpha = 0.75$.



(A) $\alpha = 0.25$ (B) $\alpha = 0.50$ (C) $\alpha = 0.75$

FIGURE 8. Three-term DGJM solutions of (27-28), when $y = 0.9$.

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5. Conclusion

In this paper, a new algorithm of DGJM has been successfully used for obtaining the numerical solutions to the time and space fractional foam drainage equation and Zakharov Kuznetsov equations (FZK(m,n,k)). The DGJM solutions are represented graphically and compared numerically with those obtained by ADM and HPM. The DGJM solutions are accurate and as well as in very good agreement with ADM and HPM solutions. Moreover, the solution procedure of the new algorithm is very simple and straightforward than ADM and HPM. The amount of computation required in the new algorithm is much less than ADM and HPM, which makes it more time-efficient.

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