

CERTAIN CUBIC REDUCTION FORMULAS INVOLVING HYPERGEOMETRIC FUNCTIONS

MOHAMMAD IDRIS QURESHI¹ AND SHAKIR HUSSAIN MALIK^{2*}

ABSTRACT. In this paper, we obtain a new general double infinite series identity (in terms of the sum of three infinite series) involving the bounded sequence of arbitrary complex numbers using Saalschütz summation theorem for terminating Clausen series. As application of our double series identity, we establish two cubic reduction formulas for Srivastava-Daoust double hypergeometric functions in terms of generalized hypergeometric function with suitable convergence conditions. By the theory of analytic continuation, our cubic reduction formula for special Srivastava-Daoust double hypergeometric function, is also valid in $-\frac{211}{25} \leq \Re(z) \leq \frac{3}{4}$ when $\Im(z) = 0$, using Mathematica software.

1. INTRODUCTION AND PRELIMINARIES

In our investigation here, we use the following standard notations:

$\mathbb{N} := \{1, 2, 3, \dots\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\} = \{0, -1, -2, -3, \dots\}$. Also, as usual, the symbols $\mathbb{C}, \mathbb{R}, \mathbb{Z}$ and $\mathbb{N}$ denote the sets of complex numbers, real numbers, integers, and positive integers, respectively.

Pochhammer symbol:

The Pochhammer symbol $(\alpha)_p$ ($\alpha, p \in \mathbb{C}$) is defined, in terms of Gamma function Γ (see, e.g., [19, p. 2 and p. 5]), by

$$\begin{aligned}
 (\alpha)_p &= \frac{\Gamma(\alpha + p)}{\Gamma(\alpha)} \quad (\alpha + p \in \mathbb{C} \setminus \mathbb{Z}_0^-, p \in \mathbb{C} \setminus \{0\}; \alpha \in \mathbb{C} \setminus \mathbb{Z}_0^-, p = 0) \\
 &= \begin{cases} 1 & (p = 0; \alpha \in \mathbb{C} \setminus \mathbb{Z}_0^-), \\ \alpha(\alpha + 1) \cdots (\alpha + n - 1) & (p = n \in \mathbb{N}; \alpha \in \mathbb{C}), \\ \frac{(-1)^k n!}{(n-k)!} & (p = k, \alpha = -n; n, k \in \mathbb{N}_0, 0 \leq k \leq n), \\ 0 & (p = k, \alpha = -n; n, k \in \mathbb{N}_0, k > n), \\ \frac{(-1)^n}{(1-\alpha)_n} & (p = -n; \alpha \in \mathbb{C} \setminus \mathbb{Z}, n \in \mathbb{N}), \end{cases}
 \end{aligned} \tag{1.1}$$

Date: Received: Jan 27, 2022; Accepted: May 16, 2022.

*Corresponding author.

2020 *Mathematics Subject Classification.* Primary 33C05, 33C20; Secondary 33C70.

Key words and phrases. Srivastava-Daoust double hypergeometric function, Saalschütz summation theorem, Hypergeometric transformation and reduction formulas, Bounded sequence, Series rearrangement technique.

it being understood that $(0)_0 := 1$ (see, e.g., [11, 25]) and assumed tacitly that the Gamma quotient exists.

Generalized hypergeometric function of one variable:

The generalized hypergeometric series (or function) ${}_pF_q$ ($p, q \in \mathbb{N}_0$), which is a natural generalization of the Gaussian hypergeometric series ${}_2F_1$, is defined by (see, e.g., [1, 6, 11, 19, 25])

$$\begin{aligned} {}_pF_q \left[\begin{matrix} (\alpha_p); \\ (\beta_q); \end{matrix} z \right] &= {}_pF_q \left[\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z \right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (\alpha_j)_n}{\prod_{j=1}^q (\beta_j)_n} \frac{z^n}{n!} \\ &= {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z). \end{aligned} \quad (1.2)$$

Here an empty product is interpreted as 1, and it is assumed that the variable z , the numerator parameters $\alpha_1, \dots, \alpha_p$, and the denominator parameters $\beta_1, \dots, \beta_q$ take on complex values, provided that

$$(\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-; j = 1, \dots, q). \quad (1.3)$$

Then, if a numerator parameter is a negative integer or zero, the ${}_pF_q$ series terminates in view of the third and fourth cases in (1.1).

With none of the numerator and denominator parameters being zero or a negative integer, the ${}_pF_q$ in (1.2)

- (i) diverges for all $z \in \mathbb{C} \setminus \{0\}$, if $p > q + 1$;
- (ii) converges for all $z \in \mathbb{C}$, if $p \leq q$;
- (iii) converges for $|z| < 1$ and diverges for $|z| > 1$ if $p = q + 1$;
- (iv) converges absolutely for $|z| = 1$, if $p = q + 1$ and $\Re(\omega) > 0$;
- (v) converges conditionally for $|z| = 1$ ($z \neq 1$), if $p = q + 1$ and $-1 < \Re(\omega) \leq 0$;
- (vi) diverges for $|z| = 1$, if $p = q + 1$ and $\Re(\omega) \leq -1$,

where

$$\omega := \sum_{j=1}^q \beta_j - \sum_{j=1}^p \alpha_j. \quad (1.4)$$

In Wolfram's Mathematica this important function ${}_pF_q$ is implemented as HypergeometricPFQ and suitable for both symbolic and numerical calculations. Double hypergeometric function of Srivastava-Daoust:

Srivastava and Daoust [20, p. 199] defined a generalization of the Kampé de Fériet function [2, p. 150] by means of the double hypergeometric series (see also

[21] and [22]):

$$\begin{aligned}
 & F_{C:D;D'}^{A:B;B'} \left(\begin{matrix} [(a_A) : \vartheta, \varphi] : [(b_B) : \psi] ; [(b'_{B'}) : \psi'] ; \\ [(c_C) : \delta, \varepsilon] : [(d_D) : \eta] ; [(d'_{D'}) : \eta'] ; \end{matrix} \quad x, y \right) \\
 &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^A (a_j)_{m\vartheta_j+n\varphi_j} \prod_{j=1}^B (b_j)_{m\psi_j} \prod_{j=1}^{B'} (b'_j)_{n\psi'_j}}{\prod_{j=1}^C (c_j)_{m\delta_j+n\varepsilon_j} \prod_{j=1}^D (d_j)_{m\eta_j} \prod_{j=1}^{D'} (d'_j)_{n\eta'_j}} \frac{x^m y^n}{m! n!}, \tag{1.5}
 \end{aligned}$$

where the coefficients

$$\left\{ \begin{matrix} \vartheta_1, \dots, \vartheta_A; \varphi_1, \dots, \varphi_A; \psi_1, \dots, \psi_B; \psi'_1, \dots, \psi'_{B'}; \delta_1, \dots, \delta_C; \\ \varepsilon_1, \dots, \varepsilon_C; \eta_1, \dots, \eta_D; \eta'_1, \dots, \eta'_{D'} \end{matrix} \right. \tag{1.6}$$

are real and positive; and for the sake of brevity, (a_A) is taken to denote the sequence of A parameters $a_1, a_2, \dots, a_A$ with similar interpretations for (b_B) , $(b'_{B'})$, and so on.

Let

$$\Delta_1 := 1 + \sum_{j=1}^C \delta_j + \sum_{j=1}^D \eta_j - \sum_{j=1}^A \vartheta_j - \sum_{j=1}^B \psi_j \tag{1.7}$$

and

$$\Delta_2 := 1 + \sum_{j=1}^C \varepsilon_j + \sum_{j=1}^{D'} \eta'_j - \sum_{j=1}^A \varphi_j - \sum_{j=1}^{B'} \psi'_j. \tag{1.8}$$

We have

- (i) If $\Delta_1 > 0$ and $\Delta_2 > 0$, then the double power series in (1.5) converges for all complex values of x and y ;
- (ii) If $\Delta_1 = 0$ and $\Delta_2 = 0$, then the double power series in (1.5) is convergent for suitably constrained values of $|x|$ and $|y|$ (for more details of this case, see [22]);
- (iii) If $\Delta_1 < 0$ and $\Delta_2 < 0$, then the double power series in (1.5) would diverge except when $x = y = 0$.

Each of the following results will be needed in our present study.

Saalschütz summation theorem for terminating Clausen series (see [11, p. 87, Theorem(29)]; see also [25, p.95, Q.N.25(i)]):

$${}_3F_2 \left[\begin{matrix} -n, \beta, \gamma; \\ \delta, 1 - n + \beta + \gamma - \delta; \end{matrix} \quad 1 \right] = \frac{(\delta - \beta)_n (\delta - \gamma)_n}{(\delta)_n (\delta - \beta - \gamma)_n}, \tag{1.9}$$

where $n \in \mathbb{N}_0$; $\beta, \gamma, \delta, 1 - n + \beta + \gamma - \delta \in \mathbb{C} \setminus \mathbb{Z}_0^-$.

Note: (Sum of Denominator parameters in ${}_3F_2$ —Sum of Numerator parameters in ${}_3F_2$) = 1

Pfaff-Kummer linear transformation [6, p.247, Eq.(9.5.1) and (9.5.2)]; see also [1,

p.68, Eq.(2.2.6)]:

$${}_2F_1 \left[\begin{matrix} \alpha, \beta; \\ \gamma; \end{matrix} z \right] = (1-z)^{-\alpha} {}_2F_1 \left[\begin{matrix} \alpha, \gamma - \beta; \\ \gamma; \end{matrix} \frac{-z}{1-z} \right], \quad (1.10)$$

where $\gamma \in \mathbb{C} \setminus \mathbb{Z}_0^-$ and $|\arg(1-z)| < \pi$.

Decomposition of unilateral infinite series :

$$\sum_{n=0}^{\infty} \Delta(n) = \sum_{n=0}^{\infty} \Delta(3n) + \sum_{n=0}^{\infty} \Delta(3n+1) + \sum_{n=0}^{\infty} \Delta(3n+2), \quad (1.11)$$

Cauchy's double series identity (see[11, p. 57, Lemma(11)]; see also [25, p.100, Lemma(2)]):

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \Theta(n, m) = \sum_{n=0}^{\infty} \sum_{m=0}^{\lfloor \frac{n}{3} \rfloor} \Theta(n-3m, m), \quad (1.12)$$

provided that both sides of identities (1.11) and (1.12) are absolutely convergent.

Short notation for a sequence of parameters:

The notation $\Delta(N; \alpha)$ represents the sequence of N number of parametrs given by

$$\frac{\alpha}{N}, \frac{\alpha+1}{N}, \frac{\alpha+2}{N}, \dots, \frac{\alpha+N-1}{N}, \quad (1.13)$$

$$\text{and } \Delta[N; (d_D)] := \Delta(N; d_1), \Delta(N; d_2), \Delta(N; d_3), \dots, \Delta(N; d_D) \quad (1.14)$$

represents the sequence of DN numbers of parameters, with similar interpretation for others.

Some useful identities involving Pochhammer symbols:

$$(n-3m)! = \frac{n! (-1)^m}{(-n)_{3m}}; 0 \leq m \leq n. \quad (1.15)$$

$$\frac{(-1)^m}{(2-c-n-3m)_m} = \frac{(c-1)_{n+2m}}{(c-1)_{n+3m}}. \quad (1.16)$$

$$(\alpha)_{m+n} = (\alpha)_m (\alpha+m)_n; (m, n \in \mathbb{N}_0, \alpha \in \mathbb{C}) \quad (1.17)$$

Product theorem of Pochhammer symbol[25, p.23, Eq.(26)]:

$$(\alpha)_{mn} = m^{mn} \prod_{j=1}^m \left(\frac{\alpha+j-1}{m} \right)_n; (m \in \mathbb{N}, n \in \mathbb{N}_0, \alpha \in \mathbb{C}) \quad (1.18)$$

It is well known that, whenever a generalized hypergeometric function reduces to the quotients of the products of the gamma function or Pochhammer symbols, the results are very important from the point of view of applications in numerous areas of physical, mathematical and statistical sciences including (for example) in series systems of symbolic computer algebra manipulation (see, for example, [24]).

Our present investigation is motivated essentially by the usefulness of several interesting and widespread developments of general double-series identities and reduction formulas for Srivastava-Daoust double hypergeometric functions (for example) Buschman-Srivastava[3, p.439], Srivastava[16] and [17], Chen-Srivastava

[4], Chen-Liu-Srivastava [5], Olver *et al.*[7], Prudnikov *et al.*[8], two transformations (one product theorem of two ${}_1F_1$ hypergeometric functions and another product theorem of two ${}_0F_2$ hypergeometric functions) of Srinivasa Ramanujan [11, p.106, Q.N. (5) and Q.N.(7)] and further generalizations of Ramanujan theorems given by Saran[12, pp.91-92]; see also [23, p.32, Eq.(50)] and Shanker-Saran[13, p.10]; see also[23, p.31, Eq. (46)] in the form of the reducibility of Kampé de Fériet function, three new transformations formulas with their applications to partial theta function identities derived by Somashekara-Vidya[15] and some generalizations of reciprocity theorems of Ramanujan established by Somashekara-Murthy- Shalini[14].

The article is organized as follows. In Section 2, we derive a new general double-series identity (2.1) by using Cauchy's double series identity (1.12), decomposition series identity (1.11), Saalschütz summation theorem ${}_3F_2(1)$ (or one-balanced summation theorem) (1.9) and series rearrangement technique. In Sections 3 and 4, by the adjustment of bounded sequence $\{\Phi(\mu)\}_{\mu=1}^{\infty}$, our general double-series identity(2.1) is used to derive two cubic reduction formulas (3.1) and (4.1) for Srivastava-Daoust double hypergeometric functions having arguments $\left(z, \frac{z^3}{27}\right)$ and $\left(\frac{z^3}{27(1-z)^2}, \frac{-z}{1-z}\right)$ with suitable convergence conditions. Finally, in Section 5, we present several concluding remarks and observations.

Remark 1.1. Any values of parameters and arguments in sections 2 to 4, leading to the results which do not make sense, are tacitly excluded.

2. USE OF SAALSCHÜTZ THEOREM IN GENERAL DOUBLE SERIES IDENTITY

In this section, we present an identity (presumably new) which transforms a double series into sum of three single series, in the following theorem.

Theorem 2.1. *Let us assume that $\{\Phi(\mu)\}_{\mu=1}^{\infty}$ be a bounded sequence of essentially arbitrary complex numbers or real numbers such that $\Phi(0) \neq 0$. Then, the following general double-series identity holds true:*

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \Phi(n+3m) \frac{(c-1)_{n+2m} z^{n+3m}}{(c-1)_{n+3m} (c)_m (27)^m n! m!} \\ &= \sum_{n=0}^{\infty} \Phi(3n) \frac{\left(\frac{3c-1}{3}\right)_{2n} \left(\frac{3c-2}{3}\right)_{2n} (c-1)_{2n} z^{3n}}{(1)_{3n} \left(\frac{3c-1}{3}\right)_n \left(\frac{3c-2}{3}\right)_n (c)_n (c-1)_{3n}} + \\ &+ z \sum_{n=0}^{\infty} \Phi(3n+1) \frac{\left(\frac{3c+1}{3}\right)_{2n} \left(\frac{3c-1}{3}\right)_{2n} (c)_{2n} z^{3n}}{(2)_{3n} \left(\frac{3c+1}{3}\right)_n \left(\frac{3c-1}{3}\right)_n (c)_n (c)_{3n}} + \\ &+ \frac{z^2}{2} \sum_{n=0}^{\infty} \Phi(3n+2) \frac{\left(\frac{3c+2}{3}\right)_{2n} \left(\frac{3c+1}{3}\right)_{2n} (c+1)_{2n} z^{3n}}{(3)_{3n} \left(\frac{3c+2}{3}\right)_n \left(\frac{3c+1}{3}\right)_n (c)_n (c+1)_{3n}}, \end{aligned} \quad (2.1)$$

$(3c \in \mathbb{C} \setminus (\mathbb{Z}_0^- \cup \{1, 2, 3\})) \text{ and } z \in \mathbb{C}$

provided that infinite series occurring on both sides of equation (2.1) are absolutely convergent. For the convergence condition on complex variable z in equation (2.1), it is necessary that actual form of sequence $\{\Phi(\mu)\}_{\mu=1}^{\infty}$ should be known.

In this connection, for the convergence condition on complex variable z see the results (3.1) and (4.1), where the value of $\Phi(\mu)$ is known.

Proof of the assertion (2.1):

Let

$$\Psi(z) := \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \Phi(n+3m) \frac{(c-1)_{n+2m} z^{n+3m}}{(c-1)_{n+3m} (c)_m (27)^m n! m!} \quad (2.2)$$

Using the identity (1.16) in equation (2.2), we get

$$\Psi(z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \Phi(n+3m) \frac{(-1)^m z^{n+3m}}{3^{3m} (2-c-n-3m)_m (c)_m n! m!}. \quad (2.3)$$

Replacing n by $n-3m$ in the right-hand side of equation (2.3) and also using Cauchy's double series identity (1.12), we obtain

$$\Psi(z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\lfloor \frac{n}{3} \rfloor} \Phi(n) \frac{(-1)^m z^n}{3^{3m} (2-c-n)_m (c)_m (n-3m)! m!}$$

Employing the identity (1.15) and the following formula (1.18), we obtain

$$\begin{aligned} \Psi(z) &= \sum_{n=0}^{\infty} \Phi(n) \frac{z^n}{n!} \sum_{m=0}^{\lfloor \frac{n}{3} \rfloor} \frac{(-n)_{3m}}{3^{3m} (2-c-n)_m (c)_m m!} \\ &= \sum_{n=0}^{\infty} \Phi(n) \frac{z^n}{n!} \sum_{m=0}^{\lfloor \frac{n}{3} \rfloor} \frac{\left(\frac{-n}{3}\right)_m \left(\frac{-n+1}{3}\right)_m \left(\frac{-n+2}{3}\right)_m}{(2-c-n)_m (c)_m m!} \\ &= \sum_{n=0}^{\infty} \left\{ \Phi(n) \frac{z^n}{n!} {}_3F_2 \left[\begin{matrix} \frac{-n}{3}, \frac{-n+1}{3}, \frac{-n+2}{3}; \\ c, 2-c-n; \end{matrix} \quad 1 \right] \right\}. \end{aligned} \quad (2.4)$$

Now using decomposition series identity (1.11) in the equation (2.4), we find

$$\begin{aligned} \Psi(z) &= \sum_{n=0}^{\infty} \Phi(3n) \frac{z^{3n}}{(3n)!} {}_3F_2 \left[\begin{matrix} -n, -n + \frac{1}{3}, -n + \frac{2}{3}; \\ c, 2-c-3n; \end{matrix} \quad 1 \right] + \\ &+ \sum_{n=0}^{\infty} \Phi(3n+1) \frac{z^{3n+1}}{(3n+1)!} {}_3F_2 \left[\begin{matrix} -n, -n - \frac{1}{3}, -n + \frac{1}{3}; \\ c, 1-c-3n; \end{matrix} \quad 1 \right] + \\ &+ \sum_{n=0}^{\infty} \Phi(3n+2) \frac{z^{3n+2}}{(3n+2)!} {}_3F_2 \left[\begin{matrix} -n, -n - \frac{2}{3}, -n - \frac{1}{3}; \\ c, -c-3n; \end{matrix} \quad 1 \right]. \end{aligned} \quad (2.5)$$

Further using Saalschütz summation theorem (1.9) in three members ${}_3F_2(1)$ of the right-hand side of equation (2.5), after long and systematic calculation, we have

$$\Psi(z) = \sum_{n=0}^{\infty} \Phi(3n) \frac{z^{3n}}{(3n)!} \left\{ \frac{\left(\frac{3c-1}{3}\right)_{2n} \left(\frac{3c-2}{3}\right)_{2n} (c-1)_{2n}}{\left(\frac{3c-1}{3}\right)_n \left(\frac{3c-2}{3}\right)_n (c)_n (c-1)_{3n}} \right\} +$$

$$\begin{aligned}
& + \sum_{n=0}^{\infty} \Phi(3n+1) \frac{z^{3n+1}}{(3n+1)!} \left\{ \frac{\left(\frac{3c+1}{3}\right)_{2n} \left(\frac{3c-1}{3}\right)_{2n} (c)_{2n}}{\left(\frac{3c+1}{3}\right)_n \left(\frac{3c-1}{3}\right)_n (c)_n (c)_{3n}} \right\} + \\
& + \sum_{n=0}^{\infty} \Phi(3n+2) \frac{z^{3n+2}}{(3n+2)!} \left\{ \frac{\left(\frac{3c+2}{3}\right)_{2n} \left(\frac{3c+1}{3}\right)_{2n} (c+1)_{2n}}{\left(\frac{3c+2}{3}\right)_n \left(\frac{3c+1}{3}\right)_n (c)_n (c+1)_{3n}} \right\}. \quad (2.6)
\end{aligned}$$

On further simplification, we get the required result (2.1).

3. FIRST APPLICATION IN GENERAL SRIVASTAVA-DAOUST FUNCTION

Using double series identity (2.1), we establish a result for reducibility of general Srivastava-Daoust double hypergeometric function containing the arguments z and $\frac{z^3}{27}$ into a sum of three generalized hypergeometric functions ${}_{3D+6}F_{3E+8} \left(\frac{64z^3}{(27)^{(2+E-D)}} \right)$ as in the following theorem.

Theorem 3.1. *The following cubic reduction formula holds true for general Srivastava-Daoust double hypergeometric function:*

$$\begin{aligned}
& F_{E+1;0;1}^{D+1;0;0} \left(\begin{matrix} [(d_D):1, 3], [c-1:1, 2] : - ; - ; \\ [(e_E):1, 3], [c-1:1, 3] : - ; [c:1]; \end{matrix} \middle| z, \frac{z^3}{27} \right) \\
& = {}_{3D+6}F_{3E+8} \left[\begin{matrix} \Delta(3; d_1), \dots, \Delta(3; d_D), \Delta(2; \frac{3c-1}{3}), \Delta(2; \frac{3c-2}{3}), \Delta(2; c-1); \\ \Delta(3; e_1), \dots, \Delta(3; e_E), \frac{1}{3}, \frac{2}{3}, \Delta(3; 3c-2), \Delta(3; c-1); \end{matrix} \middle| \frac{64z^3}{(27)^{(2+E-D)}} \right] + \\
& \quad z \frac{\prod_{i=1}^D (d_i)}{\prod_{j=1}^E (e_j)} \times \\
& \times {}_{3D+6}F_{3E+8} \left[\begin{matrix} \Delta(3; 1+d_1), \dots, \Delta(3; 1+d_D), \Delta(2; \frac{3c+1}{3}), \Delta(2; \frac{3c-1}{3}), \Delta(2; c); \\ \Delta(3; 1+e_1), \dots, \Delta(3; 1+e_E), \frac{2}{3}, \frac{4}{3}, \Delta(3; 3c-1), \Delta(3; c); \end{matrix} \middle| \frac{64z^3}{(27)^{(2+E-D)}} \right] + \\
& \quad z^2 \frac{\prod_{i=1}^D (d_i)_2}{2 \prod_{j=1}^E (e_j)_2} \times \\
& \times {}_{3D+6}F_{3E+8} \left[\begin{matrix} \Delta(3; 2+d_1), \dots, \Delta(3; 2+d_D), \Delta(2; \frac{3c+2}{3}), \Delta(2; \frac{3c+1}{3}), \Delta(2; c+1); \\ \Delta(3; 2+e_1), \dots, \Delta(3; 2+e_E), \frac{4}{3}, \frac{5}{3}, \Delta(3; 3c), \Delta(3; c+1); \end{matrix} \middle| \frac{64z^3}{(27)^{(2+E-D)}} \right], \quad (3.1)
\end{aligned}$$

where $e_1, e_2, \dots, e_E \in \mathbb{C} \setminus \mathbb{Z}_0^-$; $3c \in \mathbb{C} \setminus (\mathbb{Z}_0^- \cup \{1, 2, 3\})$ and (d_D) denotes the sequence of “ D ” number of parameters given by $d_1, d_2, d_3, \dots, d_D$ with similar interpretation for others.

When $D \leq E$ then both sides of equation (3.1) are convergent for $|z| < \infty$ and when $D = E + 1$ then both sides are convergent for some common (suitably constrained values of $|z|$ in left-hand side of (3.1) and $64|z|^3 < (27)^{(2+E-D)}$ in

right-hand side of (3.1)).

Proof of the assertion (3.1):

$$\text{Put } \Phi(\mu) = \frac{(d_1)_\mu (d_2)_\mu \cdots (d_D)_\mu}{(e_1)_\mu (e_2)_\mu \cdots (e_E)_\mu} = \frac{\prod_{i=1}^D (d_i)_\mu}{\prod_{j=1}^E (e_j)_\mu}, \quad (\mu \in \mathbb{N}_0)$$

in both sides of the double series identity (2.1), we obtain

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\prod_{i=1}^D (d_i)_{n+3m} (c-1)_{n+2m} z^{n+3m}}{\prod_{j=1}^E (e_j)_{n+3m} (c-1)_{n+3m} (c)_m (27)^m n! m!} \\ &= \sum_{n=0}^{\infty} \frac{\prod_{i=1}^D (d_i)_{3n} \left(\frac{3c-1}{6}\right)_n \left(\frac{3c+2}{6}\right)_n \left(\frac{3c-2}{6}\right)_n \left(\frac{3c+1}{6}\right)_n \left(\frac{c-1}{2}\right)_n \left(\frac{c}{2}\right)_n (64z^3)^n}{\prod_{j=1}^E (e_j)_{3n} \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{3c-1}{3}\right)_n \left(\frac{3c-2}{3}\right)_n (c)_n \left(\frac{c-1}{3}\right)_n \left(\frac{c}{3}\right)_n \left(\frac{c+1}{3}\right)_n (729)^n n!} + \\ &+ z \sum_{n=0}^{\infty} \frac{\prod_{i=1}^D (d_i)_{3n+1} \left(\frac{3c+1}{6}\right)_n \left(\frac{3c+4}{6}\right)_n \left(\frac{3c-1}{6}\right)_n \left(\frac{3c+2}{6}\right)_n \left(\frac{c}{2}\right)_n \left(\frac{c+1}{2}\right)_n (64z^3)^n}{\prod_{j=1}^E (e_j)_{3n+1} \left(\frac{2}{3}\right)_n \left(\frac{4}{3}\right)_n \left(\frac{3c+1}{3}\right)_n \left(\frac{3c-1}{3}\right)_n (c)_n \left(\frac{c}{3}\right)_n \left(\frac{c+1}{3}\right)_n \left(\frac{c+2}{3}\right)_n (729)^n n!} + \\ &+ \frac{z^2}{2} \sum_{n=0}^{\infty} \frac{\prod_{i=1}^D (d_i)_{3n+2} \left(\frac{3c+2}{6}\right)_n \left(\frac{3c+5}{6}\right)_n \left(\frac{3c+1}{6}\right)_n \left(\frac{3c+4}{6}\right)_n \left(\frac{c+1}{2}\right)_n \left(\frac{c+2}{2}\right)_n (64z^3)^n}{\prod_{j=1}^E (e_j)_{3n+2} \left(\frac{4}{3}\right)_n \left(\frac{5}{3}\right)_n \left(\frac{3c+2}{3}\right)_n \left(\frac{3c+1}{3}\right)_n (c)_n \left(\frac{c+1}{3}\right)_n \left(\frac{c+2}{3}\right)_n \left(\frac{c+3}{3}\right)_n (729)^n n!}. \end{aligned} \quad (3.2)$$

Now using the definition of double hypergeometric function (1.5) of Srivastava-Daoust in the left-hand side of equation (3.2); product theorem (1.18) of Pochhammer symbol and definition of generalized hypergeometric function (1.2) in the right-hand side of equation (3.2), we obtain the required result (3.1).

4. SECOND APPLICATION IN SPECIAL SRIVASTAVA-DAOUST FUNCTION

Using double series identity (2.1), we establish another result for reducibility of special Srivastava-Daoust double hypergeometric function containing the arguments $\frac{z^3}{27(1-z)^2}$ and $\left(\frac{z}{z-1}\right)$ into the product of binomial expansion and sum of three hypergeometric functions ${}_9F_8\left(\frac{64z^3}{27}\right)$ as in the following theorem.

Theorem 4.1. *The following cubic reduction formula holds true for special Srivastava-Daoust double hypergeometric function:*

$$F_{1:1;0}^{1:1;1} \left(\begin{matrix} [c-1:2, 1]: [b:3]; [c-1-b:1]; \\ [c-1:3, 1]: [c:1]; -; \end{matrix} ; \frac{z^3}{27(1-z)^2}, \frac{-z}{1-z} \right)$$

$$\begin{aligned}
&= (1-z)^{(c-1)} {}_9F_8 \left[\begin{matrix} \Delta(3; b), \Delta(2; \frac{3c-1}{3}), \Delta(2; \frac{3c-2}{3}), \Delta(2; c-1); \\ \frac{1}{3}, \frac{2}{3}, \Delta(3; 3c-2), \Delta(3; c-1); \end{matrix} \right. \left. \frac{64z^3}{27} \right] + \\
&+ (1-z)^{(c-1)} bz {}_9F_8 \left[\begin{matrix} \Delta(3; b+1), \Delta(2; \frac{3c+1}{3}), \Delta(2; \frac{3c-1}{3}), \Delta(2; c); \\ \frac{2}{3}, \frac{4}{3}, \Delta(3; 3c-1), \Delta(3; c); \end{matrix} \right. \left. \frac{64z^3}{27} \right] + \\
&+ \frac{(1-z)^{(c-1)} b(b+1)z^2}{2} {}_9F_8 \left[\begin{matrix} \Delta(3; b+2), \Delta(2; \frac{3c+2}{3}), \Delta(2; \frac{3c+1}{3}), \Delta(2; c+1); \\ \frac{4}{3}, \frac{5}{3}, \Delta(3; 3c), \Delta(3; c+1); \end{matrix} \right. \left. \frac{64z^3}{27} \right], \tag{4.1}
\end{aligned}$$

where $3c \in \mathbb{C} \setminus (\mathbb{Z}_0^- \cup \{1, 2, 3\})$ and both sides are convergent for some common region (suitably constrained values of $|z|$ in left-hand side of (4.1) and $|z| < \frac{3}{4}, |\arg(1-z)| < \pi, |z| < 1$ in right-hand side of (4.1)).

Proof of the assertion (4.1):

Put $\Phi(\mu) = (b)_\mu$, ($\mu \in \mathbb{N}_0$) in the both sides of double series identity (2.1) and after using (1.17) in left-hand side, we obtain

$$\begin{aligned}
&\sum_{m=0}^{\infty} \frac{(b)_{3m}(c-1)_{2m}(\frac{z^3}{27})^m}{(c-1)_{3m}(c)_m m!} \sum_{n=0}^{\infty} \frac{(b+3m)_n(c-1+2m)_n z^n}{(c-1+3m)_n n!} \\
&= \sum_{n=0}^{\infty} (b)_{3n} \frac{(\frac{3c-1}{3})_{2n} (\frac{3c-2}{3})_{2n} (c-1)_{2n} z^{3n}}{(1)_{3n} (\frac{3c-1}{3})_n (\frac{3c-2}{3})_n (c)_n (c-1)_{3n}} + \\
&+ z \sum_{n=0}^{\infty} (b)_{3n+1} \frac{(\frac{3c+1}{3})_{2n} (\frac{3c-1}{3})_{2n} (c)_{2n} z^{3n}}{(2)_{3n} (\frac{3c+1}{3})_n (\frac{3c-1}{3})_n (c)_n (c)_{3n}} + \\
&+ \frac{z^2}{2} \sum_{n=0}^{\infty} (b)_{3n+2} \frac{(\frac{3c+2}{3})_{2n} (\frac{3c+1}{3})_{2n} (c+1)_{2n} z^{3n}}{(3)_{3n} (\frac{3c+2}{3})_n (\frac{3c+1}{3})_n (c)_n (c+1)_{3n}}. \tag{4.2}
\end{aligned}$$

Using the definition (1.2) in the left-hand side of equation (4.2) and also employing the following identities (1.17) and (1.18) in the right-hand side of equation (4.2), we get

$$\begin{aligned}
&\sum_{m=0}^{\infty} \frac{(b)_{3m}(c-1)_{2m}(\frac{z^3}{27})^m}{(c-1)_{3m}(c)_m m!} {}_2F_1 \left[\begin{matrix} b+3m, c-1+2m; \\ c-1+3m; \end{matrix} \right. \left. z \right] \\
&= \sum_{n=0}^{\infty} \frac{(\frac{b}{3})_n (\frac{b+1}{3})_n (\frac{b+2}{3})_n (\frac{3c-1}{6})_n (\frac{3c+2}{6})_n (\frac{3c-2}{6})_n (\frac{3c+1}{6})_n (\frac{c-1}{2})_n (\frac{c}{2})_n (64z^3)^n}{(\frac{1}{3})_n (\frac{2}{3})_n (\frac{3c-1}{3})_n (\frac{3c-2}{3})_n (c)_n (\frac{c-1}{3})_n (\frac{c}{3})_n (\frac{c+1}{3})_n (27)^n n!} + \\
&+ bz \sum_{n=0}^{\infty} \frac{(\frac{b+1}{3})_n (\frac{b+2}{3})_n (\frac{b+3}{3})_n (\frac{3c+1}{6})_n (\frac{3c+4}{6})_n (\frac{3c-1}{6})_n (\frac{3c+2}{6})_n (\frac{c}{2})_n (\frac{c+1}{2})_n (64z^3)^n}{(\frac{2}{3})_n (\frac{4}{3})_n (\frac{3c+1}{3})_n (\frac{3c-1}{3})_n (c)_n (\frac{c}{3})_n (\frac{c+1}{3})_n (\frac{c+2}{3})_n (27)^n n!} + \\
&+ \frac{b(b+1)z^2}{2} \sum_{n=0}^{\infty} \frac{(\frac{b+2}{3})_n (\frac{b+3}{3})_n (\frac{b+4}{3})_n (\frac{3c+2}{6})_n (\frac{3c+5}{6})_n (\frac{3c+1}{6})_n (\frac{3c+4}{6})_n (\frac{c+1}{2})_n (\frac{c+2}{2})_n (64z^3)^n}{(\frac{4}{3})_n (\frac{5}{3})_n (\frac{3c+2}{3})_n (\frac{3c+1}{3})_n (c)_n (\frac{c+1}{3})_n (\frac{c+2}{3})_n (\frac{c+3}{3})_n (27)^n n!}. \tag{4.3}
\end{aligned}$$

Now using Pfaff-Kummer linear transformation (1.10) in the left-hand side of equation (4.3), after expressing into double hypergeometric function (1.5) of

Srivastava-Daoust and also using definition (1.2) in the right-hand side of equation (4.3), we find the required result (4.1).

Remark 4.1. We have verified the above reduction formula (4.1) numerically by using Mathematica software under the following conditions:

- (i) $-\frac{3}{4} < \Re(z) < \frac{3}{4}$ and $\Im(z) = 0$.
- (ii) When $\Re(z) = \frac{3}{4}$ and $\Im(z) = 0$ then $\Re(c - b) > \frac{1}{2}$.
- (iii) When $\Re(z) = -\frac{3}{4}$ and $\Im(z) = 0$ then $\Re(c - b) > -\frac{1}{2}$.

Note: By the theory of analytic continuation, the result (4.1) also valid in $-\frac{211}{25} \leq \Re(z) \leq \frac{3}{4}$ and $\Im(z) = 0$.

5. CONCLUDING REMARKS AND OBSERVATIONS

Here, in this paper, we have established a new general double-series identity (2.1) by using Cauchy's double series identity (1.12), decomposition series identity (1.11), Saalschütz summation theorem ${}_3F_2(1)$ (or one-balanced summation theorem) (1.9) and series rearrangement technique. Then among numerous applications of this general double-series identity (2.1), we obtain two cubic reduction formulas (3.1) and (4.1) for Srivastava-Daoust double hypergeometric functions having arguments $\left(z, \frac{z^3}{27}\right)$ and $\left(\frac{z^3}{27(1-z)^2}, \frac{-z}{1-z}\right)$ under appropriate convergence conditions. Therefore, we conclude our present investigation, by observing that the several other interesting and potentially useful general double-series identities, certain reduction formulas for Srivastava-Daoust double hypergeometric functions can be derived in an analogous manner. The various results, which we have presented in this article, are potentially useful in mathematical analysis and applied mathematics (see, for example, [23],[24]; see also [18]).

With a view to encouraging and motivating further researches emerging from the present investigation, we have chosen to draw the attention of the interested readers toward some related recent developments (see, for example, [9], [10], [26] and [27]) on hypergeometric functions and other families of higher transcendental functions.

Conflicts of Interest: The authors declare that they have no conflicts of interest.

Acknowledgement. The authors are very grateful to the anonymous reviewer and editor for their constructive encouraging comments and valuable suggestions which improved this paper after revision.

REFERENCES

1. G.E. Andrews, R. Askey and R. Roy, *Special Functions*, Cambridge university press, Cambridge, UK, 1999.
2. P. Appell and J. Kampé de Fériet, *Fonctions Hypergéométriques et Hypersphériques: Polynômes d' Hermite*, Gauthier-Villars, Paris, 1926.
3. R.G. Buschman and H.M. Srivastava, *Series identities and reducibility of Kampé de Fériet functions*, Math. Proc. Cambridge Philos. Soc. **91** (1982), 435-440. <https://doi.org/10.1017/S0305004100059478>
4. K.-Y. Chen and H.M. Srivastava, *Series identities and associated families of generating functions*, J. Math. Anal. Appl. **311** (2005), 582-599. <https://doi.org/10.1016/j.jmaa.2005.03.030>

5. K.-Y. Chen, S.-J. Liu and H.M. Srivastava, *Some double-series identities and associated generating-function relationships*, Appl. Math. Lett. **19** (2006), 887-893. <https://doi.org/10.1016/j.aml.2005.07.013>
6. N.N. Lebedev, *Special Functions and their Applications*, (Translated by Richard A. Silverman), Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1965.
7. F.W.J. Olver, D.W. Lozier, R.F. Boisvert and C.W. Clark (Editors), *NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge, 2010.
8. A.P. Prudnikov, Yu.A. Brychkov and O.I. Marichev, *Integrals and Series, Vol.3: More special functions*, Nauka, Moscow, 1986 (In Russian); (Translated from the Russian by G. G. Gould), Gordon and Breach Science Publishers, New York, Philadelphia, London, Paris, Montreux, Tokyo, Melbourne, 1990.
9. M.I. Qureshi, K.A. Quraishi and H.M. Srivastava, *Some hypergeometric summation formulas and series identities associated with exponential and trigonometric functions*, Integral Transforms Spec. Funct. **19** (2008), 267-276. <https://doi.org/10.1080/10652460801896024>
10. M.I. Qureshi and S.H. Malik, *A general double-series identity and its application in hypergeometric reduction formulas*, Turk. J. Math. **45** (2021), 2759-2772. <https://doi.org/10.3906/mat-2105-101>
11. E.D. Rainville, *Special Functions*, The Macmillan Company, New York 1960; Reprinted by Chelsea Publ. Co. Bronx, New York, 1971.
12. S. Saran, *Reducibility of generalized Kampé de Fériet function*, Ganita, **31**, (1980), 89-98.
13. O. Shanker and S. Saran, *Reducibility of Kampé de Fériet function*, Ganita, **21**(1), (1970), 9-16.
14. D.D. Somashekara, K.N. Murthy and S.L. Shalini, *On new forms of the reciprocity theorems*, Gulf Journal of Mathematics, **3**(1), (2015), 104-117. <https://doi.org/10.56947/gjom.v3i1.174>
15. D.D. Somashekara and K.N. Vidya, *On some transformation formulas and their applications to partial theta function identities*, Gulf Journal of Mathematics, **5**(2), (2017), 31-45. <https://doi.org/10.56947/gjom.v5i2.95>
16. H.M. Srivastava, *On the reducibility of Appell's function F_4* , Canad. Math. Bull. **16** (1973), 295-298. <https://doi.org/10.4153/CMB-1973-049-9>
17. H.M. Srivastava, *Some generalizations of Carlson's identity*, Boll. Un. Mat. Ital. **18A**(5) (1981), 138-143.
18. H.M. Srivastava, *A survey of some recent developments on higher transcendental functions of analytic number theory and applied mathematics*, Symmetry. **13** (2021), Article ID 2294, 1-22. <https://doi.org/10.3390/sym13122294>
19. H.M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier Science Publishers, Amsterdam, London and New York, 2012.
20. H.M. Srivastava and M.C. Daoust, *On Eulerian integrals associated with Kampé de Fériet's function*, Publ. Inst. Math. (Beograd) (N.S.). **9**(23) (1969), 199-202. <https://doi.org/10.1002/mana.19720530114>
21. H.M. Srivastava and M.C. Daoust, *Certain generalized Neumann expansions associated with the Kampé de Fériet's function*, Nederl. Akad. Wetensch. Proc. Ser. A, 72=Indag. Math., **31** (1969), 449-457.
22. H.M. Srivastava and M.C. Daoust, *A note on the convergence of Kampé de Fériet's double hypergeometric series*, Math. Nachr. **53** (1972), 151-159.
23. H.M. Srivastava and P.W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1985.
24. H.M. Srivastava and B.R.K. Kashyap, *Special Functions in Queuing Theory and Related Stochastic Processes*, Academic Press, New York and London, 1982.

25. H.M. Srivastava and H.L. Manocha, *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester, U.K.) John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.
26. H.M. Srivastava, M.I. Qureshi, K.A. Quraishi and R. Singh, *Applications of some hypergeometric summation theorems involving double series*, Journal of the Applied Mathematics, Statistics and Informatics. **8**(2)(2012), 37-48. <https://doi.org/10.2478/v10294-012-0013-3>
27. H.M. Srivastava, M.I. Qureshi and S. Jabee, *Some general series identities and summation theorems for Clausen's hypergeometric function with negative integer numerator and denominator parameters*, J. Nonlinear Convex Anal. **21** (2020), 805-819.

¹DEPARTMENT OF APPLIED SCIENCES AND HUMANITIES, FACULTY OF ENGINEERING AND TECHNOLOGY, JAMIA MILLIA ISLAMIA (A CENTRAL UNIVERSITY), NEW DELHI-110025, INDIA

Email address: miqureshi.delhi@yahoo.co.in

² DEPARTMENT OF APPLIED SCIENCES AND HUMANITIES, FACULTY OF ENGINEERING AND TECHNOLOGY, JAMIA MILLIA ISLAMIA (A CENTRAL UNIVERSITY), NEW DELHI-110025, INDIA

Email address: malikshakir774@gmail.com