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COMMON MULTIPLES OF PATHS AND STARS WITH COMPLETE GRAPHS

REJI THANKACHAN[1] AND SARITHA CHANDRAN[2*]

ABSTRACT. A graph G is a common multiple of two graphs H_1 and H_2 if there exists a decomposition of G into edge-disjoint copies of H_1 and also a decomposition of G into edge-disjoint copies of H_2. If G is a common multiple of H_1 and H_2 and G has q edges, then we call G a (q, H_1, H_2) graph. Our paper deals with the following question: 'Given two graphs H_1 and H_2, for which values of q does there exist a (q, H_1, H_2) graph?' when H_1 is either a path or a star with 3 or 4 edges and H_2 is a complete graph.

1. Introduction and preliminaries

All graphs considered here are finite and undirected, unless otherwise noted. The size of a graph G, denoted by $e(G)$, is its number of edges.

K_n denotes the complete graph on n vertices, and $K_{m,n}$ denotes the complete bipartite graph with vertex partitions of sizes m and n.

A *k-path*, denoted by P_k, is a path with k vertices (is a path of length $k-1$); a *k-star*, denoted by S_k, is the complete bipartite graph $K_{1,k}$.

Let G and H be graphs. A *decomposition* of G is a set of edge-disjoint subgraphs of G whose union is G. An H- *decomposition* of G is a decomposition of G into copies of H. If G has an *H-decomposition*, we say that G is *H-decomposable* or H *divides* G and write $H|G$.

Given two graphs H_1 and H_2, one may ask for a graph G that is a *common multiple of* H_1 and H_2 in the sense that both H_1 and H_2 divide G. Several authors have investigated the problem of finding *least common multiples* of pairs of graphs; that is, graphs of minimum size which are both H_1- and H_2-decomposable. The problem was introduced by Chartrand et al in [6] and they showed that every two nonempty graphs have a least common multiple. It is clear that least common multiple of two graphs may not be unique. The size of a least common multiple of two graphs H_1 and H_2 is denoted by *lcm*(H_1,H_2). Also if q_1 and q_2 are two natural numbers, their number theoretic *lcm* is denoted by $lcm(q_1, q_2)$ as usual. Clearly, the least common multiple of two graphs H_1 and H_2 , *lcm*(H_1,H_2) $\geq$ *lcm*($e(H_1)$,$e(H_2)$). The problem of finding the size of least common multiples of graphs has been studied for several pairs of graphs: cycles and stars[6, 15],

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* Corresponding author.
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paths and complete graphs [12], pairs of cycles [10], pairs of cubes [2]. Pairs of graphs having a unique least common multiple were investigated in [7] and least common multiples of digraphs were considered in [8]. This is an interesting area in decomposition of graphs like different types of domination[5, 13] in the field of domination in graphs.

If G is a common multiple of H_1 and H_2, and G has q edges, then we call G a (q, H_1, H_2) graph. An obvious necessary condition for the existence of a (q, H_1, H_2) graph is that $e(H_1)|q$ and $e(H_2)|q$. This obvious necessary condition is not always sufficient. Some necessary conditions are easy to see and others are more difficult. For example there is no $(15, K_3, K_6)$ graph as there is no K_3-decomposition of K_6. However, non-existence of a $(36, K_3, K_4)$ graph is somewhat less obvious. Hence a natural question is: *Given two graphs H_1 and H_2, for which values of q, does there exist a (q, H_1, H_2) graph?* Adams, Bryant, and Maenhaut [1] gave a complete solution to this problem in the case where H_1 is the *4-cycle* and H_2 is a complete graph; Bryant and Maenhaut [3] gave a complete solution to this problem in the case where H_1 is the complete graph K_3 and H_2 is a complete graph. A complete solution to this problem in the case where H_1 is a path and H_2 is a star, is investigated in [9].

Since $S_2 = P_3$, the obvious necessary and sufficient condition for the existence of a (q, P_3, K_n) graph is that $2q \equiv 0 \pmod{n(n-1)}$ when $\binom{n}{2}$ is even and $q \equiv 0 \pmod{n(n-1)}$ when $\binom{n}{2}$ is odd (a nontrivial connected graph G is P_3-decomposable if and only if G has even size). So in this paper we establish the necessary and sufficient condition for the existence of a (q, P_4, K_n) graph, a (q, P_5, K_n) graph, a (q, S_3, K_n) graph and a (q, S_4, K_n) graph. The graph theoretic concepts described here are, of course, suggested by their number theoretic counterparts.

The complete graph with vertex set $\{v_1, v_2,..., v_m\}$ will be denoted by $[v_1, v_2,..., v_m]$, m-path P_m with vertex set $\{v_1, v_2,..., v_m\}$ and edges $\{v_1, v_2\}$, $\{v_2, v_3\}$,..., $\{v_{m-1}, v_m\}$ will be denoted by $\langle v_1, v_2, ..., v_m\rangle$ and m-star S_m with vertex set $\{v_0, v_1, v_2,..., v_m\}$ and center at v_0 will be denoted by $[v_0; v_1, v_2,..., v_m]$. If G and H are graphs, and H is a subgraph of G, then the graph obtained by removing the edges of H from G will be denoted by $G - H$. If G_1 and G_2 are graphs, then the *union* of G_1 and G_2, denoted by $G_1 \cup G_2$, is the graph with vertex set $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. (We shall only be considering the union of edge-disjoint graphs.)

2. Common Multiples of P_4 and K_n

In this section we determine, for all positive integers n, the set of integers q for which there exists a common multiple of P_4 and K_n having precisely q edges. The following known results on the path-decomposition of complete graphs are used for the discussion.

Theorem 2.1. [14] *Let k and n be positive integers. There exists a P_{k+1}-decomposition of K_n if and only if $n \geq k + 1$ and $n(n-1) \equiv 0 \pmod{2k}$.*

Theorem 2.2. [4] *Let n and t be positive integers and let $m_1, m_2, ..., m_t$ be a sequence of positive integers. There exist t pairwise edge-disjoint paths of lengths*

$m_1, m_2, ..., m_t$ in K_n if and only if $m_i \leq n-1$ for $i = 1, 2, ..., t$ and $m_1 + m_2 + \cdots + m_t \leq \binom{n}{2}$.

Characterization for the existence of a (q, P_4, K_n) graph is given in the next theorem.

Theorem 2.3. *There exists a graph with q edges that is both P_4-decomposable and K_n-decomposable if and only if*

(1) $2q \equiv 0 \pmod{n(n-1)}$ *when* $n \equiv 0, 1, 3, 4 \pmod 6$
(2) $2q \equiv 0 \pmod{3n(n-1)}$ *when* $n \equiv 2, 5 \pmod 6$
(3) $q \neq 3$ *when* $n = 3$.

Proof. If there exists a (q, P_4, K_n) graph, then we require that 3 divides q and that $\binom{n}{2}$ divides q. Necessary conditions (1) and (2) follow immediately from this and will be referred to as the obvious necessary conditions. If $n = 3$, then $q \neq 3$ as K_3 is not P_4-decomposable.

Sufficient Conditions

If $n \equiv 0, 1, 3, 4 \pmod 6$ and $n \geq 4$, then $P_4 | K_n$ (Theorem 2.1) and hence when $n \equiv 0, 1, 3, 4 \pmod 6$, there exists a (q, P_4, K_n) graph G for all $q \equiv 0 \pmod{\binom{n}{2}}$ by taking G to be $\frac{q}{\binom{n}{2}}$ vertex-disjoint copies of K_n. If $n = 3$, it is sufficient to construct a $(6, P_4, K_3)$ graph and a $(9, P_4, K_3)$ graph as all the required graphs can be constructed as the vertex-disjoint union of the appropriate number of copies of these.
To construct a $(6, P_4, K_3)$ graph G, we let G be the union of the following two edge-disjoint copies of K_3.
[1, 2, 3] [1, 4, 5]

A P_4-decomposition of G is given by the following two edge-disjoint copies of P_4.
$\langle 2, 3, 1, 4 \rangle$ $\langle 4, 5, 1, 2 \rangle$

To construct a $(9, P_4, K_3)$ graph G, we let G be the union of the following three edge-disjoint copies of K_3.
[1, 2, 3] [1, 4, 5] [1, 6, 7]

A P_4-decomposition of G is given by the following three edge-disjoint copies of P_4.
$\langle 7, 6, 1, 5 \rangle$ $\langle 5, 4, 1, 3 \rangle$ $\langle 3, 2, 1, 7 \rangle$

Thus sufficient conditions (1)and (3) obtained.

If $n \equiv 2, 5 \pmod 6$, then $\binom{n}{2} \equiv 1 \pmod 3$ and $lcm(3, \binom{n}{2}) = 3\binom{n}{2}$. First suppose that $n = 2$. For a $(3, P_4, K_2)$ graph G, we let G be P_4, which is K_2-decomposable.

Now suppose that $n \equiv 2, 5 \pmod 6$ and $n \geq 5$. Let $\binom{n}{2} = 3r + 1$, where $r > 0$. By Theorem 2.2, K_n can be decomposed into the paths $\underbrace{P_4, P_4, ..., P_4}_{r-copies}, P_2$.

Take three copies of K_n. Consider the end vertices of P_2 in each copy of K_n in the decomposition. Identify these vertices to get a connected graph G, which is actually three K_n's joined by identification of end vertices of copies of P_2 serially. G is a $(3\binom{n}{2}, P_4, K_n)$ graph, since by the construction itself G is K_n-decomposable. G is also P_4-decomposable, since r copies of P_4 can be taken from each K_n and one copy of P_4 can be obtained by identification of the end vertices of three P_2's. Therefore there exists a $(k\binom{n}{2}, P_4, K_n)$ graph G for all $k \equiv 0 \pmod 3$, $n \equiv 2, 5 \pmod 6$.

□

3. Common Multiples of P_5 and K_n

Following theorem gives a characterization for the existence of a (q, P_5, K_n) graph.

Theorem 3.1. *There exists a graph with q edges that is both P_5-decomposable and K_n-decomposable if and only if*

(1) $2q \equiv 0 \pmod{n(n-1)}$ *when* $n \equiv 0, 1 \pmod 8$
(2) $q \equiv 0 \pmod{n(n-1)}$ *when* $n \equiv 4, 5 \pmod 8$
(3) $q \equiv 0 \pmod{2n(n-1)}$ *when* $n \equiv 2, 3, 6, 7 \pmod 8$.

If there exists a (q, P_5, K_n) graph, then we require that 4 divides q and that $\binom{n}{2}$ divides q. Necessary conditions follow immediately from this and will be referred to as the obvious necessary conditions.

Sufficient Conditions

To show that the stated necessary conditions are sufficient we consider each in turn and construct the (q, P_5, K_n) graphs required to prove Theorem 3.1. First note that if $n \equiv 0, 1 \pmod 8$, then $P_5|K_n$ (Theorem 2.1) and hence when $n \equiv 0, 1 \pmod 8$, there exists a (q, P_5, K_n) graph G for all $q \equiv 0 \pmod{\binom{n}{2}}$. (Take G be $\frac{q}{\binom{n}{2}}$ vertex-disjoint copies of K_n). Thus sufficient condition (1) obtained. We require a few lemmas to construct the graphs for the remaining congruence classes of $n \pmod 8$.

Lemma 3.2. *There exists a $(6k, P_5, K_4)$ graph for all even k.*

Proof. $lcm(4, 6) = 12$. So it is sufficient to construct a $(12, P_5, K_4)$ graph G, as all the required graphs can be constructed as the vertex-disjoint union of the appropriate number of copies of this.

To construct a $(12, P_5, K_4)$ graph G, we let G be the union of the following two edge-disjoint copies of K_4.

[1, 2, 3, 4]   [1, 5, 6, 7]

A P_5-decomposition of G is given by the following three edge-disjoint copies of P_5.

$\langle 2,4,3,1,6\rangle \qquad \langle 3,2,1,5,7\rangle \qquad \langle 4,1,7,6,5\rangle$

□

Lemma 3.3. *For all $k \equiv 0 \pmod 4$, there exists a $(k\binom{n}{2}, P_5, K_n)$ graph, when $n = 2, 3$.*

Proof. If $n = 2, 3$, then $gcd(4, \binom{n}{2}) = 1$ and hence $lcm(4, \binom{n}{2}) = 4\binom{n}{2}$.
First suppose that $n = 2$. For a $(4, P_5, K_2)$ graph G, we let G be P_5, which is K_2-decomposable.

Now suppose that $n = 3$. To construct a $(12, P_5, K_3)$ graph G, we let G be the union of the following four edge-disjoint copies of K_3.

[1, 2, 3] [3, 4, 5] [5, 6, 7] [7, 8, 9]

A P_5-decomposition of G is given by the following three edge-disjoint copies of P_5.

$\langle 1,2,3,4,5\rangle \qquad \langle 5,6,7,8,9\rangle \qquad \langle 9,7,5,3,1\rangle$

Hence, if $n = 2, 3$, we can construct a $(k\binom{n}{2}, P_5, K_n)$ graph G, for all $k \equiv 0 \pmod 4$.

□

Lemma 3.4. *For all $n \geq 5$, there exists a (q, P_4, K_n) graph if*

(1) *$q \equiv 0 \pmod{n(n-1)}$ and $n \equiv 4, 5 \pmod 8$; or*
(2) *$q \equiv 0 \pmod{2n(n-1)}$ when $n \equiv 2, 3, 6, 7 \pmod 8$.*

Proof. If $n \equiv 4, 5 \pmod 8$, then $\binom{n}{2} \equiv 2 \pmod 4$ and $lcm(4, \binom{n}{2}) = 2\binom{n}{2}$.
If $n \equiv 2, 7 \pmod 8$, then $\binom{n}{2} \equiv 1 \pmod 4$ and $lcm(4, \binom{n}{2}) = 4\binom{n}{2}$.
If $n \equiv 3, 6 \pmod 8$, then $\binom{n}{2} \equiv 3 \pmod 4$ and $lcm(4, \binom{n}{2}) = 4\binom{n}{2}$.
Consider the following cases.

Case 1: $n \equiv 4, 5 \pmod 8, n \geq 5$

Let $\binom{n}{2} = 4r + 2$, where $r > 0$.

Since $n \geq 5$, by Theorem 2.2, K_n can be decomposed into the edge-disjoint paths

$$\underbrace{P_5, P_5, ..., P_5}_{r-copies}, P_3$$

Take two copies of K_n and identify the end vertices of two P_3 in each copy of K_n to get the required graph $G = (2\binom{n}{2}, P_5, K_n)$. By the construction itself G is K_n-decomposable. G is P_5-decomposable, since G can be decomposed into $2r+1$ edge-disjoint copies of P_5. The identified vertex joined two copies of P_3 to make a P_5. Then there exists a $(k\binom{n}{2}$,P_5,$K_n)$ graph for all even k.

Case 2: $n \equiv 2, 7 \pmod 4$

Let $\binom{n}{2}$ = 4r+1, where $r > 0$. By Theorem 2.2, K_n can be decomposed into the edge-disjoint paths

$$\underbrace{P_5, P_5, ..., P_5}_{r-copies}, P_2$$

Take four copies of K_n. Consider the four edges of P_2 in each copy of K_n in the above decomposition. Concatenate these four edges at the end vertices to make a path of length 4. This is a $(4\binom{n}{2}, P_5, K_n)$ graph G. By the construction itself G is K_n-decomposable. G is P_5-decomposable, since G can be decomposed into $4r + 1$ edge-disjoint copies of P_5. Among these copies of P_5 in G, one copy of P_5 is obtained by the concatenation of four P_2's. Then there exists a $(k\binom{n}{2}, P_5, K_n)$ graph for all $k \equiv 0 \pmod 4$.

Case 3: $n \equiv 3, 6 \pmod 4$.
Let $\binom{n}{2} = 4r + 3$, where $r > 0$. By Theorem 2.2, K_n can be decomposed into the edge-disjoint paths

$$\underbrace{P_5, P_5, ..., P_5}_{r-copies}, P_4$$

Take four copies of K_n. Consider the end vertices of P_4 in each copy of K_n in the decomposition. Concatenate the four paths at these end vertices to make a path of length 12. This is the required graph $G = (4\binom{n}{2}, P_5, K_n)$. By the construction itself G is K_n-decomposable. G is P_5-decomposable, since G can be decomposed into $4r + 3$ edge-disjoint copies of P_5. Three copies of P_5 is obtained by decomposing the above path of length 12. Thus there exists a $(k\binom{n}{2}, P_5, K_n)$ graph for all $k \equiv 0 \pmod 4$. □

Lemmas 3.2, 3.3 and 3.4 allow us to construct all of the (q, P_5, K_n) graphs that we require for the sufficient conditions (2) and (3) of Theorem 3.1. Thus proof of Theorem 3.1 is completed.

4. Common Multiples of S_3 and K_n

In this section we determine, for all positive integers n, the set of integers q for which there exists a common multiple of $S_3(3 - star)$ and complete graph K_n having precisely q edges. The following known results on the star-decomposition of complete graphs are used for the discussion.

Theorem 4.1. [16] *A complete graph, K_n can be decomposed into $\frac{\binom{n}{2}}{k}$ edge disjoint stars, S_k if and only if $\binom{n}{2} \equiv 0 \pmod k$ and $n \geq 2k$.*

Theorem 4.2. [11] *Let $m_1, m_2, ..., m_l$ be non-negative integers such that $\sum_{i=1}^{l} m_i = \binom{n}{2}$ and $2m_i \leq n$ for $1 \leq i \leq l$. Then K_n can be decomposed into $S_{m_1}, S_{m_2}, ..., S_{m_l}$.*

Theorem 4.3. [6] *For every integer $l \geq 1$, $lcm(C_3, K_{1,l}) = \dfrac{3kl}{d}$, where $d = gcd(3, l)$ and $k = \lceil \frac{d(2l+3)}{9} \rceil$.*

The following theorem gives the necessary and sufficient condition for the existence of a (q, S_3, K_n) graph.

Theorem 4.4. *There exists a graph with q edges that is both S_3-decomposable and K_n-decomposable if and only if*

(1) $2q \equiv 0 \pmod{n(n-1)}$ *when* $n \equiv 0, 1, 3, 4 \pmod 6$
(2) $2q \equiv 0 \pmod{3n(n-1)}$ *when* $n \equiv 2, 5 \pmod 6$
(3) $2q \geq 3n(n-1)$ *when* $n = 3, 4$.

Proof. The given conditions are necessary for the following reasons.

If there exists a (q, S_3, K_n) graph, then we require that 3 divides q and that $\binom{n}{2}$ divides q. Conditions (1) & (2) follow immediately from this and will be referred to as the obvious necessary conditions.

If $\mathbf{n = 3}$ and there exists a (q, S_3, K_3) graph, then $q \neq 3, 6$ since $lcm(S_3, K_3) = 9$ by Theorem 4.3.

If $\mathbf{n = 4}$ and there exists a (q, S_3, K_4) graph, then $q \neq 6$, since K_4 is not S_3-decomposable, and $q \neq 12$, since two copies of K_4 intersecting in at most one vertex is not S_3-decomposable. Hence $q \geq 3\binom{n}{2}$ when $n = 3, 4$.

Sufficient Conditions

To show that the stated necessary conditions are sufficient we consider each in turn and construct the (q, S_3, K_n) graphs required to prove Theorem 4.4.

Case 1: $n = 2, 3, 4, 5$

When $n = 2$, $K_2|S_3$ and hence there exists a (q, S_3, K_2) graph G for all $q \equiv 0 \pmod 3$ (Take G to be $\frac{q}{3}$ vertex disjoint copies of S_3).

When $n = 3$, it is sufficient to construct a $(9, S_3, K_3)$ graph, a $(12, S_3, K_3)$ graph and a $(15, S_3, K_3)$ graph as all the required graphs can be constructed as the vertex-disjoint union of the appropriate number of copies of these.

For a $(9, S_3, K_3)$ graph G, we let G be the union of the following three edge-disjoint copies of K_3.

[1, 2, 3]   [2, 4, 5]   [3, 5, 6]

An S_3-decomposition of G is given by the following three edge-disjoint copies of S_3.

[2; 1, 3, 4]   [3; 1, 5, 6]   [5; 2, 4, 6]

For a $(12, S_3, K_3)$ graph G, we let G be $K_{2,2,2}$, the graph of the octahedron. It is easy to see that $K_{2,2,2}$ is S_3-decomposable and K_3-decomposable.

To construct a $(15, S_3, K_3)$ graph G, we let G be the union of the following five edge-disjoint copies of K_3.

[1, 2, A]   [1, 3, B]   [2, 4, B]   [3, 5, A]   [4, 5, C]

An S_3-decomposition of G is given by the following five edge-disjoint copies of S_3.

[1; 2, A, B]   [2; 4, A, B]   [3; 1, A, B]   [4; 5, C, B]   [5; 3, A, C]

Hence there exists a $(3k, S_3, K_3)$ graph G for all $k \geq 3$.

When $n = 4$, $lcm(3, \binom{n}{2}) = 6$. It is sufficient to construct a $(18, S_3, K_4)$ graph, a $(24, S_3, K_4)$ graph and a $(30, S_3, K_4)$ graph as all the required graphs can be constructed as the vertex-disjoint union of the appropriate number of copies of these. To construct a $(18, S_3, K_4)$ graph G, we let G be the union of the following three edge-disjoint copies of K_4.

[1, 2, 3, A] [1, 4, 5, B] [2, 4, 6, C]

An S_3-decomposition of G is given by the following six edge-disjoint copies of S_3.

[1; 2, A, B] [2; 4, C, A] [3; 2, 1, A]

[4; 1, C, B] [5; 4, 1, B] [6; 4, 2, C]

To construct a $(24, S_3, K_4)$ graph G, we let G be the union of the following four edge-disjoint copies of K_4.

[0, 2, 3, A] [2, 4, 5, B] [4, 6, 7, A] [1, 3, 7, B]

An S_3- decomposition of G is given by the following eight edge-disjoint copies of S_3.

[0; 2, 3, A] [1; 3, 7, B] [2; 4, A, B] [3; 2, A, B]

[4; 7, A, B] [5; 2, 4, B] [6; 4, 7, A] [7; 3, A, B]

To construct a $(30, S_3, K_4)$ graph G, we let G be the union of the following five edge-disjoint copies of K_4.

[0, 2, 4, A] [1, 3, 5, A] [4, 6, 8, B] [5, 7, 9, B] [2, 3, 6, 7]

An S_3- decomposition of G is given by the following ten copies of S_3.

[0; 2, 4, A] [1; 3, 5, A] [2; 4, 7, A] [3; 6, 2, A]

[4; 6, A, B] [5; 3, A, B] [6; 7, 2, B] [7; 5, 3, B]

[8; 6, 4, B] [9; 7, 5, B]

Therefore there exists a $(6k, S_3, K_4)$ graph G for all $k \geq 3$.

When $n = 5$, $lcm(3, \binom{n}{2}) = 30$. It is easy to see that K_5 can be decomposed into three edge-disjoint copies of S_3 and an S_1. Take three copies of K_5 and identify them at one vertex of S_1 in the decomposition of each K_5, to get the required graph $G = (30, S_3, K_5)$ graph. Then G is K_5-decomposable, by its construction and G can be decomposed into 10 edge-disjoint copies of S_3(one S_3 centered at the identified vertex and 9 from three copies of K_5). Therefore there exists a $(10k, S_3, K_5)$ graph G for all $k \equiv 0 \pmod 3$.

Case 2: $n \geq 6$ and n$\equiv 0, 1, 3, 4 \pmod 6$.

Then $S_3|K_n$, by Theorem 4.1. Hence in this case there exists a (q, S_3, K_n) graph G for all $q \equiv 0 \pmod{\binom{n}{2}}$(Take G to be $\frac{q}{\binom{n}{2}}$ vertex-disjoint copies of K_n).

Case 3: $n \geq 6$ and $n \equiv 2, 5 \pmod 6$.

In this case $\binom{n}{2} \equiv 1 \pmod 3$ and $lcm(3, \binom{n}{2}) = 3\binom{n}{2}$. Let $\binom{n}{2} = 3r + 1$, where $r > 0$. By Theorem 4.2, K_n can be decomposed into the edge-disjoint stars $\underbrace{S_3, S_3, ..., S_3}_{r-copies}, S_1$.

Take 3 copies of K_n. Consider the centers of S_1 in each copy of K_n in the decomposition of K_n. Identify these three vertices from 3 copies of K_n to get the required graph $G = (3\binom{n}{2}, S_3, K_n)$. By construction itself G is K_n-decomposable. Also, G can be decomposed into $3r+1$ copies of S_3 (r copies of S_3 from each K_n and one copy of S_3 obtained at the identified vertex). Therefore there exists a $(k\binom{n}{2}, S_3, K_n)$ for all $k \equiv 0 \pmod 3$. □

5. Common Multiples of S_4 and K_n

In this section we determine, for all positive integers n, the set of integers q for which there exists a common multiple of $S_4(4 - star)$ and K_n having precisely q edges. The following theorem gives necessary and sufficient condition for the existence of a (q, S_4, K_n) graph.

Theorem 5.1. *There exists a graph with q edges that is both S_4-decomposable and K_n-decomposable if and only if*

(1) $2q \equiv 0 \pmod{n(n-1)}$ *when* $n \equiv 0, 1 \pmod 8$
(2) $q \equiv 0 \pmod{n(n-1)}$ *when* $n \equiv 4, 5 \pmod 8$
(3) $q \equiv 0 \pmod{2n(n-1)}$ *when* $n \equiv 2, 3, 6, 7 \pmod 8$
(4) $q \neq 12$ *when* $n = 3$
(5) $q > n(n-1)$ *when* $n = 4, 5$.

Proof. If there exists a (q, S_4, K_n) graph, then we require that 4 divides q and that $\binom{n}{2}$ divides q. Conditions (1) - (3) follow immediately from this and will be referred to as the obvious necessary conditions. The following two lemmas establish the remaining necessary conditions (4) and (5).

Lemma 5.2. *If there exists a (q, S_4, K_3) graph, then $q \neq 12$.*

Proof. If $n = 3$, then the obvious necessary condition for the existence of a (q, S_4, K_n) graph is that $q \equiv 0 \pmod{4\binom{n}{2}}$. Suppose that $n = 3$, and there exists a $(12,\ S_4,\ K_3)$ graph. This is impossible since $lcm(S_4, K_3) = 24$ by Theorem 4.3. So $q \neq 12$. □

Lemma 5.3. *If $n = 4, 5$ and there exists a (q, S_4, K_n) graph, then $q > 2\binom{n}{2}$.*

Proof. If $n = 4, 5$, then the obvious necessary condition for the existence of a (q, S_4, K_n) graph is that $q \equiv 0 \pmod{2\binom{n}{2}}$. First suppose that $n = 4$, and there exists a $(12, S_4, K_4)$ graph G. Such a graph consists of two copies of K_4 intersecting in at most one vertex (in order for G to be K_n-decomposable), then there is only one vertex with degree greater than 4. So, if we remove a copy of S_4 from G, we can not obtain two more edge-disjoint copies of S_4's from G.

Now suppose that $n = 5$, and there exists a $(20, S_4, K_5)$ graph G. Then G consists of two copies of K_5 intersecting in at most one vertex, and hence G has eight vertices of degree 4 and one vertex of degree 8. If we remove copies of S_4's one by one from G we could obtain only three S_4's. But G must contain five S_4 's. Thus $q > 2\binom{n}{2}$ when $n = 4, 5$.

□

Sufficient Conditions

To show that the stated necessary conditions are sufficient we consider each in turn and construct the (q, S_4, K_n) graphs required to prove Theorem 5.1. First note that if $n \equiv 0, 1 \pmod 8$, then $S_4|K_n$ (Theorem 4.1) and hence when $n \equiv 0, 1 \pmod 8$, there exists a (q, S_4, K_n) graph G for all $q \equiv 0 \pmod{\binom{n}{2}}$(Take G to be $\frac{q}{\binom{n}{2}}$ vertex-disjoint copies of K_n). Thus sufficient condition (1) obtained. We require a few lemmas to construct the graphs for the remaining congruence classes of $n \pmod 8$.

Lemma 5.4. *For all $k > 1$, there exists a $(12k, S_4, K_4)$ graph.*

Proof. It is sufficient to construct a $(24, S_4, K_4)$ graph and a $(36, S_4, K_4)$ graph as all the required graphs can be constructed as the vertex-disjoint union of the appropriate number of copies of these.

To construct a $(24, S_4, K_4)$ graph G, we let G be the union of the following four edge-disjoint copies of K_4.

$[1, 2, 3, A]$ $[1, 4, 5, B]$ $[2, 4, 6, C]$ $[3, 5, 6, D]$

An S_4-decomposition of G is given by the following six edge-disjoint copies of S_4.

[1; 2, 4, A, B] [2; 3, 6, A, C] [3; 1, 5, A, D] [4; 2, 5, B, C]

[5; 1, 6, B, D] [6; 3, 4, C, D]

To construct a $(36, S_4, K_4)$ graph, we let G be the union of the following six edge-disjoint copies of K_4.

[1, 2, 3, A] [4, 5, 6, A] [7, 8, 9, A] [1, 4, 7, B]

[2, 5, 8, B] [3, 6, 9, B]

An S_4-decomposition of G is given by the following nine edge-disjoint copies of S_4.

[1; 2, 4, A, B] [2; 3, 5, A, B] [3; 1, 6, A, B] [4; 5, 7, A, B]

[5; 6, 8, A, B] [6; 4, 9, A, B] [7; 1, 8, A, B] [8; 2, 9, A, B]

[9; 3, 7, A, B]

□

Lemma 5.5. *For all $k > 1$, there exists a $(20k, S_4, K_5)$ graph.*

Proof. It is sufficient to construct a $(40, S_4, K_5)$ graph and a $(60, S_4, K_5)$ graph as all the required graphs can be constructed as the vertex-disjoint union of the appropriate number of copies of these.

To construct a $(40, S_4, K_5)$ graph G, we let G be the union of the following four edge-disjoint copies of K_5.

[0, 1, 2, 3, A] [0, 4, 5, 6, B] [1, 4, 7, 8, C] [2, 5, 7, 9, D]

An S_4-decomposition of G is given by the following ten edge-disjoint copies of S_4.

[0; 1, 2, B, A] [1; 2, 4, C, A] [2; 3, 5, D, A] [3; 0, 1, 2, A]

[4; 0, 7, C, B] [5; 0, 4, D, B] [6; 4, 5, 0, B] [7; 1, 2, D, C]

[8; 1, 7, 4, C] [9; 2, 5, 7, D]

To construct a $(60, S_4, K_5)$ graph G, we let G be the union of the following six edge-disjoint copies of K_5.

[1, 2, 3, 4, 5] [1, 6, 7, 8, 9] [2, 6, 10, 11, 12] [3, 7, 10, 13, 14]

[4, 8, 11, 13, 15] [5, 9, 12, 14, 15]

An S_4-decomposition of G is given by the following fifteen edge-disjoint copies of S_4.

[1; 2, 3, 4, 5] [2; 3, 4, 5, 6] [3; 4, 5, 7, 10] [4; 5, 8, 11, 13]

[5; 9, 12, 14, 15] [6; 1, 10, 11, 12] [7; 1, 10, 13, 14] [8; 1, 6, 7, 9]

[9; 1, 6, 7, 12] [10; 2, 11, 12, 13] [11; 2, 8, 12, 13] [12; 2, 9, 14, 15]

[13; 3, 8, 14, 15] [14; 3, 9, 10, 15] [15; 4, 8, 9, 11]

□

Lemma 5.6. *There exists a (q, S_4, K_n) graph if*

(1) $q \equiv 0 \pmod{n(n-1)}$, $q > n(n-1)$ *and* $n = 4, 5$; *or*
(2) $q \equiv 0 \pmod{2n(n-1)}$ *and* $n = 2, 6, 7$; *or*
(3) $q \equiv 0 \pmod{2n(n-1)}$, $q \neq 12$ *and* $n = 3$.

Proof. The result is true for $n = 4$ and 5 by Lemmas 5.4 and 5.5 respectively.

If $n = 2, 3, 6, 7$, then $gcd(4, \binom{n}{2}) = 1$ and hence $lcm(4, \binom{n}{2}) = 4\binom{n}{2}$.

Case 1: $n = 2$

For every $k \equiv 0 \pmod 4$ let G be $\dfrac{k}{4}$ vertex-disjoint copies of S_4, which is both S_4-decomposable and K_2- decomposable. Hence a (k,S_4,K_2) graph exists for all $k \equiv 0 \pmod 4$.

Case 2: $n = 3$

By Lemma 5.2, $q \neq 12$. It is sufficient to construct a $(24, S_4, K_3)$ graph and a $(36, S_4, K_3)$ graph as all the required graphs can be constructed as the vertex-disjoint union of the appropriate number of copies of these.

A $(24, S_4, K_3)$ graph G is obtained from $K_{2,4}$ by adding a new vertex for each of its eight edges and joining the vertex to the two vertices incident with the corresponding edge. For a $(36, S_4, K_3)$ graph G, we take $G = K_9$, which is S_4-decomposable (by Theorem 4.1) and K_3-decomposable ($K_3|K_n$ if and only if $n \equiv 1, 3 \pmod 6$).

Case 3: $n = 6$

Then $lcm(4, \binom{n}{2}) = 60$.

To construct a $(60, S_4, K_6)$ graph G, we let G be the union of the following four edge-disjoint copies of K_6.

[1, 2, 3, 4, 5, 0]   [1, 6, 7, 8, 9, A]

[1, 10, 11, 12, 13, B]   [5, 7, 13, 14, 15, C]

An S_4-decomposition of G is given by the following fifteen edge-disjoint copies of S_4.

[1; 0, 5, A, B]   [2; 0, 5, 3, 1]   [3; 0, 5, 4, 1]   [4; 0, 5, 2, 1]

[5; 0, 7, 15, C]   [6; 7, 8, 1, A]   [7; 1, 13, A, C]   [8; 9, 1, 7, A]

[9; 7, 6, 1, A]   [10; 11, 13, 1, B]   [11; 12, 13, 1, B]   [12; 13, 10, 1, B]

[13; 1, 5, B, C]   [14; 5, 7, 13, C]   [15; 7, 13, 14, C]

Hence there exists a $(15k, S_4, K_6)$ graph exists for all $k \equiv 0 \pmod 4$.

Case 4: $n = 7$

In this case $lcm(4, \binom{n}{2}) = 84$. We can easily verify that K_7 can be decomposed into five edge-disjoint copies of S_4 and one copy of S_1. To construct a $(84, S_4, K_7)$ graph G, take four copies of K_7 and identify them at the one vertex of S_1 in the decomposition of K_7. By the construction itself G is K_7-decomposable. G can be decomposed into 21 edge-disjoint copies of S_4 (five copies of S_4 from each K_7 and one copy of S_4 at the identified vertex). Therefore there exists a $(21k, S_4, K_7)$ graph for all $k \equiv 0 \pmod 4$.

□

Lemma 5.7. *For all $n \geq 8$, there exists a (q, S_4, K_n) graph if*

(1) $q \equiv 0 \pmod{n(n-1)}$ *and* $n \equiv 4, 5 \pmod 8$*; or*
(2) $q \equiv 0 \pmod{2n(n-1)}$ *when* $n \equiv 2, 3, 6, 7 \pmod 8$*;*

Proof. If $n \equiv 4, 5 \pmod 8$, then $\binom{n}{2} \equiv 2 \pmod 4$ and $lcm(4, \binom{n}{2}) = 2\binom{n}{2}$.
If $n \equiv 2, 7 \pmod 8$, then $\binom{n}{2} \equiv 1 \pmod 4$ and $lcm(4, \binom{n}{2}) = 4\binom{n}{2}$.
If $n \equiv 3, 6 \pmod 8$, then $\binom{n}{2} \equiv 3 \pmod 4$ and $lcm(4, \binom{n}{2}) = 4\binom{n}{2}$.
Consider the following cases.
Case 1: $n \equiv 4, 5 \pmod 8$, $n \geq 8$.
Let $\binom{n}{2} = 4r + 2$, where $r > 0$.
Since $n \geq 8$, by Theorem 4.2, K_n can be decomposed into the edge-disjoint stars

$$\underbrace{S_4, S_4, ..., S_4}_{r-copies}, S_2.$$

Take two copies of K_n and identify the centers of two S_2 in each copy of K_n to get the required graph $G = (2\binom{n}{2}$,S_4,$K_n)$. By the construction itself G is K_n-decomposable. G is S_4-decomposable, since G can be decomposed into $2r + 1$ edge-disjoint copies of S_4. The identified vertex can be considered as the center of one copy of S_4. Then there exists a $(k\binom{n}{2}$,S_4,$K_n)$ graph for all even k.
Case 2: $n \equiv 2, 7 \pmod 4$.
Let $\binom{n}{2} = 4r + 1$, where $r > 0$. By Theorem 4.2, K_n can be decomposed into the edge-disjoint stars

$$\underbrace{S_4, S_4, ..., S_4}_{r-copies}, S_1$$

Take four copies of K_n. Consider the centers of S_1 in each copy of K_n in the decomposition. Identify these four vertices from each copy of K_n to get the required graph $G = (4\binom{n}{2}, S_4, K_n)$. By the construction itself G is K_n-decomposable. G is S_4-decomposable, since G can be decomposed into $4r + 1$ edge-disjoint copies of S_4. The identified vertex can be considered as the center of one copy of S_4. Then there exists a $(k\binom{n}{2}, S_4, K_n)$ graph for all $k \equiv 0 \pmod 4$.
Case 3: $n \equiv 3, 6 \pmod 4$.
Let $\binom{n}{2} = 4r + 3$, where $r > 0$. By Theorem 4.2, K_n can be decomposed into the edge-disjoint stars

$$\underbrace{S_4, S_4, ..., S_4}_{r-copies}, S_3$$

Take four copies of K_n. Consider the centers of S_3 in each copy of K_n in the decomposition. Identify these four vertices from each copy of K_n to get the required graph $G = (4\binom{n}{2}, S_4, K_n)$. By the construction itself G is K_n-decomposable. G is S_4-decomposable, since G can be decomposed into $4r + 3$ edge-disjoint copies of S_4. The identified vertex can be considered as the center of three copies of S_4. Then there exists a $(k\binom{n}{2}, S_4, K_n)$ for all $k \equiv 0 \pmod 4$. □

Lemmas 5.6 and 5.7 allow us to construct all of the (q, S_4, K_n) graphs that we require for the sufficient conditions (2), (3), (4) and (5) of Theorem 5.1. □

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References

[1] P. Adams, D. Bryant and B. Maenhaut, *Common multiple of complete graphs and a 4-cycle*, Discrete Math. 275 (2004) 289–297. https://doi.org/10.1016/j.disc.2002.11.001

[2] P. Adams, D. Bryant and B. Maenhaut, Saad I. El-Zanati and C.V. Eynden,*Least common multiples of cubes*, https://www.researchgate.net/publication/43443700

[3] D. Bryant and B. Maenhaut, *Common multiples of complete graphs*, Proc. London. Math. Soc. 3(86) (2003) 302–326. https://doi.org/10.1112/S0024611502013771

[4] D. Bryant, *Packing paths in complete graphs*, J. Combin. Theory Ser. B 100 (2010) 206–215. https://doi.org/10.1016/j.jctb.2009.08.004

[5] B. Chaluvaraju and V. Chaitra, *Sign Domination in Arithmetic Graphs*, Gulf. J. Math. Vol 4, 3(2016) 49-54. https://doi.org/10.56947/gjom.v4i3.73

[6] G. Chartrand, L. Holley, G. Kubicki and M. Shultz, *Greatest common divisors and least common multiple of graphs*, Period. Math. Hungar. 27 (1993) 95–104. https://doi.org/10.1007/BF01876635

[7] G. Chartrand, G. Kubicki, C.M. Mynhardt, F. Saba, *On graphs with a unique least common multiple*, Ars Combin. 46 (1997), 177-190.

[8] G. Chartrand, C.M. Mynhardt, F. Saba, *On least common multiples of digraphs*, Utilitas Math. 49 (1996), 45-63.

[9] Zhen-Chun Chen, Tay-Woei Shyu, *Common multiples of paths and stars*, Ars Combin. 146 (2019), 115-122.

[10] O. Favaron and C. M. Mynhardt, *On the sizes of least common multiple of several pairs of graphs*, Ars. Combin 43 (1996), 181-190.

[11] Chiang Lin, Tay- Woei Shyu, *A Necessary and Sufficint condition for the Star Decomposition of Complete Graphs*, Journal of Graph Theory. Vol 23. No. 4, 361 -364 (1996). https://doi.org/10.1002/(SICI)1097-0118(199612)23:4<361::AID-JGT5>3.0.CO;2-P

[12] C.M. Mynhardt, F. Saba, *On the sizes of least common multiples of paths versus complete graphs*, Utilitas Math. 46 (1994), 117- 128.

[13] Purnima Gupta, Mukti Acharya and Deepti Jain, *2-Point Set Domination Number of a Cactus*, Gulf. J. Math. Vol 4, 3(2016), 80-89. https://doi.org/10.56947/gjom.v4i3.76

[14] M. Tarsi, *Decomposition of complete multigraph into simple paths: nonbalanced handcuffed designs*, J. Combin. Theory Ser. A, 34 (1983), pp. 60-70. https://doi.org/10.1016/0097-3165(83)90040-7

[15] P. Wang, *on the sizes of least common multiples of stars versus cycles*, Utilitas Math. 53 (1998), 231-242.

[16] Sumiyasu Yamamoto, Hideto Ikeda, Shinsei Shige-Eda, Kazuhiko Ushio and Noboru Hamada, *On claw decomposition of Complete Graphs and Complete Bigraphs*, Hiroshima Math.J., 5 (1975), 33-42. https://doi.org/10.32917/hmj/1206136782

[1] Department of Mathematics, Government College, Chittur, Palakkad, Kerala, India-678104.

Email address: rejiaran@gmail.com

[2] Department of Mathematics, Government Polytechnic College, Palakkad, Kerala, India-678551
Email address: sarithachandran.gvc@gmail.com