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EXISTENCE AND UNIQUENESS OF RENORMALIZED SOLUTIONS TO NONLINEAR MULTIVALUED PARABOLIC PROBLEM WITH HOMOGENEOUS DIRICHLET BOUNDARY CONDITIONS INVOLVING VARIABLE EXPONENT

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Abstract. In this paper, we prove the existence and the uniqueness of renormalized solution to a nonlinear multivalued parabolic problem $\beta(u)_t - div\ a(x, \nabla u) \ni f$, with homogeneous Dirichlet boundary conditions and L^1-data. The functional setting involves Lebesgue and Sobolev spaces with variable exponent. Some a-priori estimates are used to obtain our results.

1. Introduction

The purpose of this paper is to study the existence and the uniqueness of renormalized solutions to the following nonlinear parabolic problem involving the $p(\cdot)-$Laplacian

$$\begin{cases} \beta(u)_t - \text{div } a(x, \nabla u) \ni f & \text{in } Q = (0,T) \times \Omega \\ \beta(u)(t=0) = b_0 & \text{in } \Omega, \\ u = 0 & \text{on } \Sigma = (0,T) \times \partial\Omega \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded open domain with smooth boundary, T is a positive fixed final time and $f \in L^1(Q)$, β is a maximal monotone graph on $\mathbb{R}$ such that $\beta^{-1} \in \mathcal{C}^0(\mathbb{R})$. Moreover, the function $a(x, \xi)$ is monotone and coercive with respect to ξ so as to define a Leray-Lions type operator.

There are mainly two types of difficulties in studying problem (1.1). The first one resides in the the discontinuous character of β coupled with a nonlinear diffusion term. The second one is raised by the L^1-data so that finite energy solutions could not be expected.

The study of various mathematical problems with variable exponent has considerable attention in recent years. These problems have numerous applications (see [1, 4, 5, 13, 14]) and raise many difficult mathematical problems.

We recall that the notion of renormalized solutions was introduced for the first

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time by Diperna and Lions [7] in their study of the Boltzmann equation. Following [7] and using the same notion of solution, Wittbold and Zimmermann (see [15]) studied the problem

$$\begin{cases} \beta(u) - \text{div } a(x, \nabla u) - \text{div } F(u) \ni f & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where they proved the existence and the uniqueness of renormalized solution for $F : \mathbb{R} \to \mathbb{R}^N$ a locally Lipschitz continuous function, $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ a set-valued, maximal monotone mapping such that $0 \in \beta(0)$, a data $f \in L^1(\Omega)$ and $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ a Carathéodory function and $p(.) : \overline{\Omega} \to (1, \infty)$ a continuous variable exponent such that $1 < \min_{x \in \overline{\Omega}} p(x) < N$.
Recently, Kyelem *et als* [10] proved existence and uniqueness of the entropy solution to the problem

$$\begin{cases} b(u)_t - \text{div } a(x, \nabla u) = f & \text{in } Q = (0, T) \times \Omega \\ b(u)(t = 0) = b_0 & \text{in } \Omega, \\ u = 0 & \text{on } \Sigma = (0, T) \times \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded open domain with smooth boundary, T is a positive fixed final time, b is monotone and continuous function such that $b(0) = 0$, and $f \in L^1(Q)$.
For other works in this direction, see [11, 16]. The aim of our paper is to extend the results in [10] , to the case of a monotone graph β instead of a continuous and monotone function b.
The paper is organized as follows: In section 2 we make assumptions on the data, recall some basic notations and properties of Lebesgue and Sobolev spaces with variable exponents, and give the definition of renormalized solutions to problem (1.1). In section 3, we prove the existence and the uniqueness of the renormalized solution of problem (1.1).

2. Preliminaries

We study the problem (1.1) with the following assumptions on the data:

$$\begin{cases} p(.) : \Omega \longrightarrow \mathbb{R} \text{ is a continuous measurable function such that} \\ 1 < p_- \leq p_+ < +\infty, \end{cases} \tag{2.1}$$

where $p_- := \inf_{x \in \Omega} p(x)$ and $p_+ := \sup_{x \in \Omega} p(x)$.
For the vector fields $a(.,.)$, we assume that $a(.,.) : \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is Carathéodory and is the continuous derivative with respect to ξ of the mapping $A : \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}$, i.e. $a(x, \xi) = \nabla_\xi A(x, \xi)$ such that:
• The following equality holds

$$A(x, 0) = 0, \tag{2.2}$$

for almost every $x \in \Omega$.
• There exists a positive constant C_1 such that

$$|a(x, \xi)| \leq C_1 (j(x) + |\xi|^{p(x)-1}) \tag{2.3}$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$ where j is a nonnegative function in $L^{p'(\cdot)}(\Omega)$, with $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$.

• The following inequalities hold

$$(a(x,\xi) - a(x,\eta)).(\xi - \eta) > 0, \tag{2.4}$$

for almost every $x \in \Omega$ and for every $\xi, \eta \in \mathbb{R}^N$, with $\xi \neq \eta$ and

$$\frac{1}{C}|\xi|^{p(x)} \leq a(x,\xi).\xi \leq Cp(x)A(x,\xi) \tag{2.5}$$

for almost every $x \in \Omega, C > 0$ and for every $\xi \in \mathbb{R}^N$.

Remark 2.1. (1) $a(x,0) = 0$ for a.e. $x \in \Omega$. Indeed for $x \in \Omega$ fixed, denote $z = a(x,0) \in \mathbb{R}^N$. By the continuity of $a(x,.)$, we have $\lim_{\xi \to 0} a(x,\xi) = z$. Suppose now that $z \neq 0$ and choose $\xi_0 = -sz$ with $s > 0$ goes to 0; then $a(x,\xi_0).\xi_0 = -s(z + \epsilon(s)).z \leq -s\,|z|^2 + s\,|z|\,|\epsilon(s)|$, where $\lim_{s \to 0} |\epsilon(s)| = 0$. Therefore, for s sufficiently small, $-s\,|z|^2 + s\,|z|\,|\epsilon(s)| < 0$, which is a contradiction by assumption (2.5). Thus, $z = 0$.

(2) As examples of models with respect to assumptions (2.2)-(2.5) for problem (1.1), we can give the following:

(i): Set $A(x,\xi) = (1/p(x))|\xi|^{p(x)}$, $a(x,\xi) = |\xi|^{p(x)-2}\xi$, where $p(x) \geq 2$. Then we get the $p(\cdot)$-Laplace operator

$$\text{div } (|\nabla u|^{p(x)-2}\nabla u).$$

(ii): Set $A(x,\xi) = (1/p(x))\left[\left(1 + |\xi|^2\right)^{p(x)/2} - 1\right]$, $a(x,\xi) = \left(1+|\xi|^2\right)^{(p(x)-2)/2}\xi$, where $p(x) \geq 2$. Then we obtain the generalized mean curvature operator

$$\text{div } \left(\left(1 + |\nabla u|^2\right)^{(p(x)-2)/2}\nabla u\right).$$

As the exponent $p(\cdot)$ appearing in (2.3) and (2.5) depends on the variable x, we must work with Lebesgue and Sobolev spaces with variable exponents.

We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $u : \Omega \longrightarrow \mathbb{R}$ for which the convex modular

$$\rho_{p(.)}(u) := \int_\Omega |u|^{p(x)}\,dx$$

is finite. If the exponent is bounded, i.e., if $p_+ < +\infty$, then the expression

$$|u|_{p(.)} := \inf\{\lambda > 0 : \rho_{p(.)}(u/\lambda) \leq 1\}$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called the Luxembourg norm. The space $(L^{p(\cdot)}(\Omega), |.|_{p(.)})$ is a separable Banach space. Moreover, if $1 < p_- \leq p_+ < +\infty$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$. Finally, we have the Hölder type inequality:

$$\Big| \int_\Omega uv\, dx \Big| \leq \Big(\frac{1}{p_-} + \frac{1}{p_+} \Big) |u|_{p(.)} |v|_{p'(.)}, \tag{2.6}$$

for all $u \in L^{p(.)}(\Omega)$ and $v \in L^{p'(.)}(\Omega)$.
Now, let

$$W^{1,p(.)}(\Omega) := \Big\{ u \in L^{p(.)}(\Omega) : |\nabla u| \in L^{p(.)}(\Omega) \Big\},$$

which is a Banach space equipped with the following norm

$$||u||_{1,p(.)} = |u|_{p(.)} + |(|\nabla u|)|_{p(.)}.$$

We define also $W_0^{1,p(.)}(\Omega) := \overline{\mathcal{C}_c^\infty(\Omega)}^{W^{1,p(.)}(\Omega)}$. Assuming $p_- > 1$ the spaces $W^{1,p(.)}(\Omega)$ and $W_0^{1,p(.)}(\Omega)$ are separable and reflexive Banach spaces. The space $\big(W_0^{1,p(.)}(\Omega)\big)^*$ denotes the dual space of $W_0^{1,p(.)}(\Omega)$. For the interested reader, more details about Lebesgue and Sobolev spaces with variable exponent can be found in [6] (see also [9]).
An important role in manipulating the generalized Lebesgue and Sobolev spaces is played by the modular $\rho_{p(.)}$ of the space $L^{p(.)}(\Omega)$. We have the following result (cf. [8]):

Lemma 2.2. *If $u_n, u \in L^{p(.)}$ and $p_+ < +\infty$, then the following properties hold:*

(1) $|u|_{p(.)} > 1 \Longrightarrow |u|_{p(.)}^{p_-} \leq \rho_{p(.)}(u) \leq |u|_{p(.)}^{p_+}$;
(2) $|u|_{p(.)} < 1 \Longrightarrow |u|_{p(.)}^{p_+} \leq \rho_{p(.)}(u) \leq |u|_{p(.)}^{p_-}$;
(3) $|u|_{p(.)} < 1$ *(respectively* $= 1; > 1$*)* $\Longleftrightarrow \rho_{p(.)}(u) < 1$ *(respectively* $= 1; > 1$*);*
(4) $|u_n|_{p(.)} \longrightarrow 0$ *(respectively* $\longrightarrow +\infty$*)* $\Longleftrightarrow \rho_{p(.)}(u_n) \longrightarrow 0$ *(respectively* $\longrightarrow +\infty$*);*
(5) $\rho_{p(.)}\big(u/|u|_{p(.)}\big) = 1$.

Following [2], we extend a variable exponent $p : \overline{\Omega} \longrightarrow [1, +\infty)$ to $\overline{Q} = [0,T] \times \overline{\Omega}$ by setting $p(t,x) := p(x)$ for all $(t,x) \in \overline{Q}$.
We may also consider the generalized Lebesgue space

$$L^{p(.)}(Q) = \Big\{ u : Q \longrightarrow \mathbb{R} \text{ mesurable such that } \int\int_Q |u(t,x)|^{p(x)}\, d(x,t) < \infty \Big\},$$

endowed with the norm

$$||u||_{L^{p(.)}(Q)} := \inf \Big\{ \lambda > 0 : \int\int_Q \Big| \frac{u(t,x)}{\lambda} \Big|^{p(x)} d(x,t) < 1 \Big\},$$

which share the same properties as $L^{p(.)}(\Omega)$.
Now, define the spaces:

$$V = \Big\{ f \in L^{p_-}(0,T; W_0^{1,p(.)}(\Omega)); |\nabla f| \in L^{p(.)}(Q) \Big\}$$

and

$$E = \big\{ \varphi \in V \cap L^\infty(Q);\ \varphi_t \in V^* + L^1(Q) \big\}.$$

According to [12], we have $E \subset \mathcal{C}([0,T];L^1(\Omega))$.
We will assume that

$$\beta : \mathbb{R} \to 2^{\mathbb{R}} \qquad \text{is a maximal monotone graph such that} \qquad 0 \in \beta(0) \tag{2.7}$$

and denote β^{-1} the inverse function of β, defined as $\beta^{-1}(r) = s \Longleftrightarrow r \in \beta(s)$.
We also assume that

$$\beta^{-1} \in \mathcal{C}^0(\mathbb{R}), \quad \text{i.e. } \beta^{-1} \text{ is continuous on } \quad \mathbb{R}. \tag{2.8}$$

As in [3], we will write $\int \beta(r)\,dr$ every time a graph β appears under line integrals, instead of using its maximal or minimal univalued representations $\overline{\beta} = \sup\{s \in \beta(r)\}$ and $\underline{\beta} = \inf\{s \in \beta(r)\}$ (in fact $\overline{\beta} = \underline{\beta}$ almost everywhere).
In order to find more estimates for renormalized solutions and also to get useful a priori estimates of approximate solutions to the problem (3.1) below, the following integration by parts formula plays a crucial role:

Lemma 2.3. *(see Corollary 2.4, [3])*
Let $F : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function and $\beta(s)$ be a maximal monotone graph in $\mathbb{R}$. We set $G(\lambda) = \displaystyle\int^{\lambda} F'(r)\beta(r)\,dr$.
Let $w \in L^{p_-}(0,T;W_0^{1,p(.)}(\Omega))$, $w_0 \in L^\infty(\Omega)$ and $b \in E$, $b_0 \in L^\infty(\Omega) \cap (W_0^{1,p(.)}(\Omega))^*$ such that

$$b \in \beta(w) \ \text{a.e. in } \ Q, \qquad b_0 \in \beta(w_0) \ \text{a.e. in } \ \Omega.$$

Then we have

$$\begin{aligned} -\langle b_t, F(w)\psi\rangle = &\int_0^T \int_\Omega \psi_t\big(bF(w) - G(w)\big)\,dtdx \\ &+ \int_\Omega \psi(0)\big(b_0F(w_0) - G(w_0)\big)\,dx \end{aligned} \tag{2.9}$$

for any $\psi \in W^{1,\infty}(Q)$, $\psi(T) = 0$, where $\langle .,. \rangle$ denotes the dual pairing between $V^* + L^1(Q)$ and $V \cap L^\infty(Q)$.
In particular, we have

$$-\langle b_t, F(w)\psi\rangle = \int_0^T \int_\Omega \psi_t \mathcal{F}(b)\,dtdx + \int_\Omega \psi(0)\mathcal{F}(b_0)\,dx, \tag{2.10}$$

where $\mathcal{F}(s) = \int^s F(\beta^{-1})(r)\,dr$.

In order to deal with L^1-data, we will need to introduce some auxiliary truncation functions which are fundamental in the sequel.

Definition 2.4. For every $k > 0$, we set $T_k(s) = \max(-k, \min(s,k))$ the truncation function at level k, then we define

$$\theta_n(s) = \frac{1}{n}T_n(s - T_n(s)),\ h_n(s) = 1 - |\theta_n(s)|,\ S_n(s) = \int_0^s h_n(r)\,dr, \forall s \in \mathbb{R}. \tag{2.11}$$

To end this section, let now define what we understand by renormalized solution of the parabolic problem (1.1).

Definition 2.5. For $f \in L^1(Q)$ and $b_0 \in L^1(\Omega)$, a renormalized solution of problem (1.1) is a function u which verifies the following:
(R_1) u is measurable and $T_k(u) \in V$, there exists $b_u \in L^1(Q) \cap E$ such that $b_u \in \beta(u)$ a.e. in Q;
(R_2) for all S in $W^{2,\infty}(\mathbb{R})$ such that S' has a compact support,

$$\begin{aligned} -\int_\Omega \varphi(0) \int_0^{b_0} S'(\beta^{-1}(r))\, dr\, dx - \int_0^T \int_\Omega \varphi_t \varphi(0) \int_0^{b_u} S'(\beta^{-1}(r))\, dr\, dt\, dx \\ + \int_0^T \int_\Omega \left[a(x, \nabla u).\nabla S'(u)\varphi + a(x,\nabla u).\nabla(u) S"(u)\varphi \right] dtdx \\ = \int_0^T \int_\Omega f S'(u)\varphi\, dtdx, \end{aligned} \tag{2.12}$$

for any $\varphi \in \mathcal{C}_c^\infty([0,T) \times \overline{\Omega})$ such that $S'(u)\varphi \in V \cap L^\infty(Q)$;

(R_3)
$$\lim_{n\to+\infty} \frac{1}{n} \int_{\{(x,t)\in Q: n \le |u(x,t)| \le 2n\}} a(x,\nabla u).\nabla u\, dtdx = 0. \tag{2.13}$$

3. Existence and uniqueness of renormalized solution

Our main result for the existence is the following.

Theorem 3.1. *Assume (2.1)-(2.5) hold and β satisfies (2.7)-(2.8). Moreover, let $b_0 \in L^1(\Omega)$ and $f \in L^1(Q)$. Then, there exists at least a renormalized solution for problem (1.1).*

Proof. The proof of Theorem 3.1 is divided into several steps.

3.1. **Regularized problems.** As $b_0 \in L^1(\Omega)$, $f \in L^1(Q)$ and L^∞ is dense in L^1, then we can find two sequences of functions $\left(f_\epsilon\right)_{\epsilon>0} \subset L^\infty(Q)$ and $\left(b_{0,\epsilon}\right)_{\epsilon>0} \subset L^\infty(\Omega)$ strongly converging respectively to f and b_0 such that

$$||f_\epsilon||_{L^1(Q)} \le ||f||_{L^1(Q)}, \qquad ||b_{0,\epsilon}||_{L^1(\Omega)} \le ||u_0||_{L^1(\Omega)}.$$

Now, we consider the approximated problem

$$\left\{ \begin{array}{ll} \beta(u_\epsilon)_t - \operatorname{div}\, a(x,\nabla u_\epsilon) = f_\epsilon & \text{in } Q = (0,T)\times\Omega \\ \beta(u_\epsilon)(t=0) = b_{0,\epsilon} & \text{in } \Omega, \\ u_\epsilon = 0 & \text{on } \Sigma = (0,T)\times\partial\Omega. \end{array} \right. \tag{3.1}$$

We have the following result.

Proposition 3.2. *Assume that (2.1)-(2.5) hold, $b_{0,\epsilon} \in L^\infty(\Omega)$ and $f_\epsilon \in L^\infty(Q)$. Then, there exists a function u_ϵ which is a weak solution of problem (3.1).*
By a weak solution of the problem (3.1) we understand a function

$$u_\epsilon \in \left(V = \left\{ f \in L^{p_-}(0,T; W_0^{1,p(.)}(\Omega)); |\nabla f| \in L^{p(.)}(Q) \right\} \right) \cap L^\infty(Q)$$

such that there exists $b_{u_\epsilon} \in L^\infty(Q) \cap E$ satisfying $b_{u_\epsilon} \in \beta(u_\epsilon)$, and

$$-\int_0^T \int_\Omega \varphi_t b_{u_\epsilon}\, dtdx - \int_\Omega \varphi(0) b_{0,\epsilon}\, dx + \int_0^T \int_\Omega a(x, \nabla u_\epsilon).\nabla\varphi = \int_0^T \int_\Omega f_\epsilon \varphi\, dtdx, \quad (3.2)$$

for any $\varphi \in \mathcal{C}_c^\infty([0,T) \times \overline{\Omega})$.
Moreover we have

$$\|(b_{u_\epsilon} - b_{u_\eta})(t)\|_{L^1(\Omega)} \le \|f_\epsilon - f_\eta\|_{L^1(Q)} + \|b_{0,\epsilon} - b_{0,\eta}\|_{L^1(\Omega)}, \qquad \forall t \in (0,T). \quad (3.3)$$

Proof. For shortness, we drop the index ϵ.
Let $\mathcal{B}_\delta(s)$ be the Yosida approximation of the graph β, and let $\beta_\delta(s) = \mathcal{B}_\delta(s) + s$. Note that β_δ is a Lipschitz increasing function with Lipschitz inverse β_δ^{-1}.
Then, according to ([10], Proposition 2), there exists a unique weak solution u_δ of the problem

$$\begin{cases} \big(\beta_\delta(u_\delta)\big)_t - \text{div } a(x, \nabla u_\delta) = f & \text{in } Q = (0,T) \times \Omega \\ \beta_\delta(u_\delta)(t=0) = b_0 & \text{in } \Omega, \\ u_\delta = 0 & \text{on } \Sigma = (0,T) \times \partial\Omega. \end{cases} \quad (3.4)$$

As in [10], standard energy estimates imply that u_δ is bounded in $L^\infty(Q)$ and in $L^{p_-}(0,T; W_0^{1,p(.)}(\Omega))$. Thanks to (2.3), we also have that $a(x, \nabla u_\delta)$ is bounded in $L^{p'(.)}(Q)^N$. We deduce that

$$\beta_\delta(u_\delta) \text{ is bounded in } L^{p'(.)}(Q) \text{ and } \big(\beta_\delta(u_\delta)\big)_t \text{ is bounded in } L^{p'_-}(0,T; (W_0^{1,p(.)}(\Omega))^*). \quad (3.5)$$

Since we have, by Holder inequality (setting $\langle \cdot, \cdot \rangle$ the dual pairing in $W_0^{1,p(.)}(\Omega)$),

$$\begin{aligned} &\int_0^{T-h} \int_\Omega \big(\beta_\delta(u_\delta)(\tau+h) - \beta_\delta(u_\delta)(\tau)\big)\big(u_\delta(\tau+h) - u_\delta(\tau)\big)\, dxd\tau \\ &\quad = \int_0^{T-h} \int_\tau^{\tau+h} \langle \big(\beta_\delta(u_\delta)\big)_t, \big(u_\delta(\tau+h) - u_\delta(\tau)\big) \rangle\, dt \\ &\quad \le \|\big(\beta_\delta(u_\delta)\big)_t\|_{L^{p'_-}(0,T;(W_0^{1,p(.)}(\Omega))^*)} h^{\frac{1}{p_-}} \int_0^{T-h} \Big(\|u_\delta)(\tau+h)\|_{W_0^{1,p(.)}(\Omega)} + \|u_\delta)(\tau)\|_{W_0^{1,p(.)}(\Omega)} \Big)\, d\tau \end{aligned}$$

we conclude, thanks to the energy estimates as in [10],

$$\int_0^{T-h} \int_\Omega \big(\beta_\delta(u_\delta)(\tau+h) - \beta_\delta(u_\delta)(\tau)\big)\big(u_\delta(\tau+h) - u_\delta(\tau)\big)\, dxd\tau \le Ch^{\frac{1}{p_-}}. \quad (3.6)$$

Now, since β^{-1} is continuous, we have that β_δ^{-1} is a sequence of uniformly equicontinuous functions on the interval $[-L, L]$, where $L > \|u_\delta\|_\infty$. Thus, let us fix $\epsilon > 0$; there exists a value σ_ϵ such that

$$\text{if } |y - z| < \sigma_\epsilon \text{ then } \big|\beta_\delta^{-1}(y) - \beta_\delta^{-1}(z)\big| < \epsilon \text{ for any } \delta > 0.$$

Thus in the set $\{|u_\delta(\tau+h) - u_\delta(\tau)| > \epsilon\}$ we have

$$\big|\beta_\delta(u_\delta)(\tau+h) - \beta_\delta(u_\delta)(\tau)\big| \ge \sigma_\epsilon$$

so that (3.6) implies

$$\int_0^{T-h}\int_\Omega |u_\delta(\tau+h)-u_\delta(\tau)|\chi_{|u_\delta(\tau+h)-u_\delta(\tau)|>\epsilon}\,dxd\tau \le C\frac{h^{\frac{1}{p_-}}}{\sigma_\epsilon}.$$

Therefore,

$$\sup_\delta \int_0^{T-h}\int_\Omega |u_\delta(\tau+h)-u_\delta(\tau)|\,dxd\tau \le |Q|\epsilon + C\frac{h^{\frac{1}{p_-}}}{\sigma_\epsilon}$$

which gives

$$\lim_{h\to 0}\sup_\delta \int_0^{T-h}\int_\Omega |u_\delta(\tau+h)-u_\delta(\tau)|\,dxd\tau = 0. \tag{3.7}$$

We also have from Rellich theorem and the estimate in $L^{p_-}(0,T;W_0^{1,p(.)}(\Omega))$:

$$\sup_\delta \int_0^{T}\int_\Omega |u_\delta(\tau+h)-u_\delta(\tau)|\,dxd\tau \le Ch^{\frac{1}{p'_-}}. \tag{3.8}$$

We conclude, from (3.7) and (3.8), that u_δ is equi-integrable in Q, so that it strongly converges to u in $L^1(Q)$ and, up to subsequences, almost everywhere. Therefore, we have that $\beta_\delta(u_\delta)$ weakly converges, in $L^{q(.)}(Q)$ for any $q(.) < +\infty$, to a function $\chi \in \beta(u)$. Moreover from estimate (3.5) and using a classical result (Aubin's lemma) we deduce

$$\beta_\delta(u_\delta) \longrightarrow \chi \qquad \text{strongly in} \qquad \mathcal{C}^0([0,T];(W_0^{1,p(.)}(\Omega))^*).$$

This strong convergence, together with the uniform bound on $\beta_\delta(u_\delta)$, yields that

$$\beta_\delta(u_\delta)(t) \rightharpoonup \chi(t) \text{ in } L^{q(.)}(\Omega) \text{ as } \delta \to 0. \tag{3.9}$$

Moreover, standard methods (see [10]) and almost everywhere convergence of u_δ to u, yield

$$a(x,\nabla u_\delta) \rightharpoonup a(x,\nabla u) \text{ in } L^{p'(.)}(\Omega)^N \text{ as } \delta \to 0.$$

This is enough to pass to the limit and deduce that u is a weak solution of the problem (3.1).
Then, techniques made in [10] allow us to obtain (3.3). □

Now, we aim to prove that this weak solution is also a renormalized solution of the problem (1.1). Firstly, we prove an $\epsilon-$uniform a-priori-estimates in certain Bochner spaces as well as in appropriate exponent Lebesgue spaces for u_ϵ and ∇u_ϵ. Secondly, we pass to the limit in the renormalized relation as $\epsilon \longrightarrow 0$.

3.2. **A priori estimates and convergence results.** Now, let u_ϵ be a weak solution to problem (3.1) with f_ϵ and $b_{0,\epsilon}$ as data, i.e. there exists $b_{u_\epsilon} \in L^\infty(Q)\cap E$ satisfying $b_{u_\epsilon} \in \beta(u_\epsilon)$, and

$$\begin{aligned} -\int_0^T\int_\Omega \varphi_t b_{u_\epsilon}\,dtdx - \int_\Omega \varphi(0)b_{0,\epsilon}\,dx + \int_0^T\int_\Omega a(x,\nabla u_\epsilon).\nabla u_\epsilon \\ = \int_0^T\int_\Omega f_\epsilon\varphi\,dtdx \end{aligned} \tag{3.10}$$

for any $\varphi \in \mathcal{C}_c^\infty([0,T) \times \overline{\Omega})$.

Remark 3.3. Note that any weak solution, as defined above, is a renormalized solution. In fact, for any
$S \in W^{2,\infty}(\mathbb{R})$ and any $\varphi \in \mathcal{C}_c^\infty([0,T)\times\overline{\Omega})$ such that $S'(u_\epsilon)\varphi \in L^{p_-}(0,T;W_0^{1,p(.)}(\Omega))$, we can choose $S'(u_\epsilon)\varphi$ as test function in (3.10).
Since we have from (2.10)

$$\begin{aligned} -\langle (b_{u_\epsilon})_t,\ S'(u_\epsilon)\varphi\rangle &= \int_0^T \int_\Omega \varphi_t \int_0^{b_{u_\epsilon}} S'(\beta^{-1}(r))\,dr\,dx\,dt \\ &\quad + \int_\Omega \varphi(0) \int_0^{b_0^\epsilon} S'(\beta^{-1}(r))\,dr\,dx, \end{aligned}$$

and since $u_\epsilon = 0$ on $(0,T) \times \partial\Omega$, integration of the convection term gives

$$\begin{aligned} &-\int_\Omega \varphi(0) \int_0^{b_0^\epsilon} S'(\beta^{-1}(r))\,dr\,dx - \int_0^T \int_\Omega \varphi_t \int_0^{b_{u_\epsilon}} S'(\beta^{-1}(r))\,dr\,dt\,dx \\ &\quad + \int_0^T \int_\Omega \left[S'(u_\epsilon)a(x,\nabla u_\epsilon).\nabla\varphi + S"(u_\epsilon)a(x,\nabla u_\epsilon).\nabla u_\epsilon\varphi\right] dtdx \\ &\qquad = \int_0^T \int_\Omega f_\epsilon S'(u_\epsilon)\varphi\,dt\,dx, \end{aligned} \tag{3.11}$$

for any S in $W^{2,\infty}(\mathbb{R})$ and $\varphi \in \mathcal{C}_c^\infty([0,T) \times \overline{\Omega})$ such that $S'(u_\epsilon)\varphi \in V$.
Moreover, in (3.11) it is enough to have $\varphi \in W^{1,\infty}(Q)$ such that $\varphi(T) = 0$ to get the result.
We also deduce as in ([3], Remark 3.2) that

$$\|b_{u_\epsilon}\|_{L^\infty(0,T;L^1(\Omega))} \leq C\Big(\|b_0\|_{L^1(\Omega)}, \|f\|_{L^1(Q)}\Big). \tag{3.12}$$

Lemma 3.4. *Let $(u_\epsilon)_{\epsilon>0}$ be the sequence of solutions of the problem (3.1). Then we have for all $k > 0$*

$$||\nabla T_k(u_\epsilon)||_{L^{p(.)}(Q)} \leq k \max\Big\{ \big(||f||_{L^1(Q)} + ||u_0||_{L^1(\Omega)}\big)^{\frac{1}{p_-}}, \big(||f||_{L^1(Q)} + ||u_0||_{L^1(\Omega)}\big)^{\frac{1}{p^+}} \Big\} \tag{3.13}$$

and

$$\lim_{n\to+\infty} \sup_\epsilon \frac{1}{n} \int_{\{n\leq|u_\epsilon|\leq 2n\}} a(x,\nabla u_\epsilon).\nabla u_\epsilon\,dtdx = 0. \tag{3.14}$$

Moreover, we deduce from (3.12) that $(b_{u_\epsilon})_{\epsilon>0}$ is a Cauchy sequence in $L^\infty(0,T;L^1(\Omega))$; in particular, there exist u and $b_u \in \beta(u)$ such that (up to subsequences)

$$b_{u_\epsilon} \longrightarrow b_u \text{ in } L^\infty(0,T;L^1(\Omega)) \text{ and almost everywhere in } Q,$$

$$u_\epsilon \longrightarrow u \text{ a.e. in } Q,$$

and

$$T_k(u_\epsilon) \to T_k(u) \text{ in } L^{p_-}(0,T;W_0^{1,p(.)}(\Omega)) \text{ and a.e in } Q.$$

Proof. Choosing $S'(r) = T_k(r)$ and $\varphi = \min\big(\frac{(T-\delta-t)^+}{\delta}, 1\big)$ in (3.10) and discarding the positive terms, one gets

$$\begin{aligned}\int_0^{T-2\delta}\int_\Omega a(x,\nabla u_\epsilon).\nabla T_k(u_\epsilon)\,dtdx \;&\leq\; \int_0^T\int_\Omega |f_\epsilon T_k(u_\epsilon)|\,dt\,dx \\ &\quad + \int_\Omega \varphi(0)\int_0^{b_0^\epsilon} T_k(\beta^{-1}(r))\,dr\,dx \\ &\leq\; k\big(\|b_0\|_{L^1(\Omega)} + \|f\|_{L^1(Q)}\big).\end{aligned} \tag{3.15}$$

Letting $\delta \to 0$ in (3.15) and using (2.5), we obtain

$$\frac{1}{C}\int_0^T\int_\Omega |\nabla T_k(u_\epsilon)|^{p(x)}\,dtdx \leq\; k\big(\|b_0\|_{L^1(\Omega)} + \|f\|_{L^1(Q)}\big). \tag{3.16}$$

Then, according to Lemma 2.2, we deduce (3.13).
To prove (3.14), similarly we choose $S'(r) = \theta_n(r)$ to obtain

$$\begin{aligned}&\frac{1}{n}\int_{\{n\leq|u_\epsilon|\leq 2n\}} a(x,\nabla u_\epsilon).\nabla u_\epsilon\,dtdx \\ &\qquad\leq \int_{\{|u_\epsilon|>n\}} |f_\epsilon|\,dt\,dx + \int_{\{|b_0^\epsilon|>n\}} |b_0^\epsilon|\,dx.\end{aligned} \tag{3.17}$$

By Poincaré inequality in constant exponent and (3.16), we have

$$n^{p_-}\text{meas}(\{|u_\epsilon|>n\}) = \int_{\{|u_\epsilon|>n\}} |T_n(u_\epsilon)|^{p_-}\,dtdx \leq C\int_0^T\int_\Omega |\nabla T_n(u_\epsilon)|^{p_-}\,dtdx \leq Cn,$$

then,

$$\lim_{n\to+\infty}\sup_\epsilon \text{meas}(\{|u_\epsilon|>n\}) = 0.$$

Hence by equi-integrability of b_0^ϵ and f_ϵ we get from (3.17) the relation (3.14).
Finally, from (3.12) we deduce that $(b_{u_\epsilon})_{\epsilon>0}$ strongly converges in $L^\infty(0,T;L^1(\Omega))$ towards a function b_u. Since β^{-1} is continuous, this implies, up to subsequences, that u_ϵ almost everywhere converges to $u := \beta^{-1}(b_u)$.
We also deduce that $T_k(u_\epsilon)$ weakly converges to $T_k(u)$ in $L^{p_-}(0,T;W_0^{1,p(.)}(\Omega))$. $\square$

To continue our proof of Theorem 3.1, we need the following.

Proposition 3.5. *For all $k>0$,*
(i) $a(x,\nabla T_k(u_\epsilon)) \rightharpoonup a(x,\nabla T_k(u))$ in $L^{p'(.)}(Q)$,
(ii) $a(x,\nabla T_k(u_\epsilon)).\nabla T_k(u_\epsilon) \to a(x,\nabla T_k(u)).\nabla T_k(u)$ weakly in $L^1((0,\tau)\times\Omega)$ for any $\tau<T$.

Proof. The proof follows the same lines as in constant exponent case (see [3]). $\square$

Proposition 3.6. *For all $k>0$,*

(i): *$\nabla T_k(u_\epsilon) \to \nabla T_k(u)$ a.e. in Q,*

(ii): *$a(x,\nabla T_k(u_\epsilon)).\nabla T_k(u_\epsilon) \to a(x,\nabla T_k(u)).\nabla T_k(u)$ strongly in $L^1(Q)$ and a.e. in Q,*

(iii): $\nabla T_k(u_\epsilon) \to \nabla T_k(u)$ *in* $\left(L^{p(.)}(Q)\right)^N$.

Proof. **(i):** By relation (ii) of the Proposition 3.5, we deduce that

$$\limsup_{\epsilon\to 0} \int_0^T \int_\Omega (a(x,\nabla T_k(u_\epsilon)) - a(x,\nabla T_k(u))).\nabla(T_k(u_\epsilon) - (T_k(u)))\, dtdx = 0. \quad (3.18)$$

Now, set $g_\epsilon(t,x) = \big[a(x,\nabla u_\epsilon) - a(x,\nabla u)\big].\nabla\big[T_k(u_\epsilon) - T_k(u)\big] \geq 0$. $g_\epsilon(t,x) \to 0$ strongly in $L^1(\Omega)$ as $\epsilon \to 0$. Up to a subsequence, $g_\epsilon(t,x) \to 0$ a.e. in Q, which means that there exists $\omega \subset Q$ such that meas $(\omega) = 0$ and $g_\epsilon(t,x) \to 0$ in $Q\backslash\omega$.
Let $(t,x) \in Q\backslash\omega$. Using assumptions (2.3) and(2.5), it follows that the sequence $\big(\nabla T_k(u_\epsilon(t,x))\big)_{\epsilon>0}$ is bounded in $\mathbb{R}^N$ and so we can extract a subsequence which converges to some θ in $\mathbb{R}^N$.
Passing to the limit in the expression of $g_\epsilon(t,x)$, it follows that

$$0 = \big[a(x,\theta) - a(x,\nabla T_k(u))\big].\big[\theta - T_k(u)\big]$$

and it yields $\theta = \nabla T_k(u),\ \forall(t,x) \in Q\backslash\omega$.
As the limit dœsn't depend on the subsequence, the whole sequence $\big(\nabla T_k(u_\epsilon(t,x))\big)_{\epsilon>0}$ converges to θ in $\mathbb{R}^N$. This means that $\nabla T_k(u_\epsilon) \to \nabla T_k(u)$ a.e. in Q.

(ii): The continuity of $a(x,\xi)$ with respect to $\xi \in \mathbb{R}^N$ gives us

$$a(x,\nabla T_k(u_\epsilon)) \to a(x,\nabla T_k(u)) \text{ a.e. in } Q.$$

Therefore

$$a(x,\nabla T_k(u_\epsilon)).\nabla T_k(u_\epsilon) \to a(x,\nabla T_k(u)).\nabla T_k(u) \text{ a.e. in } Q.$$

Setting $z_\epsilon = a(x,\nabla T_k(u_\epsilon)).\nabla T_k(u_\epsilon)$ and $z = a(x,\nabla T_k(u)).\nabla T_k(u)$, we have

$$\begin{cases} z_\epsilon > 0, z_\epsilon \to z \text{ a.e. in } Q, z \in L^1(\Omega), \\ \displaystyle\iint_Q z_\epsilon\, dtdx \to \iint_Q z\, dtdx \end{cases}$$

and as $\displaystyle\iint_Q |z_\epsilon - z|\, dtdx = 2\iint_Q (z - z_\epsilon)^+\, dtdx + \iint_Q (z_\epsilon - z)\, dtdx$ and $(z - z_\epsilon)^+ \leq z$, it follows by using the Lebesgue dominated convergence theorem that

$$\lim_{\epsilon\to 0} \int_Q |z_\epsilon - z|\, dtdx = 0,$$

which means that

$$a(x,\nabla T_k(u_\epsilon)).\nabla T_k(u_\epsilon) \to a(x,\nabla T_k(u)).\nabla T_k(u) \text{ strongly in } L^1(Q) \text{ and a.e. in } Q.$$

(iii): We need the following lemmas.

Lemma 3.7. *(see [8]) Let $u, u_n \in L^{p(.)}(Q)$, $n = 1, 2, \ldots.$ Then the following statements are equivalent to each other:*
1) $\lim_{n\to\infty} |u_n - u|_{p(.)} = 0$*;*

2) $\lim_{n\to\infty} \rho_{p(.)}(u_n - u) = 0$;
3) u_n *converges to* u *in* Q *in measure and* $\lim_{n\to\infty} \rho_{p(.)}(u_n) = \rho_{p(.)}(u)$.

Lemma 3.8. *(Lebesgue generalized convergence theorem) Let* $(f_n)_{n\in\mathbb{N}}$ *be a sequence of measurable functions and* f *a measurable function such that* $f_n \to f$ *a.e. in* Q*. Let* $(g_n)_{n\in\mathbb{N}} \subset L^1(Q)$ *such that for all* $n \in \mathbb{N}$, $|f_n| \le g_n$ *a.e. in* Q *and* $g_n \to g$ *in* $L^1(Q)$*. Then*

$$\iint_Q f_n \, dtdx \to \iint_Q f \, dtdx.$$

Now, set $f_\epsilon = \Big|\nabla T_k(u_\epsilon)\Big|^{p(x)}$, $f = \Big|\nabla T_k(u)\Big|^{p(x)}$, $g_\epsilon = a(x, \nabla T_k(u_\epsilon)).\nabla T_k(u_\epsilon)$ and $g = a(x, \nabla T_k(u)).\nabla T_k(u)$.
We have:
• f_ϵ is a sequence of measurable functions, f is a measurable function and according to (ii), $f_\epsilon \to f$ a.e. in Q.
• Using (ii), we have $(g_\epsilon)_{\epsilon>0} \subset L^1(Q)$, $g_\epsilon \to g$ a.e. in Q, $g_\epsilon \to g$ in $L^1(Q)$ and using (2.5), we have $|f_\epsilon| \le C g_\epsilon$.
Then, by Lemma 3.8, we have $\iint_Q f_\epsilon \, dtdx \to \iint_Q f \, dtdx$, which is equivalent to say

$$\iint_Q \Big|\nabla T_k(u_\epsilon)\Big|^{p(x)} dtdx \to \iint_Q \Big|\nabla T_k(u)\Big|^{p(x)} dtdx.$$

We deduce from (i) that the sequence $\big(\nabla T_k(u_\epsilon)\big)_{\epsilon>0}$ converges to $\nabla T_k(u)$ in Q in measure. Then, by Lemma 3.7 we deduce that

$$\lim_{\epsilon\to 0} \iint_Q \Big|\nabla T_k(u_\epsilon) - \nabla T_k(u)\Big|^{p(x)} dtdx = 0,$$

which is equivalent to saying that $\nabla T_k(u_\epsilon) \to \nabla T_k(u)$ in $\big(L^{p(\cdot)}(Q)\big)^N$.
□

To finish the proof of Theorem 3.1, we pass to the limit as $\epsilon \to 0$ in (3.11) which we recall:

$$\begin{aligned} &-\int_\Omega \varphi(0) \int_0^{b_0^\epsilon} S'(\beta^{-1}(r))\, dr\, dx - \int_0^T \int_\Omega \varphi_t \int_0^{b_{u_\epsilon}} S'(\beta^{-1}(r))\, dr\, dt\, dx \\ &\quad + \int_0^T \int_\Omega \big[a(x, \nabla u_\epsilon).\nabla\varphi S'(u_\epsilon) + a(x, \nabla u_\epsilon).\nabla u_\epsilon S''(u_\epsilon)\varphi\big]\, dtdx \\ &\qquad = \int_0^T \int_\Omega f_\epsilon S'(u_\epsilon)\varphi\, dt\, dx, \end{aligned} \tag{3.19}$$

for any S in $W^{2,\infty}(\mathbb{R})$ and $\varphi \in \mathcal{C}_c^\infty([0,T) \times \Omega)$ such that $S'(u_\epsilon)\varphi \in V$.
Since S' has compact support, in all previous integrals u_ϵ can be replaced by its truncation $T_L(u_\epsilon)$ provided $\mathrm{Supp} S' \subset [-L, L]$.
Using the strong convergence of $(b_{u_\epsilon})_{\epsilon>0}$ to b_u and $(b_{0,\epsilon})_{\epsilon>0}$ to b_0, we obtain for

the first and the second terms in the left-hand side of (3.19)

$$\lim_{\epsilon\to 0}\int_\Omega \varphi(0)\int_0^{b_0^\epsilon} S'(\beta^{-1}(r))\,dr\,dx = \int_\Omega \varphi(0)\int_0^{b_0} S'(\beta^{-1}(r))\,dr\,dx \tag{3.20}$$

and

$$\lim_{\epsilon\to 0}\int_0^T\int_\Omega \varphi_t\int_0^{b_{u_\epsilon}} S'(\beta^{-1}(r))\,dr\,dt\,dx = \int_0^T\int_\Omega \varphi_t\int_0^{b_u} S'(\beta^{-1}(r))\,dr\,dt\,dx. \tag{3.21}$$

Using Lemma 3.4, Propositions 3.5 and 3.6, we have

$$a(x,\nabla T_L(u_\epsilon))S'(u_\epsilon) \to a(x,\nabla T_L(u))S'(u) \text{ in } \left(L^{p'(\cdot)}(Q)\right)^N$$

and

$$a(x,\nabla T_L(u_\epsilon)).\nabla T_L(u_\epsilon)S"(u_\epsilon) \to a(x,\nabla T_L(u)).\nabla T_L(u)S"(u) \text{ in } L^1(Q).$$

Then,

$$\begin{aligned}\lim_{\epsilon\to 0}&\int_0^T\int_\Omega \left[a(x,\nabla T_L(u_\epsilon)).\nabla\varphi S'(u_\epsilon) + a(x,\nabla T_L(u_\epsilon)).\nabla T_L(u_\epsilon)S"(u_\epsilon)\varphi\right]dtdx\\ &= \int_0^T\int_\Omega \left[a(x,\nabla T_L(u)).\nabla\varphi S'(u) + a(x,\nabla T_L(u)).\nabla T_L(u)\varphi S"(u)\right]dtdx.\end{aligned} \tag{3.22}$$

For the right-hand side of (3.19), thanks to the strong convergence of f_ϵ to f, we have

$$\lim_{\epsilon\to 0}\int_0^T\int_\Omega f_\epsilon S'(u_\epsilon)\varphi\,dt\,dx = \int_0^T\int_\Omega fS'(u)\varphi\,dt\,dx. \tag{3.23}$$

Therefore, passing to the limit in (3.19) as $\epsilon\to 0$ and using (3.20)-(3.23) we obtain

$$\begin{aligned}-\int_\Omega \varphi(0)\int_0^{b_0} S'(\beta^{-1}(r))\,dr\,dx &- \int_0^T\int_\Omega \varphi_t\int_0^{b_u} S'(\beta^{-1}(r))\,dr\,dt\,dx\\ &+\int_0^T\int_\Omega \left[a(x,\nabla u).\nabla\varphi S'(u) + a(x,\nabla u).\nabla u S"(u)\varphi\right]dtdx\\ &= \int_0^T\int_\Omega fS'(u)\varphi\,dt\,dx.\end{aligned} \tag{3.24}$$

Finally, (2.13) follows from the estimate (3.17).

$$\frac{1}{n}\int_{\{n\le|u_\epsilon|\le 2n\}} a(x,\nabla u_\epsilon).\nabla u_\epsilon\xi\,dtdx \le \int_{\{|u_\epsilon|>n\}}|f_\epsilon|\,dt\,dx + \int_{\{|b_{0,\epsilon}|>n\}}|b_{0,\epsilon}|\,dx, \tag{3.25}$$

which holds for any $\xi\in\mathcal{C}_c^\infty[0,T)$ with $\xi_t\le 0$.
Since $a(x,\nabla T_{2n}(u_\epsilon)).\nabla T_{2n}(u_\epsilon)$ weakly converges in $L^1_{loc}([0,T);L^1(\Omega))$, we can pass to the limit as $\epsilon\to 0$ in (3.25) to get

$$\frac{1}{n}\int_{\{n\le|u|\le 2n\}} a(x,\nabla u).\nabla u\xi\,dtdx \le \int_{\{|u|>n\}}|f|\,dt\,dx + \int_{\{|b_0|>n\}}|b_0|\,dx. \tag{3.26}$$

Letting ξ converge to 1 and then n go to infinity in (3.26), we obtain (2.13).
The proof of Theorem 3.1 is then complete. $\square$

Now, we deal with the uniqueness of the renormalized solution. We still let $\beta(s)$ satisfy assumptions (2.7) and (2.8). We assume in addition that the principal part of β, i.e. the function $\underline{\beta}(s) = \min\{r \in \beta(s)\}$, satisfies:

$$\text{for any } n \in \mathbb{N},\ \underline{\beta} \text{ has a finite number } M_n \text{ of discontinuity points in } [-n, n]. \tag{3.27}$$

Our main result here is the following L^1-contraction principle.

Theorem 3.9. *Let β satisfy (2.7)-(2.8) and (3.27). Assume that (2.1)-(2.5) hold. For $i = 1, 2$, assume $f_i \in L^1(Q), b_{0,i} \in L^1(\Omega)$. Let u_i be a renormalized solution of problem (1.1) with data $f_i, b_{0,i}$ and denote $b_{u_i} \in \beta(u_i)$ a corresponding selection given in the Definition 2.5. Then we have*

$$\int_\Omega \left(b_{u_1}(t) - b_{u_2}(t)\right)^+ dx \le \int_0^t \int_\Omega \left(f_1 - f_2\right)^+ dxds + \int_\Omega \left(b_{0,1} - b_{0,2}\right)^+ dx \tag{3.28}$$

for almost every $t \in (0, T)$. In particular, problem (1.1) has a unique renormalized solution u and a unique selection $b_u \in \beta(u)$.

Proof. The proof of Theorem 3.9 follows the same lines as the proof of Theorem 4.1 in [3]. □

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