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REFINEMENT OF THE UPPER BOUND OF THE RADIUS OF A CIRCULAR INTEGRAL POINTS SET

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Abstract. Let n be a product of the prime numbers whose positive integer powers are of the form $a^2 + Db^2$ where $D > 4$ is a square-free number and a, b are positive integers. For $n \leq 3072$, we obtained a refinement of the upper bound of the radius of a circular points set which was previously given in Tables 1 and 2 of [5]. In order to prove this, we showed that there are points on the circle with the radius $R = \dfrac{n\sqrt{D}}{2D}$ such that mutual distances between these points are all integers. Consequently, if n is a product of the prime numbers whose squares are of the form $a^2 + Db^2$, then we showed that there are $\tau(n)$ points on the circle with the radius $R = \dfrac{n\sqrt{D}}{2D}$ such that mutual distances between these points are all integer numbers, where $\tau(n)$ is the number of all positive divisors of n.

1. Introduction and preliminaries

A n *integral points set* in Euclidean plane is a set of n points such that all the distances between these points are integers. There are still unsolved classical problems concerning the n integral points set. For example, is there any perfect cuboid whose edges, face diagonals and space diagonals all have integer lengths ? Is there any point in a unit square all of whose distances from the corners are rational? Another interesting question is for what n do there exist n integral points set in the Euclidean plane, no three points on a line and no four on a circle ? Source for these and related problems, see [4, 6]. To date, the only known examples we have $n \leq 7$ [11].

Circular sets are very important class of integral point sets [7, 5, 13]. To continue, we need the following definition.

Definition 1.1. A planar n integral points set that is situated on a circle is said to be a *circular* n *integral points set.*

Anning and Erdős [2] that for any n there exists a circular n integral points set. Harborth [7] showed that if n is the product of prime numbers of the form $3k+1$,

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$R(n,15)$	n	$R(n,15)$	n
$=\frac{1}{\sqrt{15}}\cdot 2$	2	$\leq \frac{1}{\sqrt{15}}\cdot 2^5\cdot 17\cdot 19\cdot 23\cdot 31\cdot 47$	$\overline{177,192}$
$=\frac{1}{\sqrt{15}}\cdot 2^2$	3	$\leq \frac{1}{\sqrt{15}}\cdot 2^6\cdot 17\cdot 19\cdot 23\cdot 31\cdot 47$	$\overline{193,224}$
$=\frac{1}{\sqrt{15}}\cdot 2^3$	4	$\leq \frac{1}{\sqrt{15}}\cdot 2^7\cdot 17\cdot 19\cdot 23\cdot 31\cdot 47$	$\overline{225,256}$
$\leq \frac{1}{\sqrt{15}}\cdot 2^4$	5	$\leq \frac{1}{\sqrt{15}}\cdot 2^8\cdot 17\cdot 19\cdot 23\cdot 31\cdot 47$	$\overline{257,288}$
$\leq \frac{1}{\sqrt{15}}\cdot 2^5$	6	$\leq \frac{1}{\sqrt{15}}\cdot 2^9\cdot 17\cdot 19\cdot 23\cdot 31\cdot 47$	$\overline{289,320}$
$\leq \frac{1}{\sqrt{15}}\cdot 2^6$	7	$\leq \frac{1}{\sqrt{15}}\cdot 2^{10}\cdot 17\cdot 19\cdot 23\cdot 31\cdot 47$	$\overline{321,352}$
$\leq \frac{1}{\sqrt{15}}\cdot 2^7$	8	$\leq \frac{1}{\sqrt{15}}\cdot 2^5\cdot 17\cdot 19\cdot 23\cdot 31\cdot 47\cdot 53$	$\overline{353,384}$
$\leq \frac{1}{\sqrt{15}}\cdot 2^8$	9	$\leq \frac{1}{\sqrt{15}}\cdot 2^6\cdot 17\cdot 19\cdot 23\cdot 31\cdot 47\cdot 53$	$\overline{385,448}$
$\leq \frac{1}{\sqrt{15}}\cdot 2^4\cdot 17$	10	$\leq \frac{1}{\sqrt{15}}\cdot 2^7\cdot 17\cdot 19\cdot 23\cdot 31\cdot 47\cdot 53$	$\overline{449,512}$
$\leq \frac{1}{\sqrt{15}}\cdot 2^5\cdot 17$	11,12	$\leq \frac{1}{\sqrt{15}}\cdot 2^8\cdot 17\cdot 19\cdot 23\cdot 31\cdot 47\cdot 53$	$\overline{513,576}$
$\leq \frac{1}{\sqrt{15}}\cdot 2^6\cdot 17$	13,14	$\leq \frac{1}{\sqrt{15}}\cdot 2^9\cdot 17\cdot 19\cdot 23\cdot 31\cdot 47\cdot 53$	$\overline{578,640}$
$\leq \frac{1}{\sqrt{15}}\cdot 2^7\cdot 17$	15,16	$\leq \frac{1}{\sqrt{15}}\cdot 2^{10}\cdot 17\cdot 19\cdot 23\cdot 31\cdot 47\cdot 53$	$\overline{641,704}$
$\leq \frac{1}{\sqrt{15}}\cdot 2^8\cdot 17$	17,18	$\leq \frac{1}{\sqrt{15}}\cdot 2^5\cdot 17\cdot 19\cdot 23\cdot 31\cdot 47\cdot 53\cdot 61$	$\overline{705,768}$
$\leq \frac{1}{\sqrt{15}}\cdot 2^4\cdot 17\cdot 19$	19,20	$\leq \frac{1}{\sqrt{15}}\cdot 2^6\cdot 17\cdot 19\cdot 23\cdot 31\cdot 47\cdot 53\cdot 61$	$\overline{769,896}$
$\leq \frac{1}{\sqrt{15}}\cdot 2^5\cdot 17\cdot 19$	$\overline{21,24}$	$\leq \frac{1}{\sqrt{15}}\cdot 2^7\cdot 17\cdot 19\cdot 23\cdot 31\cdot 47\cdot 53\cdot 61$	$\overline{897,1024}$
$\leq \frac{1}{\sqrt{15}}\cdot 2^6\cdot 17\cdot 19$	$\overline{25,28}$	$\leq \frac{1}{\sqrt{15}}\cdot 2^{10}\cdot 17\cdot 19\cdot 23\cdot 31\cdot 47\cdot 53$	$\overline{1025,1056}$
$\leq \frac{1}{\sqrt{15}}\cdot 2^7\cdot 17\cdot 19$	$\overline{29,32}$	$\leq \frac{1}{\sqrt{15}}\cdot 2^9\cdot 17\cdot 19\cdot 23\cdot 31\cdot 47\cdot 53\cdot 61$	$\overline{1057,1280}$
$\leq \frac{1}{\sqrt{15}}\cdot 2^8\cdot 17\cdot 19$	$\overline{33,36}$	$\leq \frac{1}{\sqrt{15}}\cdot 2^{10}\cdot 17\cdot 19\cdot 23\cdot 31\cdot 47\cdot 53\cdot 61$	$\overline{1281,1408}$
$\leq \frac{1}{\sqrt{15}}\cdot 2^5\cdot 17\cdot 19\cdot 23$	$\overline{37,48}$	$\leq \frac{1}{\sqrt{15}}\cdot 2^{11}\cdot 17\cdot 19\cdot 23\cdot 31\cdot 47\cdot 53\cdot 61$	$\overline{1409,1536}$
$\leq \frac{1}{\sqrt{15}}\cdot 2^6\cdot 17\cdot 19\cdot 23$	$\overline{49,56}$	$\leq \frac{1}{\sqrt{15}}\cdot 2^{12}\cdot 17\cdot 19\cdot 23\cdot 31\cdot 47\cdot 53\cdot 61$	$\overline{1537,1664}$
$\leq \frac{1}{\sqrt{15}}\cdot 2^7\cdot 17\cdot 19\cdot 23$	$\overline{57,64}$	$\leq \frac{1}{\sqrt{15}}\cdot 2^8\cdot 17^2\cdot 19\cdot 23\cdot 31\cdot 47\cdot 53\cdot 61$	$\overline{1665,1728}$
$\leq \frac{1}{\sqrt{15}}\cdot 2^8\cdot 17\cdot 19\cdot 23$	$\overline{65,72}$	$\leq \frac{1}{\sqrt{15}}\cdot 2^6\cdot 17\cdot 19\cdot 23\cdot 31\cdot 47\cdot 53\cdot 61\cdot 79$	$\overline{1729,1792}$
$\leq \frac{1}{\sqrt{15}}\cdot 2^4\cdot 17\cdot 19\cdot 23\cdot 31$	$\overline{73,80}$	$\leq \frac{1}{\sqrt{15}}\cdot 2^9\cdot 17^2\cdot 19\cdot 23\cdot 31\cdot 47\cdot 53\cdot 61$	$\overline{1793,1920}$
$\leq \frac{1}{\sqrt{15}}\cdot 2^5\cdot 17\cdot 19\cdot 23\cdot 31$	$\overline{81,96}$	$\leq \frac{1}{\sqrt{15}}\cdot 2^7\cdot 17\cdot 19\cdot 23\cdot 31\cdot 47\cdot 53\cdot 61\cdot 79$	$\overline{1921,2048}$
$\leq \frac{1}{\sqrt{15}}\cdot 2^6\cdot 17\cdot 19\cdot 23\cdot 31$	$\overline{97,112}$	$\leq \frac{1}{\sqrt{15}}\cdot 2^{10}\cdot 17^2\cdot 19\cdot 23\cdot 31\cdot 47\cdot 53\cdot 61$	$\overline{2049,2112}$
$\leq \frac{1}{\sqrt{15}}\cdot 2^7\cdot 17\cdot 19\cdot 23\cdot 31$	$\overline{113,128}$	$\leq \frac{1}{\sqrt{15}}\cdot 2^8\cdot 17\cdot 19\cdot 23\cdot 31\cdot 47\cdot 53\cdot 61\cdot 79$	$\overline{2113,2304}$
$\leq \frac{1}{\sqrt{15}}\cdot 2^8\cdot 17\cdot 19\cdot 23\cdot 31$	$\overline{129,144}$	$\leq \frac{1}{\sqrt{15}}\cdot 2^9\cdot 17\cdot 19\cdot 23\cdot 31\cdot 47\cdot 53\cdot 61\cdot 79$	$\overline{2305,2560}$
$\leq \frac{1}{\sqrt{15}}\cdot 2^9\cdot 17\cdot 19\cdot 23\cdot 31$	$\overline{145,160}$	$\leq \frac{1}{\sqrt{15}}\cdot 2^{10}\cdot 17\cdot 19\cdot 23\cdot 31\cdot 47\cdot 53\cdot 61\cdot 79$	$\overline{2561,2816}$
$\leq \frac{1}{\sqrt{15}}\cdot 2^1 0\cdot 17\cdot 19\cdot 23\cdot 31$	$\overline{161,176}$	$\leq \frac{1}{\sqrt{15}}\cdot 2^{11}\cdot 17\cdot 19\cdot 23\cdot 31\cdot 47\cdot 53\cdot 61\cdot 79$	$\overline{2887,3072}$

Table 1

then there exits $3\tau(n)$ integral points set on the circle with radius $R=\frac{n\sqrt{3}}{3}$. In [5], the author found the maximum number of points with pairwise integral distances on the circle with radius $R=\frac{n\sqrt{D}}{2D}$ for $D\in\{1,2,3,7,11,19,43,67,163\}$. n integral points set has many nice and important connection to the other branches of mathematics, for example it has a nice property with the integer solutions of algebraic equations. Namely, Norman Anning presented in [1] an arrangement of a circular 12 integral points set and all the distances of this points among the solution (up to sign) of the diophantine equation

$$x^2+xy+y^2=7^2\cdot 13^2.$$

R_n	n	R_n	n
$=\frac{1}{2}$	2	$\leq \frac{1}{\sqrt{3}} \cdot 7^2 \cdot 13^2 \cdot 19 \cdot 31 \cdot 37$	$\overline{225, 288}$
$=\frac{1}{\sqrt{3}}$	3	$\leq \frac{1}{\sqrt{15}} \cdot 2^9 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$	$\overline{289, 320}$
$=\frac{1}{\sqrt{15}} \cdot 8$	4	$\leq \frac{1}{\sqrt{3}} \cdot 7^2 \cdot 13^2 \cdot 19^2 \cdot 31 \cdot 37$	$\overline{321, 324}$
$=\frac{1}{\sqrt{3}} \cdot 7$	5,6	$\leq \frac{1}{\sqrt{3}} \cdot 7^3 \cdot 13 \cdot 19 \cdot 31 \cdot 37 \cdot 43$	$\overline{325, 384}$
$\leq \frac{1}{\sqrt{15}} \cdot 64$	7	$\leq \frac{1}{\sqrt{3}} \cdot 7^2 \cdot 13^2 \cdot 19 \cdot 31 \cdot 37 \cdot 43$	$\overline{385, 432}$
$\leq \frac{1}{\sqrt{3}} \cdot 7^2$	8,9	$\leq \frac{1}{\sqrt{15}} \cdot 2^6 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47 \cdot 53$	$\overline{433, 448}$
$\leq \frac{1}{\sqrt{3}} \cdot 7 \cdot 13$	$\overline{10, 12}$	$\leq \frac{1}{\sqrt{3}} \cdot 7^4 \cdot 13 \cdot 19 \cdot 31 \cdot 37 \cdot 43$	$\overline{449, 480}$
$\leq \frac{1}{\sqrt{15}} \cdot 2^6 \cdot 17$	13,14	$\leq \frac{1}{\sqrt{15}} \cdot 2^7 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47 \cdot 53$	$\overline{481, 512}$
$\leq \frac{1}{\sqrt{3}} \cdot 7^2 \cdot 13$	$\overline{15, 18}$	$\leq \frac{1}{\sqrt{3}} \cdot 7^2 \cdot 13 \cdot 19 \cdot 31 \cdot 37 \cdot 43 \cdot 61$	$\overline{513, 576}$
$\leq \frac{1}{\sqrt{3}} \cdot 7 \cdot 13 \cdot 19$	$\overline{19, 24}$	$\leq \frac{1}{\sqrt{15}} \cdot 2^9 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47 \cdot 53$	$\overline{577, 640}$
$\leq \frac{1}{\sqrt{3}} \cdot 7^2 \cdot 13^2$	$\overline{25, 27}$	$\leq \frac{1}{\sqrt{3}} \cdot 7^2 \cdot 13^2 \cdot 19^2 \cdot 31 \cdot 37 \cdot 43$	$\overline{641, 648}$
$\leq \frac{1}{\sqrt{15}} \cdot 2^6 \cdot 17 \cdot 19$	28	$\leq \frac{1}{\sqrt{3}} \cdot 7^3 \cdot 13 \cdot 19 \cdot 31 \cdot 37 \cdot 43 \cdot 61$	$\overline{649, 768}$
$\leq \frac{1}{\sqrt{3}} \cdot 7^2 \cdot 13 \cdot 19$	$\overline{29, 36}$	$\leq \frac{1}{\sqrt{3}} \cdot 7^2 \cdot 13^2 \cdot 19 \cdot 31 \cdot 37 \cdot 43 \cdot 61$	$\overline{769, 864}$
$\leq \frac{1}{\sqrt{15}} \cdot 2^8 \cdot 17 \cdot 19$	$\overline{37, 40}$	$\leq \frac{1}{\sqrt{3}} \cdot 7^4 \cdot 13 \cdot 19 \cdot 31 \cdot 37 \cdot 47 \cdot 43 \cdot 61$	$\overline{865, 960}$
$\leq \frac{1}{\sqrt{15}} \cdot 2^5 \cdot 17 \cdot 19 \cdot 23$	$\overline{41, 48}$	$\leq \frac{1}{\sqrt{15}} \cdot 2^7 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47 \cdot 61$	$\overline{961, 1024}$
$\leq \frac{1}{\sqrt{3}} \cdot 7^2 \cdot 13^2 \cdot 19$	$\overline{49, 54}$	$\leq \frac{1}{\sqrt{3}} \cdot 7^2 \cdot 13 \cdot 19 \cdot 31 \cdot 37 \cdot 43 \cdot 61 \cdot 73$	$\overline{1025, 1152}$
$\leq \frac{1}{\sqrt{15}} \cdot 2^6 \cdot 17 \cdot 19 \cdot 23$	55,56	$\leq \frac{1}{\sqrt{15}} \cdot 2^9 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47 \cdot 53 \cdot 61$	$\overline{1153, 1280}$
$\leq \frac{1}{\sqrt{3}} \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$	$\overline{57, 72}$	$\leq \frac{1}{\sqrt{3}} \cdot 7^2 \cdot 13^2 \cdot 19^2 \cdot 31 \cdot 37 \cdot 43 \cdot 61$	$\overline{1281, 1296}$
$\leq \frac{1}{\sqrt{15}} \cdot 2^4 \cdot 17 \cdot 19 \cdot 23 \cdot 31$	$\overline{73, 80}$	$\leq \frac{1}{\sqrt{15}} \cdot 2^{10} \cdot 17 \cdot 19 \cdot 23 \cdot 3 \cdot 47 \cdot 53 \cdot 61$	$\overline{1297, 1408}$
$\leq \frac{1}{\sqrt{3}} \cdot 7 \cdot 13 \cdot 19 \cdot 31 \cdot 37$	$\overline{81, 96}$	$\leq \frac{1}{\sqrt{3}} \cdot 7^3 \cdot 13 \cdot 19 \cdot 31 \cdot 37 \cdot 43 \cdot 61 \cdot 73$	$\overline{1409, 1536}$
$\leq \frac{1}{\sqrt{3}} \cdot 7^2 \cdot 17^2 \cdot 19 \cdot 31$	$\overline{97, 108}$	$\leq \frac{1}{\sqrt{3}} \cdot 7^2 \cdot 13^2 \cdot 19 \cdot 31 \cdot 37 \cdot 43 \cdot 61 \cdot 73$	$\overline{1537, 1728}$
$\leq \frac{1}{\sqrt{15}} \cdot 2^6 \cdot 17 \cdot 19 \cdot 23 \cdot 31$	$\overline{109, 112}$	$\leq \frac{1}{\sqrt{15}} \cdot 2^6 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47 \cdot 53 \cdot 61 \cdot 79$	$\overline{1729, 1792}$
$\leq \frac{1}{\sqrt{15}} \cdot 2^7 \cdot 17 \cdot 19 \cdot 23 \cdot 31$	$\overline{113, 128}$	$\leq \frac{1}{\sqrt{3}} \cdot 7^4 \cdot 13 \cdot 19 \cdot 31 \cdot 37 \cdot 43 \cdot 61 \cdot 73$	$\overline{1793, 1920}$
$\leq \frac{1}{\sqrt{3}} \cdot 7^2 \cdot 13 \cdot 19^2 \cdot 31 \cdot 37$	$\overline{129, 144}$	$\leq \frac{1}{\sqrt{15}} \cdot 2^7 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47 \cdot 53 \cdot 61 \cdot 79$	$\overline{1921, 2048}$
$\leq \frac{1}{\sqrt{15}} \cdot 2^9 \cdot 17 \cdot 19 \cdot 23 \cdot 31$	$\overline{145, 160}$	$\leq \frac{1}{\sqrt{3}} \cdot 7^2 \cdot 13 \cdot 19 \cdot 31 \cdot 37 \cdot 43 \cdot 61 \cdot 73 \cdot 89$	$\overline{2049, 2304}$
$\leq \frac{1}{\sqrt{3}} \cdot 7 \cdot 13 \cdot 19 \cdot 31 \cdot 37 \cdot 43$	$\overline{161, 192}$	$\leq \frac{1}{\sqrt{3}} \cdot 7^2 \cdot 13^2 \cdot 19^2 \cdot 31 \cdot 37 \cdot 43 \cdot 61 \cdot 73$	$\overline{2305, 2592}$
$\leq \frac{1}{\sqrt{3}} \cdot 7^2 \cdot 13^2 \cdot 19 \cdot 31 \cdot 37$	$\overline{193, 216}$	$\leq \frac{1}{\sqrt{15}} \cdot 2^{10} \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47 \cdot 53 \cdot 61 \cdot 79$	$\overline{2593, 2816}$
$\leq \frac{1}{\sqrt{15}} \cdot 2^6 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$	$\overline{217, 224}$	$\leq \frac{1}{\sqrt{3}} \cdot 7^3 \cdot 13 \cdot 19 \cdot 31 \cdot 37 \cdot 43 \cdot 61 \cdot 73 \cdot 89$	$\overline{2817, 3072}$

Table 2

Very recently in [9], a generalization of this result has been proved by Halbeisen and Hungerbühler for $3 \cdot 2^n$ integral points set. Their result gives an affirmative answer to Anning's conjecture. Note also that from the proof of Lemma 1 in [5], with the help of roots of diophantine equation $x^2 + Dy^2 = m^2$ one can construct a circular n integral points set lies on the circle with radius $R = \frac{m\sqrt{D}}{D}$.

In this paper we generalize the results in [5] when n is a product of the prime numbers whose positive integer powers are of the form $a^2 + Db^2$ with $D > 4$ is a square-free number and a, b positive integers. As in an application, we improved the results in [5] about the upper bound for the minimum radius of the circle with n integral point set for $n \leq 3072$, see Table 1 and Table 2. For both tables, we denote by R_n the minimum value of the radius of the circle of a circular n integral points set. And denote by $R(n; D)$ the minimum value of the radius $R = \frac{n\sqrt{D}}{2D}$ of the circle of a circular n integral points set.

2. Main results

Lemma 2.1. *Let D be a square free number and greater than 1. Assume that for every positive integer $s \geq 3$ there exists a, b odd integers such that $a^2 + b^2 D = 2^s$. If $\left(\dfrac{a + b\sqrt{D}i}{2}\right)^n = \dfrac{x_n + y_n\sqrt{D}i}{2}$ and $x_n, y_n \in \mathbb{Q}$, then x_n, y_n are odd integers for all $n \geq 1$.*

Proof. Since $x_1 = a, y_1 = b$ are odd integers $a^2 \equiv b^2 \equiv 1 \pmod 8$. By the assumption $a^2 + b^2 D = 2^s$ and $s \geq 3$ we have $D \equiv -1 \pmod 8$. It follows that $a^2 - b^2 D \equiv 2 \pmod 8$. Hence $x_2 = \dfrac{a^2 - b^2 D}{2}$ and $y_2 = abD$ are odd integers.

The sequences $\{x_n\}, \{y_n\}$ are defined by the recurrence formula

$$\begin{aligned} x_{n+2} &= ax_{n+1} - 2^{n-2}x_n; \\ y_{n+2} &= ay_{n+1} - 2^{n-2}y_n \end{aligned}$$

for $n \geq 1$. Thus by induction we see that x_n, y_n are odd integers for $n \geq 1$. □

Lemma 2.2. *If $\cos\omega \in \mathbb{Q}$ and $2\cos\omega \notin \mathbb{Z}$, then $\cos n\omega \notin \mathbb{Z}$ for all positive integer $n \geq 1$.*

Proof. Since $\cos\omega$ is a rational number, there exists integers a, c such that $\cos\omega = \dfrac{a}{c}$ with $gcd(a, c) = 1$. By contradiction we assume that there exists positive integers n such that $\cos n\omega$ is an integer. That is, $\cos n\omega = k$, where $k \in \{0, \pm 1\}$.

First we assume that there exists odd prime number such that $p \mid c$. We consider the Chebyshev polynomial of the first kind $T_n(\cos\omega) = \cos n\omega$. On the other hand, this polynomial can be written as $T_n(x) = 2^{n-1}x^n + c_1 x^{n-1} + \cdots + c_n$, where $c_1, \ldots, c_n \in \mathbb{Z}$. Hence

$$T_n\left(\frac{a}{c}\right) = 2^{n-1}\left(\frac{a}{c}\right)^n + c_1\left(\frac{a}{c}\right)^{n-1} + \cdots + c_n = k.$$

Multiplying both sides by c^n we have

$$2^{n-1}a^n + c_1 a^{n-1}c + \cdots + c_n c^n = kc^n.$$

Since $p > 2$ by $\pmod{p}$ on the last equation gives a contradiction.

Next we suppose that there are no prime numbers such that $p \mid c$. That is, $c = 2^s$ with s is a positive integer. Thus $\cos\omega = \dfrac{a}{2^s}$. It follows that $\sin\omega = \dfrac{b\sqrt{D}}{2^s}$, where D is a square free number and b is an integer. Since $2\cos\omega \notin \mathbb{Z}$ we must have $s \geq 2$. Hence $a^2 + b^2 D = 2^{2s}$ and $2s \geqslant 4$. Since $\cos n\omega \neq 0$ and $\sin n\omega \neq 0$ by Lemma 2.1 we have $\cos n\omega \notin \mathbb{Z}$. □

From now on, we use the following notations and the following assumptions holds. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ with $p_1 < p_2 < \dots < p_k$ prime numbers. Suppose that there exists positive integers $m_1, \dots, m_k$ and $x_1, \dots, x_k, y_1, \dots, y_k$ such that $p_j^{m_j} = x_j^2 + y_j^2 D$ for $j = 1, 2, \dots, k$. For a positive integer ℓ if $(x + y\sqrt{D}i)^\ell = x_\ell + y_\ell\sqrt{D}i$ and $x_\ell, y_\ell \in \mathbb{Z}$, then $p_j^{\ell m_j} = x_\ell^2 + y_\ell^2 D$. Let s_j be the smallest possible positive integer such that $p_j^{2s_j} = a_j^2 + b_j^2 D$, where $a_j, b_j \in \mathbb{N}$ for $j = 1, 2, \dots, k$. We denote by $\omega_j = \arg(a_j + b_j\sqrt{D}i)$, the principal argument of $(a_j + b_j\sqrt{D}i)$ for $j = 1, 2, \dots, k$.

Lemma 2.3. *Under the above notations and assumptions, if we choose the infinitely many points $\xi_{(t_1,\dots,t_k)} = R \prod_{1\leqslant j\leqslant k} e^{2it_j\omega_j}$ on the circle with radius $R = \dfrac{n\sqrt{D}}{2D}$, where $t_j \in \mathbb{Z}$ for $j = 1, 2, \dots, n$, then these points are mutually distinct .*

Proof. Assume, for the sake of contradiction, the points $\xi_{(t_1,t_2,\dots,t_k)} = R \prod_{1\leqslant j\leqslant k} e^{2it_j\omega_j}$ and $\xi_{(t'_1,t'_2,\dots,t'_k)} = R \prod_{1\leqslant j\leqslant k} e^{2it'_j\omega_j}$ are the same, where $0 \leqslant t_j \leqslant s_j$ and $0 \leqslant t'_j \leqslant s_j$. It follows that

$$\frac{\xi_{(t_1,t_2,\dots,t_k)}}{\xi_{(t'_1,t'_2,\dots,t'_k)}} = \prod_{j=1}^{k} e^{2i(t_j - t'_j)\omega_j} = e^{i\sum_{j=1}^{k} 2(t_j - t'_j)\omega_j} = 1. \tag{2.1}$$

Let $|t_j - t'_j| = m_j$ for $j = 1, 2, \dots, k$. Since not all the numbers $m_1, \dots, m_k$ are simultaneously zero it follows that at least one of them is a positive integer. Suppose $m_d > 0$. From (2.1) we have

$$\sum_{j=1}^{k} \pm 2m_j\omega_j = 2\pi r \text{ for some } r \in \mathbb{Z}.$$

We may write it as follows

$$2\pi r \mp 2m_d\omega_d = \sum_{j\neq d} \pm 2m_j\omega_j.$$

Hence

$$\cos 2m_d\omega_d = \cos\sum_{j\neq d} \pm 2m_j\omega_j = \cos\sum_{j\neq d} 2m_j\,(\pm\omega_j).$$

Equivalently, we have

$$\begin{aligned} &\frac{1}{2}\left[\left(\frac{a_d+b_d\sqrt{D}i}{p_d}\right)^{2m_d}+\left(\frac{a_d-b_d\sqrt{D}i}{p_d}\right)^{2m_d}\right] \\ = &\frac{1}{2}\left[\prod_{j\neq d}\left(\frac{a_j\pm b_j\sqrt{D}i}{p_j}\right)^{2m_j}+\prod_{j\neq=d}\left(\frac{a_j\mp b_j\sqrt{D}i}{p_j}\right)^{2m_j}\right]. \end{aligned} \tag{2.2}$$

For abbreviation, we set $A_j+B_j\sqrt{D}i:=\left(a_j+b_j\sqrt{D}i\right)^{2m_j}$ for $j=1,\dots,k$ and $A+B\sqrt{D}i:=\prod_{j\neq d}\left(A_j+B_j\sqrt{D}i\right)$. Here A_j, B_j, A and B are integers. With this notation, (2.2) is equivalent to the following

$$\frac{A_d}{p_d^{2m_ds_d}}=\frac{A}{\prod_{j\neq d}p_j^{2m_js_j}}.$$

Thus,

$$A_d\prod_{j\neq d}p_j^{2m_js_j}=p_d^{2m_ds_d}A. \tag{2.3}$$

By Lemma 2.2 we have $A_d\neq 0,\pm 1$. And since $\cos 2m_d\omega_d=\dfrac{A_d}{p_d^{2m_ds_d}}$ we have $|A_d|<p_d^{2m_ds_d}$. Therefore by $\mod p_d^{2m_ds_d}$ on equation (2.3) gives an obvious contradiction. This completes the proof. □

Lemma 2.4. *Under the above notations and assumptions, the following holds*

$$\sin\left[\sum_{j=1}^{k}(\pm m_j\omega_j)\right]=\frac{S\sqrt{D}}{\prod\limits_{j=1}^{k}p_j^{m_js_j}}$$

where S is an integer.

Proof. Let $\prod\limits_{j=1}^{k}(a_j\pm b_j\sqrt{D}i)^{m_j}=T+S\sqrt{D}i$, where $T,S\in\mathbb{Z}$. Since $p_j^{s_j}=\left|a_j+b_j\sqrt{D}i\right|$ for $j=1,\dots,k$ the assertion is deduced from the following

$$\begin{aligned} \prod_{j=1}^{k}\left(\frac{a_j}{p_j}\pm i\frac{b_j}{p_j}\sqrt{D}\right)^{m_j} &= \prod_{j=1}^{k}(\cos\omega_j\pm i\sin\omega_j)^{m_j} \\ &= \prod_{j=1}^{k}(\cos m_j\omega_j\pm i\sin m_j\omega_j) \\ &= \cos\left[\sum_{j=1}^{k}m_j\omega_j\right]\pm i\sin\left[\sum_{j=1}^{k}m_j\omega_j\right] \\ &= \cos\left[\sum_{j=1}^{k}m_j\omega_j\right]+i\sin\left[\sum_{j=1}^{k}(\pm m_j\omega_j)\right]. \end{aligned}$$

□

We are now able to state and prove our main result:

Theorem 2.5. *Under the above notations and assumptions,*

a) *If n is an odd integer, then there is a circular $\prod_{j=1}^{k}\left(\left\lfloor\frac{\alpha_j}{s_j}\right\rfloor+1\right)$ integral points set on the circle with radius $R=\frac{n\sqrt{D}}{2D}$.*

b) *If n is an even integer, that is $p_1=2$, then there is a circular $\left(\left\lfloor\frac{\alpha_1}{s_1-1}\right\rfloor+1\right)\prod_{j=2}^{k}\left(\left\lfloor\frac{\alpha_j}{s_j}\right\rfloor+1\right)$ integral points set on the circle with radius $R=\frac{n\sqrt{D}}{2D}$.*

Proof. By the above assumptions we have $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ with $p^{2s_j}=a_j^2+Db_j^2$ and $\arg(a_j+b_j\sqrt{D}i)=\omega_j$.

a) We have the following factorization

$$p^{2s_j}=(a_j+b_j\sqrt{D}i)(a_j-b_j\sqrt{D}i).$$

We choose the points

$$\xi_{(t_1,\dots,t_k)}=R\prod_{1\leqslant j\leqslant k}e^{2it_j\omega_j},\quad 0\leqslant t_j\leqslant\left\lfloor\frac{\alpha_j}{s_j}\right\rfloor,1\le j\le k$$

and

$$\xi_{(t'_1,\dots,t'_k)}=R\prod_{1\leqslant j\leqslant k}e^{2it'_j\omega_j},\quad 0\leqslant t'_j\leqslant\left\lfloor\frac{\alpha_j}{s_j}\right\rfloor,1\le j\le k$$

on the circle centered at the origin of complex plane with radius R. By the law of sines we have

$$\left|\xi_{(t_1,t_2,\dots,t_k)}-\xi_{(t'_1,t'_2,\dots,t'_k)}\right|=2R\cdot\left|\sin\left(\sum_{j=1}^{k}\left(t_j-t'_j\right)\omega_j\right)\right|.$$

Set $\left|t_j-t'_j\right|=m_j$ for $j=1,2,\dots,k$. Then $\prod_{1\le j\le k}\left(a_j\pm b_j\sqrt{D}i\right)^{m_j}=T+S\sqrt{D}i$ for some integers T,S. Thus by Lemma 2.4 we have,

$$\sin\left[\sum_{j=1}^{k}(\pm m_j\omega_j)\right]=\frac{S\sqrt{D}}{\prod_{1\le j\le k}p_j^{m_js_j}}.$$

It follows that

$$\left|\xi_{(t_1,t_2,\dots,t_k)}-\xi_{(t'_1,t'_2,\dots,t'_k)}\right|=2R\cdot\frac{|S|\sqrt{D}}{\prod_{j=1}^{k}p_j^{m_js_j}}=\frac{2n|S|}{\prod_{j=1}^{k}p_j^{m_js_j}}.$$

Since $m_js_j\leqslant\alpha_j$ for $j=1,2,\dots,k$ we see that $\left|\xi_{(t_1,t_2,\dots,t_k)}-\xi_{(t'_1,t'_2,\dots,t'_k)}\right|$ is a positive integer. Therefore, we conclude that all the distances between the points

$$\xi_{(t_1,\ldots,t_k)} = R \prod_{1\leqslant j\leqslant k} e^{2it_j\omega_j}, \quad 0 \leqslant t_j \leqslant \left\lfloor \frac{\alpha_j}{s_j} \right\rfloor, 1 \leq j \leq k$$

are positive integers. By Lemma 2.3 the above points are all distinct and form a circular $\prod_{j=1}^{k}\left(\left\lfloor \frac{\alpha_j}{s_j}\right\rfloor + 1\right)$ integral points set that lies on the circle centered at the origin with radius $R = \dfrac{n\sqrt{D}}{2D}$.

b) As in the proof of part a) we consider the following points

$$\xi_{(t_1,\ldots,t_k)} = R \prod_{1\leqslant j\leqslant k} e^{2it_j\omega_j}, \quad 0 \leqslant t_1 \leqslant \left\lfloor \frac{\alpha_1}{s_1-1} \right\rfloor \text{ and } 0 \leqslant t_j \leqslant \left\lfloor \frac{\alpha_j}{s_j} \right\rfloor, 2 \leqslant j \leqslant k$$

and

$$\xi_{(t_1',\ldots,t_k')} = R \prod_{1\leqslant j\leqslant k} e^{2i{t_j}'\omega_j}, \quad 0 \leqslant t_1 \leqslant \left\lfloor \frac{\alpha_1}{s_1-1} \right\rfloor \text{ and } 0 \leqslant {t_j}' \leqslant \left\lfloor \frac{\alpha_j}{s_j} \right\rfloor, 2 \leqslant j \leqslant k.$$

By the law of sines we have

$$\left|\xi_{(t_1,t_2,\ldots,t_k)} - \xi_{(t_1',t_2',\ldots,t_k')}\right| = 2R \cdot \left|\sin\left(\sum_{j=1}^{k}\left(t_j - t_j^{'}\right)\omega_j\right)\right|.$$

Set $\left|t_j - t_j'\right| = m_j$ for $j = 1, 2, \ldots, k$. Then by Lemma 2.1 we have

$$\cos m_1\omega_1 = \frac{x_{m_1}}{2^{m_1(s_1-1)+1}} \text{ and } \sin m_1\omega_1 = \frac{y_{m_1}\sqrt{D}}{2^{m_1(s_1-1)+1}}$$

, where x_{m_1}, y_{m_1} odd integers. Let $\prod_{2\leqslant j\leqslant k}(a_j \pm b_j\sqrt{D}i)^{m_j} = T + S\sqrt{D}i$, for some integers T, S. Thus

$$\begin{aligned}\prod_{1\leqslant j\leqslant k}(a_j \pm b_j\sqrt{D}i)^{m_j} &= 2^{m_1-1}(x_{m_1} + y_{m_1}\sqrt{D}i)(T + S\sqrt{D}i)\\ &= 2^{m_1-1(}T_1 + S_1\sqrt{D}i),\end{aligned}$$

where $T_1, S_1 \in \mathbb{Z}$. Hence

$$\sin(\sum_{0\leqslant j\leqslant k}(\pm m_j\omega_j)) = \frac{S_1\sqrt{D}}{2^{m_1(s_1-1)+1}\prod_{2\leqslant j\leqslant k} p_j^{m_js_j}}.$$

From all the above

$$|\xi_{(t_1,\ldots,t_k)} - \xi_{(t_1',\ldots,t_k')}| = 2R \cdot \frac{|S_1|\sqrt{D}}{2^{|t_1-{t_1}'|(s_1-1)+1}\prod_{2\leqslant j\leqslant k} p_j^{|t_j-t_j^{'}|s_j}}.$$

In the last equation,

$$|t_1 - t_1^{'}|(s_1-1) \leqslant \alpha_1$$

and

$$|t_j - t_j^{'}|s_j \leqslant \alpha_j, \quad 2 \leqslant j \leqslant k.$$

Thus $|\xi_{(t_1,\dots,t_k)} - \xi_{(t'_1,\dots,t'_k)}|$ is a positive integer. This means all the distances between the points

$$\xi_{(t_1,\dots,t_k)} = R\prod_{1\leqslant j\leqslant k} e^{2it_j\omega_j}, \quad 0\leqslant t_1\leqslant \left\lfloor \frac{\alpha_1}{s_1-1}\right\rfloor \text{ and } 0\leqslant t_j\leqslant \left\lfloor\frac{\alpha_j}{s_j}\right\rfloor, 2\leqslant j\leqslant k$$

are positive integers. By Lemma 2.3 the above points are all distinct and form a circular $\left(\left\lfloor \frac{\alpha_1}{s_1-1}\right\rfloor+1\right)\prod_{j=2}^{k}\left(\left\lfloor\frac{\alpha_j}{s_j}\right\rfloor+1\right)$ integral points set that lies on the circle centered at the origin with radius $R=\dfrac{n\sqrt{D}}{2D}$.

This completes the proof. □

Before we present the consequences of Theorem 2.5, we consider the following two examples.

Example 2.6. In this example, we consider the case when $D=5$. Since

$$3^2 = 2^2+5\cdot 1^2, 7^2 = 2^2+5\cdot 3^2 \tag{2.4}$$

we may choose $n=3^2\cdot 7=63$ in part a) of Theorem 2.5. Thus, $\alpha_1=2,\alpha_2=1$. On the other hand, (2.4) implies that $s_1=1,s_2=1$. Hence, by part a) of Theorem 2.5 there is a circular $\tau(63)=(2+1)(1+1)=6$ integral points set on the circle with radius $R=\frac{63\sqrt{5}}{10}$. The configuration of these points are given in the following figure.

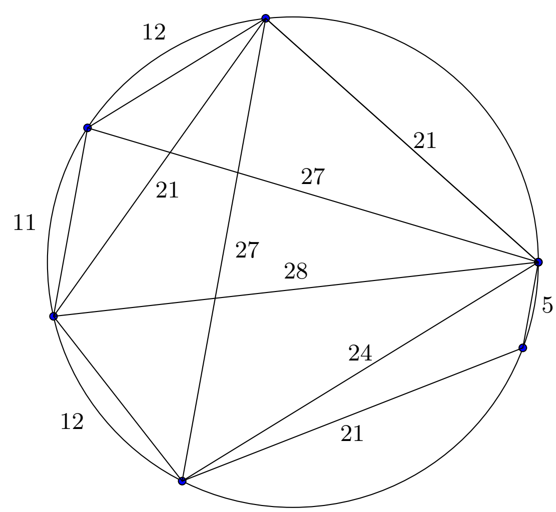

Fig.1. A circular 6 integral points set

Example 2.7. In this example, we consider the case when $D=6$. Since

$$5^2 = 1^2+6\cdot 2^2, 7^2=5^2+6\cdot 2^2, 11^2=5^2+6\cdot 4^2 \tag{2.5}$$

we can choose $n=5\cdot 7\cdot 11=385$ in part a) of Theorem 2.5. Thus, $\alpha_1=\alpha_2=\alpha_3=1$. On the other hand, (2.5) implies that $s_1=s_2=s_3=1$. Therefore, by part a) of Theorem 2.5 there is a circular $\tau(385)=(1+1)(1+1)(1+1)=8$ integral points set on the circle with radius $R=\frac{385\sqrt{6}}{12}$. The exact configuration of these points given in Fig.2.

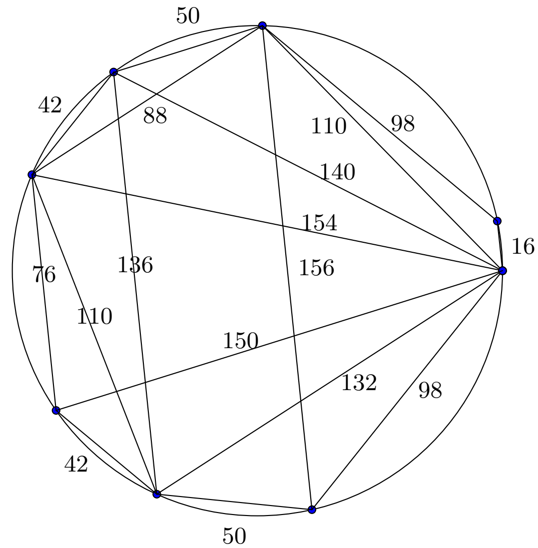

Fig.2. A circular 8 integral points set

Recall the following well known result from the elementary number theory. If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, then the number of all distinct positive divisor of n is $\tau(n) := (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_k + 1)$, where $p_1 < p_2 < \cdots < p_k$ prime numbers.

Corollary 2.8. *Let $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, where all the prime squares $p_j^2, j = 1, \ldots, k$ are of the form $a^2 + Db^2$, where $D > 4$ is a square-free number. Then there is a circular $\tau(n)$ integral points set on the circle with the radius $R = \dfrac{n\sqrt{D}}{2D}$.*

Proof. By assumption $p_j^2 = a^2 + Db^2 > D > 4$, so $p_j > 2$ for all $j = 1, 2, \ldots, k$. Thus, n is an odd integer. It follows that $s_1 = s_2 = \cdots = s_k = 1$. Now the assertion is deduced from part a) of Theorem 2.5. $\square$

Corollary 2.9. *Let $n = 2^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where all the prime squares $p_j^2, j = 2, \ldots, 3$ are of form $a^2 + 15b^2$. Then there is a circular $\tau(n)$ integral points set on the circle with radius $R = \frac{n\sqrt{15}}{15}$.*

Proof. Since $2^2 < 15, 2^{2 \cdot 2} = 1^2 + 1^2 \cdot 15$ and $p_j^2 = {a_j}^2 + {b_j}^2 D$ for $j = 2, 3, \ldots, k$ we must have

$$s_1 = 2 \text{ and } s_2 = \cdots = s_k = 1.$$

Now we consider the points $\xi_{(t_1, \ldots, t_k)} = R \prod\limits_{1 \leqslant j \leqslant k} e^{2it_j\omega_j}$, $0 \leqslant t_j \leqslant \alpha_j$. Then by part b) of Theorem 2.5 these points form a circular $\tau(n)$ integral points set on the circle with radius $R = \frac{n\sqrt{15}}{15}$. $\square$

To illustrate Corollary 2.9, we give a configuration of a circular 14 integral points set for $n = 1088 = 2^6 \cdot 17$.

Example 2.10. In Fig.3, we construct a circular $\tau(1088) = (6+1)(1+1)$ integral points set on the circle with radius $R = \frac{1088\sqrt{15}}{15}$.

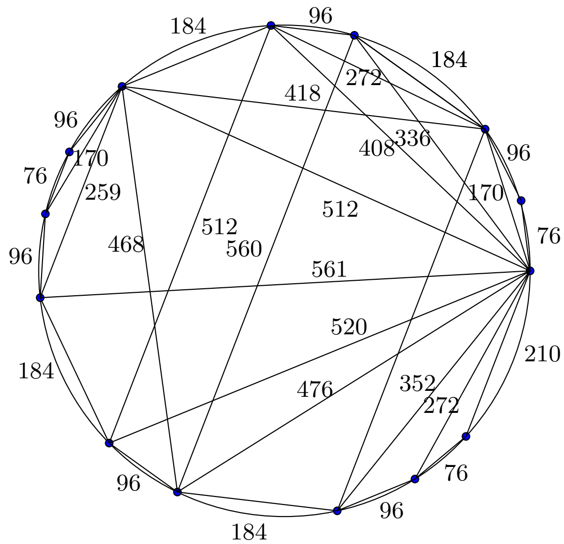

Fig.3.

2.1. **An upper bound of $R(n, 15)$ for $n \leq 3072$.** In order to obtain an refinement of upper bound for $R(n, 15)$ we consider the primes whose squares are of the form $a^2 + 15b^2$. From elementary number theory we know that the even powers of $3, 5, 7, 11, 13$ can not be written in the form $a^2 + 15b^2$.

Let $p_1 = 2 < p_2 < p_3 < \ldots$ be the prime numbers of the form $a^2 + 15b^2$. Since $n \leq 3072$ we only consider the following primes $p_2 = 17, p_3 = 19, p_4 = 23, p_5 = 31, p_6 = 47, p_7 = 53, p_8 = 61, p_9 = 79$. Let $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, where $p_1 < p_2 < \cdots < p_k < p_{k+1}$ for $k = 1, 2, \ldots, 8$. We also choose $\alpha_1, \ldots, \alpha_k$ with the following constraints from Theorem 6 in [5]:

i) If $p_i^m < p_j$, then $m\alpha_j \leq \alpha_i$;
ii) If $p_{k+1} < p_i^{m+1}$, then $\alpha_i \leq 2m$;
iii) If $p_i p_j > p_{k+1}$ and $\alpha_i \geq 3$, then $\alpha_j = 1$;
iv) If $p_i p_j > p_1 p_{k+1}$ and $\alpha_i = 2, \alpha_j = 2$, then $\alpha_1 \geq 8$.

Here m is the largest positive integer such that $p_i^m < p_j$, $i < j$. With this primes let us consider the radius $R(n, 15) = \frac{\sqrt{15}}{15} p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$. For $n \leq 3072$ in Table 1 we obtain an upper bound $R(n, 15)$. We also comparing the Table 2 of [5] with the Table 1, we improved the upper bound of R_n for $n \leq 3072$ in Table 2.

Theorem 2.11. *Under the above assumptions, for $n \leq 3072$ the upper bound of $R(n, 15)$ in Table 1 is better than the one given in Table 1 of* [5]. *The upper bound of R_n for $n \leq 3072$ in Table 2 is better than the one given in Table 2 of* [5].

Proof. First, we obtain an upper bound of $R(n, 15)$ for small values of n. Suppose $n = 2^{\alpha_1}$. Then $R(n, 15) = \frac{1}{\sqrt{15}} \cdot 2^{\alpha_1}$ and $k = 1$. Observe that $2^4 = p_1^4 < p_2 = 17$. So $m = 4$. By the constraint $i)$ we have $\alpha_1 \geq 0$. On the other hand, by the constraint $ii)$ we have $\alpha_1 \leq 8 = 2m$ because $17 < 2^5$. Hence, $0 \leq \alpha_1 \leq 8$.

Therefore, we have the following upper bounds

$$\begin{aligned} R(2,15) &\le \frac{1}{\sqrt{15}}\cdot 2, \quad R(3,15) \le \frac{1}{\sqrt{15}}\cdot 2^5,\\ R(4,15) &\le \frac{1}{\sqrt{15}}\cdot 2^3, \quad R(5,15) \le \frac{1}{\sqrt{15}}\cdot 2^4,\\ R(6,15) &\le \frac{1}{\sqrt{15}}\cdot 2^5, \quad R(7,15) \le \frac{1}{\sqrt{15}}\cdot 2^6,\\ R(8,15) &\le \frac{1}{\sqrt{15}}\cdot 2^7 \text{ and } \quad R(9,15) \le \frac{1}{\sqrt{15}}\cdot 2^8. \end{aligned}$$

Next, assume $n = 2^{\alpha_1}\cdot 17^{\alpha_2}$, $\alpha_2 \ge 1$. Then $R = \frac{1}{\sqrt{15}}\cdot 2^{\alpha_1}\cdot 17^{\alpha_2}$ and $k = 2$. Since $2^4 = p_1^4 < p_2 = 17$ by the constraint $i)$ we have $4 \le 4\alpha_2 \le \alpha_1$. On the other hand, since $p_3 = 19 < 2^5 = p_1^5$ by the constraint $ii)$ we have $\alpha_1 \le 8$. Thus, $4 \le \alpha_1 \le 8$. Also, since $2\cdot 17 > 19$ by the constraint $iii)$ we must have $\alpha_2 = 1$. Therefore, we have the following upper bounds

$$\begin{aligned} R(10,15) &\le \frac{1}{\sqrt{15}}\cdot 2^4\cdot 17, \quad R(12,15) \le \frac{1}{\sqrt{15}}\cdot 2^5\cdot 17,\\ R(14,15) &\le \frac{1}{\sqrt{15}}\cdot 2^6\cdot 17, \quad R(16,15) \le \frac{1}{\sqrt{15}}\cdot 2^7\cdot 17,\\ \text{and } R(18,15) &\le \frac{1}{\sqrt{15}}\cdot 2^8\cdot 17. \end{aligned}$$

By the exactly same way we find the upper bounds of $R(n,15)$ for the larger values of n as shown in Table 1 and Table 2. Finally, from Table 1 and Table 2 in [5], we have an upper bound of $R(n,1)$ and $R(n,3)$ for $n \le 3072$. Hence we conclude that $R_n \le \min\{R(n,3), R(n,15)\}$ for $n \le 3072$. ☐

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