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FRACTIONAL PRODUCT CORDIAL LABELING OF CERTAIN GRAPH OPERATIONS

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Abstract. Fractional product cordial labeling assigns distinct whole numbers and unit fractions to graph vertices and labels each edge by rounding up the product of its end values. We prove that the number of edges labeled one takes every integer value between its minimum and maximum, and that its average exceeds half the number of edges. A graph admits such a labeling exactly when some labeling gives at least half of its edges, rounded down, a label other than one. We settle trees, unicyclic graphs, and subdivision, switching, duplication, line and middle graphs of paths, cycles, stars and related families.

Keywords. Cordial labeling, fractional product cordial labeling, subdivision graph, duplication, middle graph
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1. Introduction

Graph labelings assign numbers to the vertices or edges of a graph subject to arithmetic conditions. The subject goes back to Rosa [9], who introduced graceful valuations, and it has grown into a large literature that is catalogued in Gallian's dynamic survey [2]. Cahit [1] introduced cordial labeling as a weakening of graceful and harmonious labeling: a map $f\colon V(G) \to \{0,1\}$ is cordial if, after each edge uv is given the induced label $|f(u) - f(v)|$, the numbers of vertices carrying 0 and 1 differ by at most one, and so do the numbers of edges carrying 0 and 1. Replacing $|f(u) - f(v)|$ by the product $f(u)f(v)$ gives product cordial labeling, introduced by Sundaram, Ponraj and Somasundaram [11]. Product cordial labeling has been studied on graphs obtained by many operations, including tensor products [13], further graph families [12], and duplication of elements in crown, helm and wheel graphs [4]. Several other variants, such as H-cordial and generalized cordial labelings, are surveyed in [2]; see also [3, 6].

Ponraj and Sutharson [7] recently introduced a variant in which the vertex labels are not residues but integers and unit fractions. We recall their definition.

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For a graph $G = (V, E)$ of order p and size q put

$$M_p = \begin{cases} \{1, 2, \ldots, \frac{p}{2}\} \cup \{\frac{1}{2}, \frac{1}{3}, \ldots, \frac{2}{p+2}\}, & p \text{ even}, \\ \{1, 2, \ldots, \frac{p-1}{2}\} \cup \{\frac{1}{2}, \frac{1}{3}, \ldots, \frac{2}{p+3}\}, & p \text{ odd}. \end{cases}$$

A bijection $\chi\colon V(G) \to M_p$ assigns to each edge xy the label $\lceil \chi(x)\chi(y) \rceil$. Let $\Pi_\chi(1)$ and $\Pi_\chi(0)$ denote the numbers of edges labeled 1 and not labeled 1, respectively. The map χ is a *fractional product cordial labeling* (*FP-cordial labeling*) if $|\Pi_\chi(0) - \Pi_\chi(1)| \leq 1$, and G is *FP-cordial* if it admits one.

In [7] the authors determine which paths, cycles, complete graphs, stars, wheels, ladders, books with triangular pages, combs, double combs and bistars are FP-cordial, together with the subdivisions of stars and bistars. They also show that every graph is a subgraph of a connected FP-cordial graph. They list the investigation of other graph families as an open problem. We verified every one of the statements in [7] that concerns a specific family against exhaustive computer search for small orders and found no discrepancy.

The proofs in [7] proceed family by family and describe a labeling case by case. Our first contribution is a structural theory that makes such case analysis largely unnecessary. We prove that the set of values taken by $\Pi_\chi(1)$ over all bijections χ is an interval of integers (Theorem 3.4), that the average of $\Pi_\chi(1)$ exceeds $q/2$ (Lemma 3.5), and hence that G is FP-cordial if and only if some labeling makes at least $\lfloor q/2 \rfloor$ edges receive a label other than 1 (Theorem 3.6). The maximum possible number of such edges is $(\lfloor p/2 \rfloor - 1)^2$ (Lemma 3.1), which gives a clean necessary condition on the size of an FP-cordial graph.

The second contribution is the application of this criterion to graph operations that have not been examined for FP-cordiality:

- **Subdivision.** Every tree of order p is FP-cordial if and only if $p \notin \{3, 5\}$ (Theorem 4.1), and every connected unicyclic graph is FP-cordial if and only if its order is at least 6 (Theorem 4.2). For a connected graph G with n vertices and $m \geq n + 6$ edges the subdivision graph $S(G)$ is FP-cordial (Theorem 5.2). Together with a finite computer-assisted check this settles $S(K_n)$, $S(K_{r,s})$, $S(W_n)$ and the subdivided gear graph for all parameters (Theorem 5.3).
- **Multiple subdivision.** We treat the graphs $S_k(G)$ obtained by subdividing every edge k times and the graphs obtained by subdividing a single edge k times (Theorems 6.1 and 6.2).
- **Vertex switching** in cycles, paths, wheels and complete graphs (Theorems 7.1–7.5).
- **Duplication** of a vertex or an edge in paths and cycles, and duplication of all vertices of trees and cycles (Theorems 8.1–8.5).
- **Line and middle graphs** of paths, cycles and stars (Theorems 9.1 and 9.2–9.4).

In every family the answer is complete: we state exactly which members are FP-cordial and which are not. Table 1 at the end of Section 10 collects the results.

All constructions used in the proofs were implemented and checked by computer on large ranges of the parameters, and the negative statements were confirmed by an exhaustive search for small orders. The computer checks supplement the proofs and are not used in them, with one exception: a short list of small base graphs in Section 5 is handled by explicit labelings, which are given in the appendix and can be verified by hand.

2. Preliminaries

All graphs are finite, simple and undirected. A graph of order p and size q is a (p, q) graph. For $S \subseteq V(G)$ we write $e(S)$ for the number of edges of the induced subgraph $G[S]$, and $N(v)$ for the neighborhood of v. We write P_n, C_n, K_n, $K_{r,s}$ and $K_{1,n}$ for the path, cycle, complete graph, complete bipartite graph and star. We use the following operations, all of which are standard.

Definition 2.1. The *subdivision graph* $S(G)$ is obtained from G by inserting a new vertex of degree 2 on every edge. For an integer $k \geq 1$, the *k-fold subdivision* $S_k(G)$ replaces every edge by a path with k internal vertices, so $S_1(G) = S(G)$. If $e = uv$ is a single edge, the graph $G^{(e,k)}$ obtained from G by *subdividing e exactly k times* is formed by deleting e and adding a path $u\, w_1 w_2 \cdots w_k\, v$ whose internal vertices $w_1, \ldots, w_k$ are new.

Definition 2.2. For a vertex v of G, the graph G_v obtained by *switching* v is formed by deleting all edges incident with v and joining v to every vertex that was not adjacent to it in G (Seidel switching of a single vertex [10]).

Definition 2.3. The *duplication of a vertex* v adds a new vertex v' with $N(v') = N(v)$. The *duplication of an edge* $e = uv$ adds a new edge $e' = u'v'$ with $N(u') = (N(u) \setminus \{v\}) \cup \{v'\}$ and $N(v') = (N(v) \setminus \{u\}) \cup \{u'\}$. The graph obtained by duplicating every vertex of G (using the neighborhoods in G, so that no two new vertices are adjacent) is the *shadow graph* $D_2(G)$. These are the notions of duplication used in the cordial labeling literature; see for instance [4, 8].

Definition 2.4. The *line graph* $L(G)$ has the edges of G as vertices, two of them adjacent when they share an end vertex [14]. The *middle graph* $M(G)$ has vertex set $V(G) \cup E(G)$; an edge $e \in E(G)$ is adjacent to its two end vertices and to every edge of G that shares an end vertex with e, and no two vertices of $V(G)$ are adjacent [5].

The remaining graphs are the wheel $W_n = C_n + K_1$ (order $n + 1$, size $2n$), the gear graph G_n obtained from W_n by inserting a vertex on every rim edge (order $2n+1$, size $3n$), and trees such as caterpillars and banana trees, which are introduced where they are used.

Notation for labelings. For a (p, q) graph put

$$a = \lfloor p/2 \rfloor, \qquad b = \lceil p/2 \rceil = p - a.$$

Then $M_p = \{1, 2, \ldots, a\} \cup \{\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{b+1}\}$: indeed $\frac{1}{b+1} = \frac{2}{p+2}$ for p even and $\frac{1}{b+1} = \frac{2}{p+3}$ for p odd. We call the elements $1, \ldots, a$ the *integer labels* and the elements $\frac{1}{j}$ $(2 \leq j \leq b + 1)$ the *fractional labels*. Note that $a \leq b$.

Because all labels are positive, $\lceil x \rceil = 1$ holds exactly when $x \leq 1$. Therefore

$$\text{the edge } xy \text{ is labeled } 1 \iff \chi(x)\chi(y) \leq 1. \tag{2.1}$$

We call an edge with $\chi(x)\chi(y) > 1$ a 0-*edge*. Thus $\Pi_\chi(0)$ is the number of 0-edges and $\Pi_\chi(1) = q - \Pi_\chi(0)$. Since $\Pi_\chi(0) + \Pi_\chi(1) = q$, the labeling χ is FP-cordial if and only if $\Pi_\chi(1) \in \{\lfloor q/2 \rfloor, \lceil q/2 \rceil\}$, equivalently $\Pi_\chi(0) \in \{\lfloor q/2 \rfloor, \lceil q/2 \rceil\}$.

Example 2.5. Label the vertices of C_6 in cyclic order by $1, 2, 3, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$. The products along the edges are $2, 6, \frac{3}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{4}$, so the edge labels are $2, 6, 2, 1, 1, 1$. Hence $\Pi_\chi(0) = \Pi_\chi(1) = 3$ and C_6 is FP-cordial.

3. A structure theory for FP-cordial labeling

Throughout this section G is an arbitrary graph of order $p \geq 2$ and size q, $a = \lfloor p/2 \rfloor$ and $M = M_p$. By a labeling we mean a bijection $\chi \colon V(G) \to M$.

3.1. **Heavy pairs.** Call a pair $\{\alpha, \beta\}$ of distinct elements of M *heavy* if $\alpha\beta > 1$. By (2.1), an edge is a 0-edge exactly when its end labels form a heavy pair.

Lemma 3.1. *Let* H_p *be the graph on* M_p *whose edges are the heavy pairs.*

(i) Any two distinct integer labels form a heavy pair.

(ii) An integer label k *and a fractional label* $\frac{1}{j}$ *form a heavy pair if and only if* $j < k$*. In particular the labels* 1 *and* 2 *lie in no heavy pair with a fractional label.*

(iii) Two fractional labels never form a heavy pair.

(iv) H_p *has exactly* $(a-1)^2$ *edges.*

Consequently every labeling of every graph of order p *has* $\Pi_\chi(0) \leq (a-1)^2$.

Proof. (i) Distinct positive integers have product at least $1 \cdot 2 = 2 > 1$. (ii) We have $k \cdot \frac{1}{j} > 1$ exactly when $j < k$, and every fractional label has $j \geq 2$. (iii) A product of two numbers in $(0, 1)$ is less than 1. (iv) There are $\binom{a}{2}$ heavy pairs of integers. For $3 \leq k \leq a$ the fractional partners of k are $\frac{1}{2}, \ldots, \frac{1}{k-1}$; these are labels because $k - 1 \leq a - 1 \leq b + 1$. This gives $\sum_{k=3}^{a}(k-2) = \binom{a-1}{2}$ further pairs. The total is $\binom{a}{2} + \binom{a-1}{2} = (a-1)^2$. The final assertion holds because the 0-edges of G are mapped injectively onto edges of H_p. □

Corollary 3.2. *If a* (p, q) *graph is FP-cordial, then* $\lfloor q/2 \rfloor \leq (\lfloor p/2 \rfloor - 1)^2$*, that is,* $q \leq 2(\lfloor p/2 \rfloor - 1)^2 + 1$*. If* $p \geq 6$ *and equality* $\lfloor q/2 \rfloor = (\lfloor p/2 \rfloor - 1)^2$ *holds, then an FP-cordial graph must contain a triangle.*

Proof. An FP-cordial labeling has $\Pi_\chi(0) \geq (q-1)/2$, hence $\Pi_\chi(0) \geq \lfloor q/2 \rfloor$, and $\Pi_\chi(0) \leq (a-1)^2$ by Lemma 3.1. If equality holds, all edges of H_p must be images of edges of G, so G contains a copy of H_p. For $a \geq 3$ the integers $1, 2, 3$ are pairwise heavy, so H_p contains a triangle. □

For $p = 3$ and $p = 5$ the bound $(a-1)^2$ equals 0 and 1 respectively, so these orders are particularly restrictive.

3.2. **The values of $\Pi_\chi(1)$ form an interval.** Write the elements of M in increasing order as $\ell_1 < \ell_2 < \cdots < \ell_p$, that is,

$$\tfrac{1}{b+1} < \cdots < \tfrac{1}{3} < \tfrac{1}{2} < 1 < 2 < \cdots < a.$$

Two labelings are *adjacent* if one is obtained from the other by exchanging the labels ℓ_i and ℓ_{i+1} between the two vertices that carry them, for some i.

Lemma 3.3. *If χ and χ' are adjacent labelings, then $|\Pi_\chi(1) - \Pi_{\chi'}(1)| \le 1$.*

Proof. Let $\alpha = \ell_i < \beta = \ell_{i+1}$, and let u, w be the vertices with $\chi(u) = \alpha$, $\chi(w) = \beta$, so that $\chi'(u) = \beta$ and $\chi'(w) = \alpha$. Only edges at u or w can change their status. The edge uw, if present, does not change, because its product is $\alpha\beta$ in both labelings. Consider an edge uz with $z \notin \{u, w\}$ and $\gamma = \chi(z)$. Its product goes from $\alpha\gamma$ to $\beta\gamma$, so it can change from "labeled 1" to "not labeled 1" only if $\alpha\gamma \le 1 < \beta\gamma$, that is, only if $\frac{1}{\gamma} \in [\alpha, \beta)$. It can never change in the opposite direction, since $\alpha < \beta$. Symmetrically, an edge wz can only change from "not labeled 1" to "labeled 1", and then only if $\frac{1}{\gamma} \in [\alpha, \beta)$.

We claim that $\frac{1}{\gamma} \in [\alpha, \beta)$ forces $\gamma = 1/\alpha$. The numbers $\frac{1}{\gamma}$ for $\gamma \in M$ are $\frac{1}{k}$ for $1 \le k \le a$, all of which lie in M because $a \le b+1$, and the integers $2, 3, \ldots, b+1$. Those integers that exceed a are larger than every element of M, hence larger than β. All remaining values $\frac{1}{\gamma}$ are elements of M, and since α, β are consecutive in M, the only element of M in $[\alpha, \beta)$ is α. Thus $\frac{1}{\gamma} = \alpha$.

Hence at most one vertex z (the one with label $1/\alpha$, if that label differs from α and β) is involved. If $z \sim u$, the edge uz goes from product 1 to product $\beta/\alpha > 1$, so $\Pi(1)$ drops by 1; if $z \sim w$, the edge wz goes from product $\beta/\alpha > 1$ to product 1, so $\Pi(1)$ rises by 1. The total change is $[z \sim w] - [z \sim u] \in \{-1, 0, 1\}$. □

Theorem 3.4. *For every graph G of order p, the set $\{\Pi_\chi(1) : \chi$ a labeling of $G\}$ consists of all integers between its minimum and its maximum.*

Proof. Any two bijections $V(G) \to M$ differ by a permutation of M, and every permutation is a product of transpositions of consecutive elements ℓ_i, ℓ_{i+1} (for instance by bubble sort). So any two labelings are joined by a sequence of adjacent labelings. By Lemma 3.3 the value of $\Pi_\chi(1)$ changes by at most 1 along this sequence, so it passes through every integer between the values at the two ends. □

3.3. **The mean value and the characterization.**

Lemma 3.5. *If $q \ge 1$, the average of $\Pi_\chi(1)$ over all labelings of G is strictly greater than $q/2$. In particular some labeling has $\Pi_\chi(1) > q/2$.*

Proof. For a fixed edge xy and a uniformly random labeling, $\{\chi(x), \chi(y)\}$ is a uniformly random 2-subset of M. By Lemma 3.1 the probability that it is heavy is $(a-1)^2/\binom{p}{2}$. So the expected number of 0-edges is $q(a-1)^2/\binom{p}{2}$, and it suffices to show $(a-1)^2 < \frac{1}{2}\binom{p}{2} = \frac{p(p-1)}{4}$. For $p = 2a$ this reads $(2a-2)^2 < 2a(2a-1)$, that is $4a^2 - 8a + 4 < 4a^2 - 2a$, which holds for $a \ge 1$. For $p = 2a+1$ it reads $4(a-1)^2 < (2a+1)(2a)$, that is $4a^2 - 8a + 4 < 4a^2 + 2a$, which is true for $a \ge 1$.

Hence the expected number of 0-edges is below $q/2$ and the expected number of edges labeled 1 exceeds $q/2$. □

Theorem 3.6. *A graph G with q edges is FP-cordial if and only if some labeling χ has $\Pi_\chi(0) \geq \lfloor q/2 \rfloor$, that is, at least $\lfloor q/2 \rfloor$ edges not labeled 1.*

Proof. Necessity was observed in the proof of Corollary 3.2. For sufficiency we may assume $q \geq 1$. Let χ_0 satisfy $\Pi_{\chi_0}(0) \geq \lfloor q/2 \rfloor$, so $\Pi_{\chi_0}(1) \leq \lceil q/2 \rceil$. By Lemma 3.5 there is χ_1 with $\Pi_{\chi_1}(1) > q/2$, hence $\Pi_{\chi_1}(1) \geq \lfloor q/2 \rfloor$. If $\Pi_{\chi_0}(1) \geq \lfloor q/2 \rfloor$, then χ_0 is FP-cordial. Otherwise $\Pi_{\chi_0}(1) < \lfloor q/2 \rfloor \leq \Pi_{\chi_1}(1)$, and Theorem 3.4 provides a labeling χ with $\Pi_\chi(1) = \lfloor q/2 \rfloor$, which is FP-cordial. □

Theorem 3.6 says that FP-cordiality is a statement about how many edges of G can simultaneously be placed on edges of the fixed graph H_p: G is FP-cordial if and only if G and H_p have a common subgraph with $\lfloor q/2 \rfloor$ edges that is spanning on the side of G. The proof is constructive, since the labeling χ is found by walking from χ_0 to χ_1 along adjacent labelings.

3.4. **A construction lemma.** All positive results below rest on the following labeling.

Lemma 3.7. *Let G be a graph of order p, let $I \subseteq V(G)$ with $|I| = a = \lfloor p/2 \rfloor$, and let $u \in I$ have r neighbors outside I. Put $c = \min\{r, \max\{a-2, 0\}\}$. Then there is a labeling χ with $\Pi_\chi(0) \geq e(I) + c$.*

Proof. Give the vertices of I the integer labels $1, \ldots, a$, with u receiving a. Choose c neighbors $w_2, \ldots, w_{c+1}$ of u outside I and give w_j the label $\frac{1}{j}$; this is possible because $c+1 \leq a-1 \leq b+1$ when $c \geq 1$. Give the remaining vertices the remaining fractional labels arbitrarily. Every edge inside I joins two distinct integers, so it is a 0-edge by Lemma 3.1(i). Every edge uw_j has product $a/j > 1$ since $j \leq c+1 \leq a-1$, so it is a 0-edge as well. These edges are distinct, which gives $\Pi_\chi(0) \geq e(I) + c$. □

Corollary 3.8. *If G has a set I of $\lfloor p/2 \rfloor$ vertices and a vertex $u \in I$ with $e(I) + c \geq \lfloor q/2 \rfloor$, where c is as in Lemma 3.7, then G is FP-cordial.*

Proof. Combine Lemma 3.7 with Theorem 3.6. □

For a connected graph, a set I of a vertices that induces a connected subgraph is obtained from the first a vertices of a breadth-first ordering, and it has $e(I) \geq a-1$. If $a < p$, connectivity also gives a vertex $u \in I$ with a neighbor outside I. When $a \geq 3$ this yields $c \geq 1$. We call such a pair (I, u) a *connected pair*, and record the resulting bound.

Lemma 3.9. *Let G be connected of order $p \geq 6$. Then G has a labeling with $\Pi_\chi(0) \geq \lfloor p/2 \rfloor = a$.*

Proof. Take a connected pair (I, u). Then $e(I) \geq a-1$ and $c \geq 1$ because $a \geq 3$. □

4. Trees and unicyclic graphs

Theorem 4.1. *A tree of order p is FP-cordial if and only if $p \notin \{3, 5\}$.*

Proof. A tree has $q = p - 1$ edges. For $p = 3$ we have $\lfloor q/2 \rfloor = 1 > 0 = (a-1)^2$, and for $p = 5$ we have $\lfloor q/2 \rfloor = 2 > 1 = (a-1)^2$. By Corollary 3.2 such trees are not FP-cordial.

Let $p \notin \{3, 5\}$. If $p \in \{1, 2\}$ then $\lfloor q/2 \rfloor = 0$ and any labeling works by Theorem 3.6. If $p = 4$ then $a = 2$, $\lfloor q/2 \rfloor = 1$, and we take I to be the two end vertices of an edge, so $e(I) = 1$. Now let $p \geq 6$. By Lemma 3.9 there is a labeling with $\Pi_\chi(0) \geq a$. If p is even then $\lfloor q/2 \rfloor = \lfloor (p-1)/2 \rfloor = \frac{p}{2} - 1 = a - 1 < a$. If p is odd then $\lfloor q/2 \rfloor = \frac{p-1}{2} = a$. In both cases Theorem 3.6 shows that the tree is FP-cordial. □

Theorem 4.2. *A connected graph of order p with exactly p edges is FP-cordial if and only if $p \geq 6$.*

Proof. If $p \in \{3, 4, 5\}$ then $\lfloor q/2 \rfloor = \lfloor p/2 \rfloor = 1, 2, 2$ exceeds $(a-1)^2 = 0, 1, 1$, so Corollary 3.2 excludes these graphs. If $p \geq 6$, Lemma 3.9 gives $\Pi_\chi(0) \geq a = \lfloor p/2 \rfloor = \lfloor q/2 \rfloor$, and Theorem 3.6 applies. □

Theorem 4.1 contains the tree results of [7] as special cases and extends them to all trees. We record the consequences.

Corollary 4.3. *(a) P_n is FP-cordial iff $n \notin \{3, 5\}$; C_n is FP-cordial iff $n \geq 6$; $K_{1,n}$ is FP-cordial iff $n \notin \{2, 4\}$.*

(b) The comb $P_n \odot K_1$ (order $2n$) and the bistar $B_{n,n}$ (order $2n + 2$) are FP-cordial for all n; the double comb $P_n \odot 2K_1$ (order $3n$) is FP-cordial iff $n \geq 2$.

(c) The banana tree $B(n, k)$, obtained from n copies of $K_{1,k}$ by joining one leaf of each copy to a new root, has order $n(k + 1) + 1$. It is FP-cordial unless $(n, k) \in \{(1, 1), (2, 1), (1, 3)\}$.

(d) Every caterpillar and every spider (a tree with one vertex of degree at least 3) of order other than 3 and 5 is FP-cordial.

Proof. (a) For C_n apply Theorem 4.2; the rest is Theorem 4.1. (b), (c), (d) follow from Theorem 4.1 by computing the orders. For (c), $n(k+1)+1 \in \{3, 5\}$ means $n(k + 1) \in \{2, 4\}$, which occurs only for $(n, k) \in \{(1, 1), (2, 1), (1, 3)\}$. □

5. Subdivision graphs

Ponraj and Sutharson treat $S(K_{1,n})$ and $S(B_{n,n})$ [7]. Both are trees, and so is the subdivision of any tree. Indeed, if T has n vertices then $S(T)$ is a tree with $2n - 1$ vertices.

Proposition 5.1. *Let T be a tree of order $n \geq 2$. Then $S(T)$ is FP-cordial if and only if $n \notin \{2, 3\}$, that is, $T \notin \{K_2, P_3\}$.*

Proof. $S(T)$ is a tree of order $2n-1$, and $2n-1 \in \{3, 5\}$ exactly when $n \in \{2, 3\}$. Apply Theorem 4.1. □

In particular $S(T)$ is FP-cordial for every caterpillar and every banana tree with at least four vertices. For graphs with cycles we need a different argument. For a connected graph G with n vertices and m edges, $S(G)$ has order $p = n+m$ and size $q = 2m$, so FP-cordiality means that at least m edges of $S(G)$ must be 0-edges. We write the subdivision vertex of an edge e again as e.

Theorem 5.2. *Let G be a connected graph with n vertices and $m \geq n+6$ edges. Then $S(G)$ is FP-cordial.*

Proof. Let $p = n+m$, $a = \lfloor p/2 \rfloor$, $b = p-a$ and $t = a-n = \lfloor (m-n)/2 \rfloor \geq 3$. By Theorem 3.6 it suffices to exhibit a labeling of $S(G)$ with at least m 0-edges.

An injection from vertices to incident edges. Since G is connected and $m \geq n$, it has a spanning unicyclic subgraph U. Let C be its cycle. Orient C cyclically; every vertex of C selects its outgoing cycle edge. Every other vertex v selects the first edge of the shortest path in U from v to C. This defines a map φ from $V(G)$ to $E(G)$ with $v \in \varphi(v)$, and φ is injective, since a non-cycle vertex selects the edge to its parent, which no other vertex selects, and the cycle vertices select distinct cycle edges. Put $\Phi = \varphi(V(G))$, so $|\Phi| = n$.

The set I. Since $m-n \geq t+1$, we can choose a set $E_0 \subseteq E(G) \setminus \Phi$ with $|E_0| = t$ and one further edge $e^* \in E(G) \setminus (\Phi \cup E_0)$. Let $I = V(G) \cup E_0$, so $|I| = n+t = a$, and let F be the set of remaining subdivision vertices, with $|F| = m-t = b$.

The labeling. Give the vertices of E_0 the integer labels $1, \ldots, t$ and the vertices $v_1, \ldots, v_n$ of G the labels $t+1, \ldots, t+n = a$, with v_i receiving $t+i$. Give e^* the label $\frac{1}{2}$, give $\varphi(v_i)$ the label $\frac{1}{i+2}$ for $i = 1, \ldots, n$, and give the remaining vertices of F the remaining fractional labels arbitrarily. These choices use distinct fractions, and $|F| \geq n+1$ because $b = m-t \geq \frac{m+n}{2} \geq n+3$.

Counting 0-edges.

(1) Each $e \in E_0$ has both neighbors in $V(G)$, and both are integer labels. These $2t$ edges are 0-edges.

(2) The edge joining v_i and $\varphi(v_i)$ has product $\frac{t+i}{i+2} > 1$ because $t \geq 3$. These n edges are 0-edges, and they are distinct from those in (1) because $\varphi(v_i) \in F$.

(3) The vertex e^* has two neighbors in $V(G)$, whose labels are at least $t+1 \geq 4$, so both edges at e^* have product greater than 2. They are not among the edges of (2), since $e^* \notin \Phi$.

Altogether there are at least $2t+n+2$ 0-edges. Since $2t \geq m-n-1$, this number is at least $m+1 \geq m$. □

If $m = n$, then $S(G)$ is connected and unicyclic of order $2n \geq 6$, hence FP-cordial by Theorem 4.2.

We now apply Theorem 5.2 to the families named in the introduction. The gear graph G_n has $2n+1$ vertices and $3n$ edges.

Theorem 5.3. *(a) $S(K_n)$ is FP-cordial if and only if $n \geq 3$.*

(b) For $1 \leq r \leq s$, $S(K_{r,s})$ is FP-cordial if and only if $(r,s) \notin \{(1,1),(1,2)\}$.

(c) $S(W_n)$ is FP-cordial for every $n \geq 3$.

(d) $S(G_n)$ is FP-cordial for every $n \geq 3$.

Proof. For the large cases we check the hypothesis $m \geq n_G + 6$ of Theorem 5.2, writing n_G for the order of the base graph. For K_n it reads $\frac{n(n-3)}{2} \geq 6$, which holds for $n \geq 6$. For $K_{r,s}$ it reads $rs - r - s \geq 6$, that is $(r-1)(s-1) \geq 7$. For W_n it reads $2n - (n+1) = n - 1 \geq 6$, and for G_n it reads $3n - (2n+1) = n - 1 \geq 6$; both hold for $n \geq 7$.

The remaining cases are finite. The graphs $S(K_2) = P_3$, $S(K_{1,1}) = P_3$ and $S(K_{1,2}) = P_5$ are not FP-cordial, while $S(K_3) = C_6$ (Example 2.5), $S(K_{2,2}) = C_8$ and $S(K_{1,s})$ for $s \geq 3$ (a tree of order $2s + 1 \geq 7$) are FP-cordial. For $S(K_4)$, $S(K_5)$, $S(K_{2,s})$ with $3 \leq s \leq 7$, $S(K_{3,3})$, $S(K_{3,4})$, $S(W_n)$ with $4 \leq n \leq 6$ (note $W_3 = K_4$) and $S(G_n)$ with $3 \leq n \leq 6$, an explicit FP-cordial labeling is given in Table 2 in the Appendix. These lists and Theorem 5.2 cover every pair $r \leq s$: for $r = 2$, $s \leq 7$ is listed and $s \geq 8$ gives $(r-1)(s-1) \geq 7$; for $r = 3$, $s = 3, 4$ are listed and $s \geq 5$ gives $2(s-1) \geq 8$; for $r \geq 4$, $(r-1)(s-1) \geq 9$. □

Remark 5.4. Exhaustive search shows that $S(G)$ is FP-cordial for every connected graph G on at most 7 vertices except K_2 and P_3 (995 graphs checked). This suggests that the hypothesis $m \geq n + 6$ in Theorem 5.2 can be removed; see Conjecture 5.5 below.

Conjecture 5.5. *For every connected graph G other than K_2 and P_3, the subdivision graph $S(G)$ is FP-cordial.*

6. Multiple subdivisions

Recall that $S_k(G)$ subdivides every edge k times.

Theorem 6.1. *Let $k \geq 1$.*

(a) $S_k(P_n)$ $(n \geq 2)$ is FP-cordial unless $(k, n) \in \{(1, 2), (1, 3), (3, 2)\}$.
(b) $S_k(C_n)$ $(n \geq 3)$ is FP-cordial for every k.
(c) $S_k(K_{1,n})$ is FP-cordial unless $(k, n) \in \{(1, 1), (1, 2), (3, 1)\}$.

Proof. $S_k(P_n)$ is a path of order $(k+1)(n-1)+1$, $S_k(C_n)$ is a cycle of length $(k+1)n \geq 6$, and $S_k(K_{1,n})$ is a tree of order $(k+1)n+1$. The orders $(k+1)(n-1)+1$ and $(k+1)n+1$ equal 3 or 5 exactly for the listed pairs. Apply Corollary 4.3(a), Theorem 4.2 and Theorem 4.1. □

We now consider subdividing a single edge k times, and ask when this preserves FP-cordiality. For trees and unicyclic graphs the answer is exact. For general connected graphs we obtain a threshold in k.

Theorem 6.2. *Let G be a connected (p, q) graph with $p \geq 3$, let $e \in E(G)$, $k \geq 1$, and let $G' = G^{(e,k)}$, which has order $p + k$.*

(a) If G is a tree, then G' is FP-cordial if and only if $p + k \notin \{3, 5\}$.
(b) If G is unicyclic $(q = p)$, then G' is FP-cordial if and only if $p + k \geq 6$.
(c) If $k \geq p$, then G' is FP-cordial, whatever G is.

Proof. (a) G' is a tree of order $p+k$; apply Theorem 4.1. (b) G' is connected and unicyclic of order $p + k$; apply Theorem 4.2.

(c) If G is a tree then $p+k \geq 2p \geq 6$ and (a) applies. So let $q \geq p$ and put $N = p+k \geq 2p$, $a = \lfloor N/2 \rfloor \geq p$ and $s = q-p \geq 0$. The graph G' has N vertices and $q+k = q+N-p$ edges. Let $w_1, \ldots, w_k$ be the subdividing vertices, in order from u to v where $e = uv$, and put $t = a-p \geq 0$. Then $t \leq k-1$, because $a \leq N-1$. Let

$$I = V(G) \cup \{w_1, \ldots, w_t\}.$$

Then $|I| = a$, and $G'[I]$ contains all edges of G except e together with the t edges of the path $u\, w_1 \cdots w_t$. Thus $e(I) \geq q-1+t$. The last vertex z of this path ($z = w_t$, or $z = u$ if $t = 0$) has the neighbor $w_{t+1} \notin I$, and $a \geq p \geq 3$, so Lemma 3.7 gives a labeling with at least $q-1+t+1 = q+a-p = a+s$ 0-edges. On the other hand

$$\left\lfloor \frac{q+k}{2} \right\rfloor = \left\lfloor \frac{N+s}{2} \right\rfloor \leq \left\lfloor \frac{N}{2} \right\rfloor + \left\lceil \frac{s}{2} \right\rceil \leq a+s.$$

By Theorem 3.6, G' is FP-cordial. □

Remark 6.3. Subdividing an edge can destroy or create FP-cordiality when k is small. For instance, P_4 is FP-cordial but $P_4^{(e,1)} = P_5$ is not, and C_5 is not FP-cordial but $C_5^{(e,1)} = C_6$ is. By Theorem 6.2(a),(b), a tree or unicyclic graph that is FP-cordial stays FP-cordial after k subdivisions of one edge unless the new order is 3 or 5 (trees) or less than 6 (unicyclic graphs).

7. Vertex switching

In this section the vertices of P_n and C_n are $v_0, \ldots, v_{n-1}$ in order. For $n \geq 3$ the switched graph $(C_n)_v$ is the same for every v, whereas for P_n the position of v matters. We write $Z(p) = (\lfloor p/2 \rfloor - 1)^2$.

Theorem 7.1. *For $n \geq 3$, $(C_n)_v$ is FP-cordial if and only if $n \neq 5$.*

Proof. Take $v = v_0$. Then v_0 is adjacent to $v_2, \ldots, v_{n-2}$ and the path $v_1 v_2 \cdots v_{n-1}$ is untouched. So $p = n$ and $q = (n-2)+(n-3) = 2n-5$. For $n = 3$ we have $q = 1$, hence $\lfloor q/2 \rfloor = 0$ and Theorem 3.6 gives FP-cordiality. For $n = 4$ the graph is $K_{1,3}$ with center v_2, a tree of order 4 (Theorem 4.1). For $n = 5$, $q = 5$ and $\lfloor q/2 \rfloor = 2 > 1 = Z(5)$, so Corollary 3.2 excludes it.

Let $n \geq 6$ and $a = \lfloor n/2 \rfloor$. Take $I = \{v_0, v_2, v_3, \ldots, v_a\}$, which has a elements. Then v_0 is adjacent to the $a-1$ other vertices of I, and the path edges $v_j v_{j+1}$ ($2 \leq j < a$) contribute $a-2$ more, so $e(I) \geq 2a-3$. The target is $\lfloor q/2 \rfloor = n-3$. If n is even, $2a-3 = n-3$ and we are done by Corollary 3.8. If n is odd, then $n \geq 7$, $2a-3 = n-4$, and $u = v_0$ has the neighbors $v_{a+1}, \ldots, v_{n-2}$ outside I, at least one of them, while $a \geq 3$; so $c \geq 1$ and $e(I)+c \geq n-3$. □

Theorem 7.2. *Let $n \geq 2$ and let v be an end vertex of P_n. Then $(P_n)_v$ is FP-cordial if and only if $n \notin \{3, 4, 5, 7\}$.*

Proof. Take $v = v_0$. Then v_0 loses the edge $v_0 v_1$ and gains $v_2, \ldots, v_{n-1}$, so $p = n$, $q = (n-2)+(n-2) = 2n-4$. For $n = 2$ we get $q = 0$ and the graph is FP-cordial. For $n = 3, 4, 5, 7$ we have $\lfloor q/2 \rfloor = n-2 = 1, 2, 3, 5$ and $Z(n) = 0, 1, 1, 4$, which is smaller, so Corollary 3.2 excludes these.

Now let $n = 6$ or $n \geq 8$, $a = \lfloor n/2 \rfloor \geq 3$, and $I = \{v_0, v_2, \ldots, v_a\}$. As in the previous proof, $e(I) \geq 2a-3$, and $u = v_0$ has the neighbors $v_{a+1}, \ldots, v_{n-1}$ outside I. The target is $\lfloor q/2 \rfloor = n-2$. For even n, $2a-3 = n-3$ and $c \geq 1$ gives $n-2$. For odd $n \geq 9$ we have $a \geq 4$, $2a-3 = n-4$, and u has $n-1-a = \frac{n-1}{2} \geq 2$ outside neighbors, so $c = \min\{r, a-2\} \geq 2$ and $e(I) + c \geq n-2$. □

Theorem 7.3. *Let $n \geq 3$ and let $v = v_i$ with $1 \leq i \leq n-2$. Then $(P_n)_v$ is FP-cordial if and only if $n \neq 5$.*

Proof. The vertex v_i loses v_{i-1}, v_{i+1} and gains the $n-3$ vertices outside $\{v_{i-1}, v_i, v_{i+1}\}$. So $p = n$ and $q = 2n-6$. For $n = 3$ we have $q = 0$. For $n = 4$ we have $q = 2 > 0$ and $a = 2$, and taking I to be the two ends of an edge gives $e(I) = 1 = \lfloor q/2 \rfloor$. For $n = 5$, $q = 4$, $\lfloor q/2 \rfloor = 2 > 1 = Z(5)$.

Let $n \geq 6$, $a = \lfloor n/2 \rfloor$. The vertices $v_0, \ldots, v_{i-2}$ and $v_{i+2}, \ldots, v_{n-1}$ form two segments with $n-3$ vertices in total; the longer one has at least $\lceil (n-3)/2 \rceil \geq a-1$ vertices. Choose $a-1$ consecutive vertices from it and let I consist of these and v_i. All of them are neighbors of v_i, and consecutive ones are adjacent, so $e(I) \geq (a-1) + (a-2) = 2a-3$. The target is $\lfloor q/2 \rfloor = n-3$. For even n we have $2a-3 = n-3$. For odd $n \geq 7$ we have $2a-3 = n-4$, while $u = v_i$ has $(n-3) - (a-1) = \frac{n-3}{2} \geq 2$ neighbors outside I and $a \geq 3$, so $c \geq 1$. □

Theorem 7.4. *$(K_n)_v$ is FP-cordial if and only if $n \leq 4$.*

Proof. $(K_n)_v = K_{n-1} \cup K_1$ has $q = \binom{n-1}{2}$. For $n \leq 4$ the graphs are $2K_1$, $K_2 \cup K_1$, $K_3 \cup K_1$; they have $q = 0, 1, 3$ and $\lfloor q/2 \rfloor = 0, 0, 1$, and an edge-set I of size a with $e(I) \geq \lfloor q/2 \rfloor$ exists (for $K_3 \cup K_1$ take $a = 2$ and two vertices of the triangle). For $n \geq 5$ we show $q > 2(\lfloor n/2 \rfloor - 1)^2 + 1$. For even n this reads $(n-1)(n-2) > (n-2)^2 + 2$, i.e. $n > 4$. For odd n it reads $(n-1)(n-2) > (n-3)^2 + 2$, i.e. $n > 3$. Apply Corollary 3.2. □

The same density argument shows that K_n itself is FP-cordial if and only if $n \leq 2$, since $\binom{n}{2} > 2(\lfloor n/2 \rfloor - 1)^2 + 1$ for all $n \geq 3$: for even n this reads $n^2 - n > (n-2)^2 + 2$, i.e. $n > 2$, and for odd n it reads $n^2 - n > (n-3)^2 + 2$, i.e. $5n > 11$. This recovers the result for complete graphs in [7].

Theorem 7.5. *For $n \geq 3$, $(W_n)_v$ with v the hub is FP-cordial if and only if $n \neq 4$.*

Proof. Switching the hub, which is adjacent to every other vertex, leaves it isolated, so $(W_n)_v = C_n \cup K_1$ with $p = n+1$, $q = n$, $a = \lfloor (n+1)/2 \rfloor$. For $n = 4$, $\lfloor q/2 \rfloor = 2 > 1 = Z(5)$. For $n = 3$, $a = 2$, $\lfloor q/2 \rfloor = 1$, and two adjacent cycle vertices give $e(I) = 1$. For $n \geq 5$ take I to be a consecutive vertices of the cycle, so $e(I) = a-1$. If n is odd, $a - 1 = \frac{n-1}{2} = \lfloor q/2 \rfloor$. If $n \geq 6$ is even, $a = \frac{n}{2} \geq 3$, $e(I) = \frac{n}{2} - 1$, and an end vertex of the segment has a neighbor outside I, so $c \geq 1$ and $e(I) + c \geq \frac{n}{2} = \lfloor q/2 \rfloor$. □

Remark 7.6. Switching a rim vertex of W_n gives a graph with $p = n+1$ and $q = 3n-6$. Exhaustive search shows that it is FP-cordial for $n \in \{3, 5, 7, 9, 10, 11, 12\}$ and not FP-cordial for $n \in \{4, 6, 8\}$. For $n = 4, 6$ this follows from the density condition, but for $n = 8$ the density condition does not exclude the graph, so a structural argument would be needed. We leave the general case open.

8. Duplication

Theorem 8.1. *Let $n \geq 2$ and let G be obtained from P_n by duplicating a vertex v. Then G is FP-cordial if and only if $n \geq 5$, or $n = 3$ and v is an end vertex.*

Proof. Let v' be the new vertex. If v is an end vertex, v' is a pendant vertex attached to the neighbor of v, so G is a tree of order $n+1$, FP-cordial if and only if $n+1 \notin \{3,5\}$, that is $n \notin \{2,4\}$ (Theorem 4.1). If v is an interior vertex ($n \geq 3$), v, v' and their two common neighbors form a 4-cycle, and G is connected unicyclic of order $n+1$; by Theorem 4.2 it is FP-cordial if and only if $n \geq 5$. Collecting the cases gives the statement. □

Theorem 8.2. *Let G be obtained from C_n ($n \geq 3$) by duplicating a vertex. Then G is FP-cordial if and only if $n = 5$ or $n \geq 7$.*

Proof. Let $v = v_0$ be duplicated by v', which is adjacent to v_1 and v_{n-1}. Then $p = n+1$, $q = n+2$, $a = \lfloor (n+1)/2 \rfloor$.

Non-FP-cordial cases. For $n = 3$ we have $p = 4$, $q = 5$, $\lfloor q/2 \rfloor = 2 > 1 = Z(4)$. For $n = 4$, $p = 5$, $q = 6$ and $3 > 1 = Z(5)$. For $n = 6$ we have $p = 7$, $q = 8$, $\lfloor q/2 \rfloor = 4 = Z(7)$. Here G is bipartite (the two vertices v_0, v' lie on the same side), so it has no triangle, and Corollary 3.2 excludes it.

$n = 5$. Here $p = 6$, $a = 3$, $\lfloor q/2 \rfloor = 3$. Take $I = \{v_1, v_0, v_2\}$ and $u = v_1$. Then $e(I) = 2$, and u has the neighbor v' outside I, so $c = 1$ and $e(I) + c = 3$.

$n \geq 7$. Put $I = \{v', v_{n-1}, v_0, v_1, \ldots, v_{a-3}\}$, which has $1 + (a-1) = a$ vertices; this is a valid set because $a \geq 4$. The $a-1$ vertices $v_{n-1}, v_0, \ldots, v_{a-3}$ are consecutive on the cycle, which gives $a-2$ edges, and v' is adjacent to v_{n-1} and v_1, which gives two more. Hence $e(I) \geq a$. If n is odd, $a = \frac{n+1}{2} = \lfloor q/2 \rfloor$. If $n \geq 8$ is even, $a = \frac{n}{2}$, $\lfloor q/2 \rfloor = \frac{n}{2} + 1 = a + 1$, and $u = v_{a-3}$ has the neighbor $v_{a-2} \notin I$, so $c \geq 1$ because $a \geq 4$. □

Theorem 8.3. *Let $n \geq 2$ and let G be obtained from P_n by duplicating an edge. Then G is FP-cordial if and only if $n \neq 3$.*

Proof. For $n = 2$ the graph is $2K_2$ with $q = 2$, $a = 2$, and one edge as I gives $e(I) = 1 = \lfloor q/2 \rfloor$. Let $n \geq 3$ and $e = v_i v_{i+1}$. If e is a pendant edge, say $e = v_0 v_1$, then u' is adjacent only to v', and v' is adjacent to v_2 and u'; thus G is P_n with a pendant path of length 2 attached at v_2, a tree of order $n+2$, FP-cordial if and only if $n + 2 \neq 5$, that is $n \neq 3$ (for $n = 3$ every edge is pendant). If e is interior (this requires $n \geq 4$), G is connected with $n+2$ vertices and $n+2$ edges, so it is unicyclic, and $n + 2 \geq 6$ gives FP-cordiality by Theorem 4.2. □

Theorem 8.4. *Let G be obtained from C_n ($n \geq 3$) by duplicating an edge. Then G is FP-cordial if and only if $n \notin \{3,5\}$.*

Proof. Let $e = v_0 v_1$ be duplicated by $u'v'$, where $u' \sim v_{n-1}, v'$ and $v' \sim v_2, u'$. Then $p = n+2$, $q = n+3$, $a = \lfloor (n+2)/2 \rfloor$.

For $n = 3$, $p = 5$, $q = 6$ and $\lfloor q/2 \rfloor = 3 > 1 = Z(5)$. For $n = 5$ we have $p = 7$, $q = 8$, $\lfloor q/2 \rfloor = 4 = Z(7)$. We check that G has no triangle: a triangle must contain one of the new edges $u'v'$, $u'v_4$, $v'v_2$, since C_5 has none. The vertices u' and v' have no common neighbor, as $N(u') = \{v', v_4\}$ and $N(v') = \{u', v_2\}$; $v_4 \not\sim v'$ and $v_2 \not\sim u'$. So by Corollary 3.2 the graph is not FP-cordial.

Let n be even, $n \geq 4$. Then $a = \frac{n+2}{2}$ and $\lfloor q/2 \rfloor = \frac{n+2}{2} = a$. Since G is connected and $a \geq 3$, Lemma 3.9 (whose proof uses only $a \geq 3$) gives a labeling with $\Pi_\chi(0) \geq a$.

Let n be odd, $n \geq 7$. Then $a = \frac{n+1}{2} \geq 4$ and $\lfloor q/2 \rfloor = \frac{n+3}{2} = a+1$. Take $I = \{v_2, v_3, \ldots, v_{a+1}\}$, a path of a vertices contained in the original cycle (note $a+1 \leq n-1$), so $e(I) \geq a-1$. The vertex $u = v_2$ has the two neighbors v_1 and v' outside I, and $a-2 \geq 2$, so $c = 2$ and $e(I)+c \geq a+1$. Corollary 3.8 applies. □

We turn to duplicating all vertices.

Theorem 8.5. *(a) For every tree T of order $n \geq 2$, the shadow graph $D_2(T)$ is FP-cordial.*

(b) For every $n \geq 3$, $D_2(C_n)$ is FP-cordial.

Proof. Write x' for the duplicate of x. If $uv \in E(G)$ then $D_2(G)$ contains the edges uv, $u'v$, uv' and not $u'v'$, so $D_2(G)$ has $3|E(G)|$ edges and order $2|V(G)|$.

(a) Here $p = 2n$, $a = n$, $q = 3(n-1)$. For $n = 2$, $D_2(T) = P_4$, which is FP-cordial. Let $n \geq 3$ and $h = \lfloor n/2 \rfloor$. Choose a connected set A of h vertices of T (the first h vertices in a breadth-first order), so $e_T(A) = h-1$.

If $n = 2h$ is even, let $I = A \cup A'$. Every edge of $T[A]$ yields three edges inside I, so $e(I) = 3(h-1)$. The target is $\lfloor q/2 \rfloor = \lfloor (6h-3)/2 \rfloor = 3h-2$. A vertex $u \in A$ has a neighbor $w \notin A$ in T (as T is connected and $A \neq V(T)$), and $w, w' \notin I$ are neighbors of u in $D_2(T)$. So $c \geq 1$ since $a = n \geq 3$, and $e(I)+c \geq 3h-2$.

If $n = 2h+1$ is odd, choose $x \in V(T) \setminus A$ adjacent to some $u \in A$, and let $I = A \cup A' \cup \{x\}$, so $|I| = 2h+1 = n$. The edges inside I include $3(h-1)$ from $T[A]$ together with xu and xu', so $e(I) \geq 3h-1$. The target is $\lfloor 6h/2 \rfloor = 3h$. The vertex u has the neighbor $x' \notin I$, so $c \geq 1$, and $e(I)+c \geq 3h$.

(b) Here $p = 2n$, $a = n$, $q = 3n$, so the target is $\lfloor 3n/2 \rfloor$. For $n = 3$ take $I = \{v_0, v_1, v_2\}$, which induces a triangle ($e(I) = 3$); $u = v_0$ has the outside neighbors v_1' and v_2', and $a = 3$ gives $c = 1$, so $e(I)+c = 4 = \lfloor 9/2 \rfloor$. Let $n \geq 4$.

If n is even, let $A = \{v_0, \ldots, v_{n/2}\}$ and $I = A \cup \{v_1', \ldots, v_{n/2-1}'\}$, which has $(\frac{n}{2}+1)+(\frac{n}{2}-1) = n$ vertices. It contains the $\frac{n}{2}$ path edges inside A, and each v_j' is adjacent to both v_{j-1} and v_{j+1} in A, adding $2(\frac{n}{2}-1)$ edges. So $e(I) = \frac{3n}{2}-2$. The vertex $u = v_0$ has the two outside neighbors v_{n-1} and v_{n-1}', and $a-2 = n-2 \geq 2$, so $c = 2$ and $e(I)+c = \frac{3n}{2} = \lfloor 3n/2 \rfloor$.

If n is odd, let $A = \{v_0, \ldots, v_{(n-1)/2}\}$ and $I = A \cup \{v_1', \ldots, v_{(n-1)/2}'\}$, of size $\frac{n+1}{2} + \frac{n-1}{2} = n$. There are $\frac{n-1}{2}$ path edges in A, $2(\frac{n-1}{2}-1)$ edges at the twins v_j' with $j < \frac{n-1}{2}$, and one edge at the last twin $v_{(n-1)/2}'$, whose other neighbor $v_{(n+1)/2}$ lies outside A. So $e(I) = \frac{3n-5}{2}$. The vertex $u = v_0$ has the outside neighbors v_{n-1} and v_{n-1}', and $a-2 = n-2 \geq 3$, so $c = 2$ and $e(I)+c = \frac{3n-1}{2} = \lfloor 3n/2 \rfloor$. □

9. Line and middle graphs

Theorem 9.1. *(a) $L(P_n) \cong P_{n-1}$ is FP-cordial if and only if $n \notin \{4, 6\}$ (for $n \geq 2$).*

(b) $L(C_n) \cong C_n$ is FP-cordial if and only if $n \geq 6$.

(c) $L(K_{1,n}) \cong K_n$ is FP-cordial if and only if $n \leq 2$.

Proof. The isomorphisms are standard. Apply Corollary 4.3(a) to P_{n-1} and C_n (the path of order $n-1$ is excluded exactly when $n-1 \in \{3,5\}$), and the observation after Theorem 7.4 to K_n. □

For the middle graph we use the following description. Let $V(P_n) = \{v_0, \ldots, v_{n-1}\}$ and $E(P_n) = \{e_0, \ldots, e_{n-2}\}$ with $e_i = v_i v_{i+1}$. Then $M(P_n)$ has the edges $e_i v_i$, $e_i v_{i+1}$ and $e_i e_{i+1}$, so $p = 2n-1$ and $q = 2(n-1) + (n-2) = 3n-4$. Arrange the vertices in the sequence $w_0 = v_0,\ w_1 = e_0,\ w_2 = v_1,\ w_3 = e_1, \ldots$; consecutive terms are adjacent, and in addition $w_{2i+1} \sim w_{2i+3}$.

Theorem 9.2. *$M(P_n)$ $(n \geq 2)$ is FP-cordial if and only if $n \geq 4$.*

Proof. For $n = 2$, $M(P_2) \cong P_3$. For $n = 3$ we have $p = 5$, $q = 5$ and $\lfloor q/2 \rfloor = 2 > 1 = Z(5)$. Let $n \geq 4$, so $a = n-1 \geq 3$, and take $I = \{w_1, \ldots, w_{n-1}\}$. This segment contains $n-2$ consecutive pairs, and its terms with odd index, namely $e_0, e_1, \ldots,$ are $\lfloor n/2 \rfloor$ in number, giving $\lfloor n/2 \rfloor - 1$ chords $w_{2i+1}w_{2i+3}$. No other edges join two vertices of I, since v_i is adjacent only to e_{i-1}, e_i. Hence $e(I) = (n-2) + \lfloor n/2 \rfloor - 1 = n + \lfloor n/2 \rfloor - 3$. The target is $\lfloor q/2 \rfloor = \lfloor (3n-4)/2 \rfloor = n + \lfloor n/2 \rfloor - 2$. The vertex $u = w_1 = e_0$ has the neighbor $v_0 \notin I$, and $a \geq 3$, so $c \geq 1$ and $e(I) + c \geq n + \lfloor n/2 \rfloor - 2$. □

For C_n with $E = \{e_0, \ldots, e_{n-1}\}$, $e_i = v_i v_{i+1}$ (indices mod n), the middle graph $M(C_n)$ has $p = 2n$, $q = 2n + n = 3n$ and the cyclic sequence $w_{2i} = v_i$, $w_{2i+1} = e_i$, again with consecutive terms adjacent and $w_{2i+1} \sim w_{2i+3}$.

Theorem 9.3. *$M(C_n)$ is FP-cordial for every $n \geq 3$.*

Proof. Here $a = n$ and the target is $\lfloor 3n/2 \rfloor = n + \lfloor n/2 \rfloor$. Take $I = \{w_1, \ldots, w_n\}$. It has $n-1$ consecutive pairs, and $\lceil n/2 \rceil$ odd-indexed terms $e_0, e_1, \ldots,$ giving $\lceil n/2 \rceil - 1$ chords. The closing chord $e_{n-1}e_0$ is not inside I because $e_{n-1} = w_{2n-1} \notin I$, and v_i is adjacent only to e_{i-1}, e_i. So $e(I) = n + \lceil n/2 \rceil - 2$, and the deficit is $\lfloor n/2 \rfloor - \lceil n/2 \rceil + 2$, which equals 2 for even n and 1 for odd n. The vertex $u = e_0 = w_1$ has the two neighbors $v_0 = w_0$ and $e_{n-1} = w_{2n-1}$ outside I. For odd $n \geq 3$ we need $c \geq 1$, which holds since $a - 2 \geq 1$. For even $n \geq 4$ we need $c = 2$, which holds since $a - 2 \geq 2$. □

For the star $K_{1,n}$ with center c, leaves $x_1, \ldots, x_n$ and edges $e_i = cx_i$, the middle graph $M(K_{1,n})$ has order $2n+1$. Its edges are $e_i c$, $e_i x_i$ and $e_i e_j$ for all $i < j$, so $q = 2n + \binom{n}{2} = \frac{n(n+3)}{2}$. It is the complete graph on $\{c, e_1, \ldots, e_n\}$ with a pendant vertex x_i attached at each e_i.

Theorem 9.4. *$M(K_{1,n})$ is FP-cordial if and only if $n \geq 3$.*

Proof. For $n = 1$, $M(K_{1,1}) \cong P_3$. For $n = 2$, $p = 5$, $q = 5$, and $\lfloor q/2 \rfloor = 2 > 1 = Z(5)$. Let $n \geq 3$, so $a = n$, and take $I = \{e_1, \ldots, e_n\}$, which induces K_n: $e(I) = \binom{n}{2}$. The vertex $u = e_1$ has the outside neighbors c and x_1, and $c = \min\{2, n-2\}$. The target is $\lfloor n(n+3)/4 \rfloor$. For $n = 3$: $e(I) + c = 3 + 1 = 4 = \lfloor 9/2 \rfloor$. For $n = 4$: $e(I) + c = 6 + 2 = 8 \geq 7$. For $n \geq 5$, $\binom{n}{2} \geq \frac{n(n+3)}{4}$ because $2(n-1) \geq n+3$. □

10. Conclusion

We characterized FP-cordial graphs by a counting condition: G is FP-cordial exactly when some bijection onto M_p makes at least $\lfloor q/2 \rfloor$ edges have a product exceeding 1. This rests on three facts: the values of $\Pi_\chi(1)$ fill an interval, their average exceeds $q/2$, and at most $(\lfloor p/2 \rfloor - 1)^2$ edges can be 0-edges. Each proof then reduces to choosing $\lfloor p/2 \rfloor$ vertices that induce many edges. In particular, all trees except those of order 3 and 5, and all connected unicyclic graphs of order at least 6, are FP-cordial, which contains the results of [7] on paths, cycles, stars, combs, bistars and subdivided stars. Table 1 summarizes the remaining results.

Graph	FP-cordial if and only if
Tree of order p	$p \notin \{3, 5\}$ (Theorem 4.1)
Connected unicyclic graph of order p	$p \geq 6$ (Theorem 4.2)
$S(T)$, T a tree of order $n \geq 2$	$n \notin \{2, 3\}$ (Proposition 5.1)
$S(G)$, G connected with $m \geq n + 6$	always (Theorem 5.2)
$S(K_n)$; $S(K_{r,s})$ with $r \leq s$	$n \geq 3$; $(r, s) \notin \{(1, 1), (1, 2)\}$ (Theorem 5.3)
$S(W_n)$; $S(G_n)$	$n \geq 3$, always (Theorem 5.3)
$S_k(P_n)$; $S_k(K_{1,n})$	$(k, n) \notin \{(1, 2), (1, 3), (3, 2)\}$; $(k, n) \notin \{(1, 1), (1, 2), (3, 1)\}$ (Theorem 6.1)
$S_k(C_n)$	always (Theorem 6.1)
One edge of a tree or unicyclic graph subdivided k times	new order $\notin \{3, 5\}$ (tree); new order ≥ 6 (unicyclic) (Theorem 6.2)
$(C_n)_v$; $(W_n)_v$ with v the hub	$n \neq 5$; $n \neq 4$ (Theorems 7.1, 7.5)
$(P_n)_v$, v an end vertex; v interior	$n \notin \{3, 4, 5, 7\}$; $n \neq 5$ (Theorems 7.2, 7.3)
$(K_n)_v$	$n \leq 4$ (Theorem 7.4)
Vertex duplication in P_n; in C_n	$n \geq 5$, or $n = 3$ with v an end vertex; $n = 5$ or $n \geq 7$ (Theorems 8.1, 8.2)
Edge duplication in P_n; in C_n	$n \neq 3$; $n \notin \{3, 5\}$ (Theorems 8.3, 8.4)
Shadow graph $D_2(T)$; $D_2(C_n)$	always ($n \geq 2$); always ($n \geq 3$) (Theorem 8.5)
$L(P_n)$; $L(C_n)$; $L(K_{1,n})$	$n \notin \{4, 6\}$; $n \geq 6$; $n \leq 2$ (Theorem 9.1)
$M(P_n)$; $M(C_n)$; $M(K_{1,n})$	$n \geq 4$; always ($n \geq 3$); $n \geq 3$ (Theorems 9.2–9.4)

Table 1. Summary of the results.

Two questions remain open. The first is Conjecture 5.5, which asserts that the hypothesis $m \geq n+6$ in Theorem 5.2 is unnecessary. The second is the switching

of a rim vertex of a wheel (Remark 7.6), for which only computer evidence is available.

Appendix A. Explicit labelings of the small subdivision graphs

Table 2 lists, for each small graph left open by Theorem 5.2, a labeling $\chi\colon V \to M_p$ with $\Pi_\chi(0) \in \{\lfloor q/2 \rfloor, \lceil q/2 \rceil\}$; the number of 0-edges is given in the fourth column. The vertices of the subdivision graph are named as follows. In $S(K_n)$, $v_1, \ldots, v_n$ are the original vertices and e_{ij} $(i < j)$ is the vertex inserted on $v_i v_j$. In $S(K_{r,s})$, $u_1, \ldots, u_r$ and $w_1, \ldots, w_s$ are the original vertices of the two parts and e_{ij} is the vertex inserted on $u_i w_j$. In $S(W_n)$, c is the hub, $v_1, \ldots, v_n$ the rim vertices (indices modulo n), s_i is the vertex inserted on cv_i and t_i the vertex inserted on $v_i v_{i+1}$. In $S(G_n)$, c is the hub, $v_1, \ldots, v_n$ the rim vertices of the wheel, x_i the vertex of the gear between v_i and v_{i+1}, and a_i, b_i, d_i are the vertices inserted on cv_i, $v_i x_i$ and $x_i v_{i+1}$ respectively.

Table 2: Labelings of the small subdivision graphs used in Theorem 5.3.

Graph	p	q	$\Pi_\chi(0)$	Labels
$S(K_4)$	10	12	6	$v_1, \ldots, v_4$: 1/3, 1/6, 1/2, 1; $e_{12}, e_{13}, \ldots, e_{3,4}$: 1/5, 1/4, 4, 5, 2, 3
$S(K_5)$	15	20	10	$v_1, \ldots, v_5$: 2, 1, 1/4, 7, 1/2; $e_{12}, e_{13}, \ldots, e_{4,5}$: 1/7, 5, 6, 3, 1/5, 1/6, 4, 1/8, 1/9, 1/3
$S(K_{2,3})$	11	12	6	u_1, u_2: 1/2, 5; $w_1, \ldots, w_3$: 1/7, 1/4, 1; $e_{11}, e_{12}, \ldots, e_{23}$: 1/6, 4, 3, 1/5, 1/3, 2
$S(K_{2,4})$	14	16	8	u_1, u_2: 3, 1/7; $w_1, \ldots, w_4$: 1, 4, 1/8, 7; $e_{11}, e_{12}, \ldots, e_{24}$: 6, 5, 1/5, 1/2, 1/4, 1/3, 2, 1/6
$S(K_{2,5})$	17	20	10	u_1, u_2: 8, 3; $w_1, \ldots, w_5$: 1/9, 1/3, 2, 1/10, 1/4; $e_{11}, e_{12}, \ldots, e_{25}$: 5, 1/5, 1, 1/8, 1/7, 4, 1/2, 1/6, 6, 7
$S(K_{2,6})$	20	24	12	u_1, u_2: 1/4, 3; $w_1, \ldots, w_6$: 1/9, 10, 1, 8, 9, 1/8; $e_{11}, e_{12}, \ldots, e_{26}$: 6, 1/3, 7, 1/5, 1/6, 5, 4, 1/7, 2, 1/11, 1/10, 1/2
$S(K_{2,7})$	23	28	14	u_1, u_2: 8, 11; $w_1, \ldots, w_7$: 1/7, 1/10, 7, 1, 1/11, 1/9, 1/8; $e_{11}, e_{12}, \ldots, e_{27}$: 6, 4, 1/13, 1/12, 3, 1/6, 1/2, 1/5, 1/4, 10, 9, 2, 1/3, 5
$S(K_{3,3})$	15	18	9	$u_1, \ldots, u_3$: 1/9, 7, 1/4; $w_1, \ldots, w_3$: 1, 1/2, 1/3; $e_{11}, e_{12}, \ldots, e_{33}$: 1/6, 5, 1/7, 3, 1/5, 4, 2, 1/8, 6
$S(K_{3,4})$	19	24	12	$u_1, \ldots, u_3$: 3, 7, 4; $w_1, \ldots, w_4$: 1/10, 1/2, 1/9, 8; $e_{11}, e_{12}, \ldots, e_{34}$: 6, 1/11, 1, 2, 1/3, 1/6, 1/7, 1/5, 1/8, 9, 5, 1/4

Graph	p	q	$\Pi_\chi(0)$	Labels
$S(W_4)$	13	16	8	c: 5; $v_1,\ldots,v_4$: 1, 6, 3, 1/6; $s_1,\ldots,s_4$: 1/8, 1/2, 4, 1/4; $t_1,\ldots,t_4$: 1/3, 1/5, 1/7, 2
$S(W_5)$	16	20	10	c: 3; $v_1,\ldots,v_5$: 1, 7, 4, 8, 1/9; $s_1,\ldots,s_5$: 1/4, 1/8, 6, 1/2, 1/7; $t_1,\ldots,t_5$: 2, 1/5, 1/3, 5, 1/6
$S(W_6)$	19	24	12	c: 3; $v_1,\ldots,v_6$: 1/10, 1/2, 7, 8, 9, 1/7; $s_1,\ldots,s_6$: 6, 5, 1, 1/6, 1/9, 2; $t_1,\ldots,t_6$: 1/3, 4, 1/5, 1/11, 1/4, 1/8
$S(G_3)$	16	18	9	c: 2; $v_1,\ldots,v_3$: 1, 7, 4; $x_1,\ldots,x_3$: 8, 1/7, 1/9; $a_1,\ldots,a_3$: 1/4, 3, 6; $b_1,\ldots,b_3$: 1/5, 1/6, 5; $d_1,\ldots,d_3$: 1/3, 1/8, 1/2
$S(G_4)$	21	24	12	c: 1/9; $v_1,\ldots,v_4$: 1/11, 6, 1, 2; $x_1,\ldots,x_4$: 1/3, 1/6, 9, 1/2; $a_1,\ldots,a_4$: 1/5, 1/7, 1/12, 3; $b_1,\ldots,b_4$: 1/10, 1/4, 8, 7; $d_1,\ldots,d_4$: 5, 10, 1/8, 4
$S(G_5)$	26	30	15	c: 13; $v_1,\ldots,v_5$: 12, 1/11, 6, 1/8, 10; $x_1,\ldots,x_5$: 1/5, 1/4, 1/12, 3, 8; $a_1,\ldots,a_5$: 1/13, 11, 2, 1/14, 1/2; $b_1,\ldots,b_5$: 1/6, 7, 1/3, 9, 5; $d_1,\ldots,d_5$: 1/10, 1/9, 4, 1/7, 1
$S(G_6)$	31	36	18	c: 1/10; $v_1,\ldots,v_6$: 12, 1/13, 1/3, 8, 10, 1/15; $x_1,\ldots,x_6$: 1/8, 13, 4, 3, 1, 15; $a_1,\ldots,a_6$: 1/4, 2, 11, 1/7, 7, 1/6; $b_1,\ldots,b_6$: 1/11, 1/14, 1/17, 9, 6, 1/12; $d_1,\ldots,d_6$: 1/5, 1/9, 1/2, 14, 1/16, 5

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