Gulf Journal of Mathematics
Vol 24, Issue 2 (2026) 1-15

ISSN: 2309-4966
https://doi.org/10.56947/q838m239

DYNAMICS ON PARA-KÄHLER STRUCTURES AND THEIR LIFTS

BERTUEL TANGUE NDAWA[1*], FERDINAND NGAKEU[2], NASSER SAIPELE NANSIDI[3] and THOMAS BOUETOU BOUETOU[4]

Abstract. Let M be a smooth manifold equipped with a para-Kähler structure. We define an action of the diffeomorphism group on the space of para-Kähler structures and prove that this action preserves the associated affine structures. Although this result is well known, we provide a proof based on this action, showing in particular that an isometry commutes with the Levi-Civita connection. We then construct several natural lifts of this action to the tangent bundle, the cotangent bundle, and the Whitney sum. In some cases, these lifted actions induce natural dynamical systems on a distinguished class of 3-webs.

Keywords. Connection, distribution, para-Kähler, 3-web
2020 Mathematics Subject Classification. Primary 53C15, 53D12

To Professor JOËL TOSSA

1. Introduction

An almost paracomplex (product) structure on a $2n$-dimensional manifold M, with $n > 0$, is a $(1,1)$-tensor field F satisfying $F^2 = \mathrm{Id}$, and such that the eigendistributions $F^{\pm}$ associated with the eigenvalues ± 1 both have dimension n. If g is a neutral metric on M that is anti-invariant under F, equivalently, if F is an anti-isometry of g (g-compatible), namely $g(FX, FY) = -g(X, Y)$ for all vector fields X, Y on M, then the pair (g, F) is called an almost para-Kähler (para-Hermitian) structure. If, in addition, the Levi-Civita connection $\nabla = \nabla^g$ of g satisfies $\nabla F = 0$, then (g, F) is called a para-Kähler structure, and the triple (M, F, g) is called a para-Kähler manifold.

Para-Kähler structures have been extensively studied in the literature. One of the main reasons for this interest is that such structures correspond bijectively to bi-Lagrangian structures, namely triples $(\omega, \mathcal{F}_1, \mathcal{F}_2)$, where ω is a symplectic form and $\mathcal{F}_1, \mathcal{F}_2$ are two transverse Lagrangian foliations defined by $F^{\pm}$. The tensors g, ω, and F are related by $g(\cdot, \cdot) = \omega(F\cdot, \cdot)$. Consequently, any two of these tensors uniquely determine the remaining one. A para-Kähler structure also induces a

Date: Received: Oct 6, 2026; Revised: Nov 4, 2026; Accepted: Nov 10, 2026.
* Corresponding author
© The Author(s) 2026. This article is licensed under a Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License. To view a copy of the licence, visit https://creativecommons.org/licenses/by-nc-nd/4.0/.

deformed Lie bracket on vector fields, defined by $[X,Y]_F = [FX,Y]+[X,FY]-F[X,Y]$ for all vector fields $X, Y \in \mathfrak{X}(M)$. This bracket gives rise to an associated cohomology theory, which is isomorphic to the de Rham cohomology via F.

In [3], the author studies a lift of the action of the symplectic group to the space of bi-Lagrangian structures, under the assumption that the underlying manifold is parallelizable. The main result is based on a lifting procedure on TM and T^*M for bi-Lagrangian manifolds. The action considered there is induced by the map $(\psi, X) \longmapsto \psi_* X$, that is, by the natural action of the diffeomorphism group on the space of vector fields. This construction is further extended in [5] to Whitney sums $E = TM \oplus T^*M$ in the setting of parallelizable manifolds. Moreover, in [4], the author introduces several distinct lifts of the same action to TM and T^*M.

In the present work, we adopt an approach different from those developed in the aforementioned papers. More precisely, by studying the natural action of the diffeomorphism group on the space of para-Kähler structures, we obtain a unified framework that generalizes the previous constructions. In particular, the results of [3, 4, 5] may be recovered as special cases of the present setting. Furthermore, the lifting of distributions from M to TM and T^*M induces, in some cases, dynamics on an associated class of Lagrangian 3-webs.

Before explaining our results more precisely and proving them, we first recall some definitions, fix the notation, and state the known results that will be needed throughout the paper.

2. Notation and Preliminaries

2.1. **General Conventions.** Throughout this paper, all objects are assumed to be smooth unless otherwise stated. Let $k \in \mathbb{N}^*$. For simplicity, we write $[k]$ for the set $[k] = \{1, 2, \dots, k\}$.

We use the Einstein summation convention: whenever an index appears once as an upper index and once as a lower index in a product, summation over its corresponding range is understood. For instance,

$$\lambda^i \xi_i = \sum_{i=1}^{n} \lambda^i \xi_i, \quad X^i \partial_{y^i} = X^i \frac{\partial}{\partial y^i} = \sum_{i=1}^{n} X^i \frac{\partial}{\partial y^i}.$$

For $k \in \mathbb{N}$, the notation I_k denotes the identity matrix of size k, while Id_S, or simply Id when no confusion can arise, denotes the identity map on a set S.

The matrix representation of a linear map f with respect to a given basis will be denoted by (f).

2.2. **Symplectic manifolds.** Symplectic manifolds have been studied since the eighteenth century. Among the fundamental examples in symplectic geometry, the cotangent bundle T^*M carries a natural symplectic structure induced by the tautological one-form. In this section, we recall this construction and state its functoriality under cotangent lifts.

Let M be an m-dimensional smooth manifold, and let ${}^*\pi : T^*M \longrightarrow M$ denote the canonical projection. The tautological one-form, also called the Liouville

one-form, is the one-form θ on T^*M defined by

$$\theta_{(x,\alpha_x)}(v) = \alpha_x\big(T_{(x,\alpha_x)}{}^*\pi(v)\big), \quad (x,\alpha_x) \in T^*M, \quad v \in T_{(x,\alpha_x)}T^*M.$$

Its exterior derivative $\omega_{\text{can}} := d\theta$ is called the canonical symplectic form on T^*M, or equivalently the Liouville two-form.

Let $(U, x^1, \ldots, x^m)$ be a local coordinate chart on the manifold M, and let $(T^*U, x^1, \ldots, x^m, \xi_1, \ldots, \xi_m)$ be the induced cotangent coordinate chart on T^*M. Then the tautological one-form and the canonical symplectic form are locally given by

$$\theta = \sum_{i=1}^{m} \xi_i \, dx^i, \quad \omega_{\text{can}} = d\theta = \sum_{i=1}^{m} d\xi_i \wedge dx^i. \tag{2.1}$$

Proposition 2.1. *Let $\psi : M_1 \longrightarrow M_2$ be a diffeomorphism. Its cotangent lift is the map*

$$\widehat{\psi} : T^*M_1 \longrightarrow T^*M_2, \quad (x,\alpha_x) \longmapsto \big(\psi(x), (\psi^{-1})^*\alpha_x\big).$$

Then $\widehat{\psi}$ is a symplectomorphism from $(T^*M_1, d\theta_1)$ to $(T^*M_2, d\theta_2)$, where $d\theta_1$ and $d\theta_2$ denote the canonical symplectic forms on T^*M_1 and T^*M_2, respectively.

The cotangent lifts $\widehat{\text{Diff}}(M)$ of diffeomorphisms of the manifold M form a subgroup of $\text{Diff}(T^*M)$. *More precisely, the set*

$$\widehat{\text{Diff}}(M) := \Big\{ \widehat{\psi} : T^*M \longrightarrow T^*M \ ; \ \psi \in \text{Diff}(M) \Big\}$$

is a subgroup of $\text{Diff}(T^*M)$.

2.3. **Para-Kähler Structures.** Throughout this work, M denotes a smooth manifold of dimension $2n$, and $\text{Met}_n = \text{Met}_n(M)$ denotes the set of pseudo-Riemannian metrics on M of neutral signature (n,n). Let $g \in \text{Met}_n$. We denote by Iso_g the isometry group of g, namely $\text{Iso}_g = \{\psi \in \text{Diff}(M) \ : \ \psi^* g = g\}$.

An almost product structure on M is a tensor field $F \in \Gamma(\text{End}(TM))$ satisfying $F^2 = \text{Id}$ and $F \neq \text{Id}$. The eigenbundles associated with the eigenvalues $+1$ and -1 are denoted by

$$F^+ := \ker(F - \text{Id}), \quad F^- := \ker(F + \text{Id}).$$

If

$$\text{rank}\, F^+ = \text{rank}\, F^- = n,$$

then F is called an almost paracomplex (par-Hermitian) structure. In some cases, we will identify F with the pair (F^-, F^+). Moreover, viewing F as a C^∞-linear map from $\mathfrak{X}(M)$ to itself, it follows that $\mathfrak{X}(M) = F^- \oplus F^+$.

We denote by $\mathcal{P}c = \mathcal{P}c(M)$ the set of almost paracomplex structures on M:

$$\mathcal{P}c(M) = \big\{ F \in \Gamma(\text{End}(TM)) \ \big| \ F^2 = \text{Id}, \quad \text{rank}\, F^+ = \text{rank}\, F^- = n \big\}.$$

If, in addition, F is compatible with the metric g in the sense that

$$g(FX, FY) = -g(X,Y), \quad \forall X, Y \in \mathfrak{X}(M),$$

then F is said to be g-compatible. Equivalently, F is an anti-isometry of g.

In local coordinates, this condition is expressed by

$$(F)^t(g)(F) = -(g).$$

We denote by $\mathcal{P}c_g = \mathcal{P}c(M,g)$ the set of g-compatible almost paracomplex structures:

$$\mathcal{P}c_g = \{F \in \mathcal{P}c(M) \mid g(FX,FY) = -g(X,Y) \quad \forall X,Y \in \mathfrak{X}(M)\}.$$

For each $F \in \mathcal{P}c_g$, the associated fundamental 2-form ω is defined by

$$\omega(X,Y) := g(FX,Y), \quad X,Y \in \mathfrak{X}(M).$$

Equivalently, since $F^2 = \mathrm{Id}$, one obtains

$$g(X,Y) = \omega(FX,Y), \quad X,Y \in \mathfrak{X}(M).$$

Thus, any two of the tensors g, F, and ω determine the third. In this sense, these objects may be regarded as mutually associated, and we write formally

$$g = (\omega, F), \quad F = (g,\omega), \quad \omega = (g,F).$$

Locally, the associated matrices satisfy

$$(\omega) = (F)^t(g), \quad (g) = (F)^t(\omega).$$

Since both g and ω are non-degenerate, each tensor $\eta \in \{g,\omega\}$ induces a vector bundle isomorphism $\eta^\flat : TM \longrightarrow T^*M, \quad (x, X_x) \longmapsto \big(x, \eta_x(X_x,\cdot)\big)$. We denote the inverse isomorphism by $\eta^\sharp : T^*M \longrightarrow TM$.

We denote by ∇^g, or simply by ∇ when no confusion can arise, the Levi-Civita connection associated with the pseudo-Riemannian metric g. It is the unique torsion-free linear connection which is compatible with g; that is, $\nabla g = 0$. In local coordinates $(x^1,\dots,x^{2n})$, its Christoffel symbols are given by

$$\Gamma_{ij}^k = \frac{1}{2} g^{k\ell}\left(\frac{\partial g_{j\ell}}{\partial x^i} + \frac{\partial g_{i\ell}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^\ell}\right),$$

where $(g^{k\ell})$ denotes the inverse matrix of $(g_{k\ell})$.

Since ω is nondegenerate, it is symplectic if and only if $d\omega = 0$.

The almost para-Hermitian structure (g,F) is called a para-Kähler structure if

$$\nabla^g F = 0.$$

Equivalently, $\nabla^g \omega = 0$. Moreover, this condition is equivalent to the simultaneous vanishing of $d\omega$ and of the Nijenhuis tensor N_F of F, where

$$N_F(X,Y) = [FX,FY] - F[FX,Y] - F[X,FY] + F^2[X,Y], \quad X,Y \in \mathfrak{X}(M).$$

In this case, the triple (M,g,F) is called a para-Kähler manifold.

We denote by

$$\mathcal{P}k = \mathcal{P}k(M) := \{(g,F) \ : \ g \in \mathrm{Met}_n(M),\ F \in \mathcal{P}c_g(M),\ \nabla F = 0\}$$

the set of para-Kähler structures on M.

A para-Kähler structure $(g,F) \in \mathcal{P}k(M)$ is said to be affine if the Levi-Civita connection ∇^g is flat. Such a structure is characterized by the following result, which follows directly from [2, Theorem 2].

Remark 2.2. A para-Kähler structure (g, F) on M is said to be affine if and only if, for every point $x \in M$, there exists a local coordinate system $(p^1, \dots, p^n, q^1, \dots, q^n)$ defined on a neighbourhood of x such that

$$F^+ = \left\langle \frac{\partial}{\partial p^1}, \dots, \frac{\partial}{\partial p^n} \right\rangle_{C^\infty(M)}, \quad F^- = \left\langle \frac{\partial}{\partial q^1}, \dots, \frac{\partial}{\partial q^n} \right\rangle_{C^\infty(M)}.$$

Moreover, in these coordinates, the symplectic form ω and the metric g are given respectively by

$$\omega = \sum_{i=1}^{n} dq^i \wedge dp^i, \quad g = -\sum_{i=1}^{n} \left(dp^i \otimes dq^i + dq^i \otimes dp^i \right),$$

and all Christoffel coefficients of the Levi-Civita connection ∇^g vanish.

Equivalently, in such a coordinate chart, the matrices of the fundamental 2-form ω, the paracomplex structure F, and the metric tensor g are respectively given by

$$(\omega) = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, \quad (F) = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}, \quad (g) = \begin{pmatrix} 0 & -I_n \\ -I_n & 0 \end{pmatrix}. \tag{2.2}$$

2.4. **Prolongations of geometric objects to the tangent bundle.** We recall in this subsection some basic facts concerning complete, vertical, and horizontal lifts to the tangent bundle. For further details on complete and vertical lifts, we refer the reader to [6, 7], and for horizontal lifts to [1]. If ζ is a geometric object on a manifold M, we denote by ζ^v, ζ^c, and ζ^h its vertical, complete, and horizontal lifts to TM, respectively.

Let $\pi : TM \to M$ be the natural projection. For $f \in C^\infty(M)$, its vertical, horizontal, and complete lifts are defined by

$$f^v = f \circ \pi = f^h, \quad f^c = df,$$

where df is viewed as a fiberwise linear function on TM.

For $X \in \mathfrak{X}(M)$, write locally

$$X = X^i \frac{\partial}{\partial x^i}, \quad \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma_{ij}^k \frac{\partial}{\partial x^k}.$$

With respect to the induced coordinates $(x^1, \dots, x^n, y^1, \dots, y^n)$ on TM, one has

$$\begin{aligned} X^v &= X^i \frac{\partial}{\partial y^i}, \\ X^h &= X^i \frac{\partial}{\partial x^i} - X^i \Gamma_{ij}^k y^j \frac{\partial}{\partial y^k}, \\ X^c &= X^i \frac{\partial}{\partial x^i} + \frac{\partial X^i}{\partial x^j} y^j \frac{\partial}{\partial y^i}. \end{aligned} \tag{2.3}$$

Moreover, for all $X, Y \in \mathfrak{X}(M)$, one has

$$\begin{aligned} &[l][X^c, Y^c] = [X, Y]^c, \quad [X^v, Y^v] = 0, \quad [X^c, Y^v] = [X, Y]^v, \\ &[X^h, Y^v] = (\nabla_X Y)^v, \quad [X^h, Y^h] = [X, Y]^h - \big(R(X, Y)y\big)^v, \\ &[X^c, Y^h] = [X, Y]^h - \left((\mathcal{L}_X \nabla)(Y, y)\right)^v, \end{aligned} \tag{2.4}$$

where R is the curvature tensor of ∇, $\mathcal{L}_X\nabla$ denotes the Lie derivative of the connection ∇ with respect to X, and $y \in T_xM$ denotes the tangent vector corresponding to a point $(x, y) \in TM$.

Remark 2.3. Let $\mathcal{D}$ be a distribution on M. For $a, b \in \{c, h, v\}$, $d \in \{c, h\}$, and , $\alpha, \beta \in \mathbb{R} \setminus \{0\}$ we define the distributions $\mathcal{D}^{ab}$ and $\mathcal{D}^{v(d)}_{\alpha,\beta}$ on TM by

$$\Gamma(\mathcal{D}^{ab}) = \left\langle X^a,\ X^b;\ X \in \Gamma(\mathcal{D}) \right\rangle_{C^\infty(TM)}. \tag{2.5}$$

$$\Gamma\left(\mathcal{D}^{v(d)}_{\alpha,\beta}\right) = \left\langle X^{v(d)} := \alpha X^v + \beta X^d;\ X \in \Gamma(\mathcal{D}) \right\rangle_{C^\infty(TM)}. \tag{2.6}$$

If $\mathcal{D}^a$ and $\mathcal{D}^b$ are transverse, then their intersection is trivial, and hence

$$\Gamma\big(\mathcal{D}^{ab}\big) = \Gamma\big(\mathcal{D}^a\big) \oplus \Gamma\big(\mathcal{D}^b\big). \tag{2.7}$$

Assume now that $\mathcal{D}$ is involutive. Then, by the bracket relations (2.4), the distributions $\mathcal{D}^c$, $\mathcal{D}^v$, and $\mathcal{D}^{cv}$ are involutive.

For fixed constants α, β satisfying $\alpha\beta \neq 0$, the distribution $\mathcal{D}^{v(c)}_{\alpha,\beta}$ is involutive when $\mathcal{D}$ is the trivial distribution.

The involutivity of distributions involving horizontal lifts depends, in general, on the connection ∇. For example, $\mathcal{D}^{hv}$ is involutive if and only if

$$\nabla_X Y \in \Gamma(\mathcal{D}) \quad \text{and} \quad R(X, Y)y \in \mathcal{D}$$

for all $X, Y \in \Gamma(\mathcal{D})$ and all $y \in TM$, where the second condition is understood fiberwise, namely $R(X, Y)y \in \mathcal{D}_{\pi(y)}$.

Similarly, the involutivity of $\mathcal{D}^{ch}$ imposes additional conditions involving both the curvature of ∇ and the tensor $\mathcal{L}_X\nabla$. In particular, if ∇ is flat along $\mathcal{D}$ and the vector fields in $\Gamma(\mathcal{D})$ are affine with respect to ∇; that is,

$$\mathcal{L}_X\nabla = 0, \quad X \in \Gamma(\mathcal{D}),$$

then the bracket relations simplify accordingly.

2.5. **Conormal prolongations of distributions to the cotangent bundle.** Let $\mathcal{D}$ be a distribution on an m-dimensional manifold M. The conormal bundle associated with $\mathcal{D}$ is defined as the annihilator of $\mathcal{D}$ in T^*M, namely

$$\begin{aligned} N^*\mathcal{D} &= \{(x, \alpha_x) \in T^*M :\ \alpha_x(v) = 0,\ \forall v \in \mathcal{D}_x,\ x \in M\} \\ &= \bigcup_{x\in M} \{x\} \times N^*_x\mathcal{D}, \end{aligned}$$

where

$$N^*_x\mathcal{D} = \{\alpha_x \in T^*_xM :\ \alpha_x(v) = 0,\ \forall v \in \mathcal{D}_x\}.$$

Equivalently, its space of smooth sections is given by

$$\Gamma(N^*\mathcal{D}) = \{\alpha \in \Omega^1(M) :\ \alpha(X) = 0,\ \forall X \in \Gamma(\mathcal{D})\}.$$

Remark 2.4. If $\mathcal{D}$ has constant rank r, then $N^*\mathcal{D}$ is a vector subbundle of T^*M of rank $\operatorname{rank}(N^*\mathcal{D}) = m - r$. We define the associated subbundle $N^{*t}\mathcal{D} := \mathcal{D} \oplus N^*\mathcal{D}$, whose space of smooth sections is $\Gamma(N^{*t}\mathcal{D}^N) = \Gamma(\mathcal{D}) \oplus \Gamma(N^*\mathcal{D})$.

3. Statements and proofs of results

3.1. **Action of diffeomorphism group on the set of paracomplex structure.** We begin this section with the following observation.

Remark 3.1. Let $\psi, \varphi : M \longrightarrow M$ be two diffeomorphisms. Recall that $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$.

For every connection $\nabla \in \mathrm{Conn}(M)$ and every diffeomorphism ψ of M, ∇^ψ defined by

$$(\nabla^\psi)_X Y = \psi_* \left(\nabla_{\psi_*^{-1} X} \psi_*^{-1} Y \right), \quad X, Y \in \mathfrak{X}(M)$$

is a connection. Moreover, for all diffeomorphisms $\psi, \varphi : M \to M$, one has $\nabla^{\psi\circ\varphi} = (\nabla^\varphi)^\psi$.

The first result concerns the push-forward of para-Kähler structures. The precise statement is as follows:

Theorem 3.2. *Let (M, g, F) be a para-Kähler manifold. Define the map*

$$\mathcal{A} : \mathrm{Diff}(M) \times \mathcal{P}k \longrightarrow \mathcal{P}k, \quad (\psi, (g, F)) \longmapsto \left((\psi^*)^{-1} g,\ F^\psi := \psi_* \circ F \circ \psi_*^{-1} \right). \tag{3.1}$$

Then $\mathcal{A}$ is a left action. Moreover, $\nabla^{(\psi^*)^{-1} g}$ is the pushforward connection ∇^g. As a consequence, the restriction of $\mathcal{A}$ to affine par-Kähler is well defined.

Proof.

Lemma 3.3. *Let (M, g, F) be a para-Kähler manifold. Let N be a manifold diffeomorphic to M. Then, for any diffeomorphism $\psi : M \longrightarrow N$, the pair $\left((\psi^*)^{-1} g,\ F^\psi \right)$ defines a para-Kähler structure on N. Moreover, one has $\nabla = \nabla^g$ if and only if $\nabla^{(\psi^*)^{-1} g} = \nabla^\psi$. Furthermore, the structure (g, F) is affine if and only if the structure $\left((\psi^*)^{-1} g,\ F^\psi := \psi_* \circ F \circ \psi_*^{-1} \right)$ is affine.*

Proof. Set $g_N := (\psi^{-1})^* g$. Since $\psi^{-1} : N \to M$ is a diffeomorphism and g is a neutral metric, the form g_N is a neutral metric as well.

We now show that F^ψ is an almost paracomplex structure.

$$\begin{aligned} F^\psi \circ F^\psi &= \psi_* \circ F \circ \psi_*^{-1} \circ \psi_* \circ F \circ \psi_*^{-1} \\ &= \psi_* \circ F^2 \circ \psi_*^{-1} \\ &= \mathrm{Id}. \end{aligned}$$

Moreover, since ψ is a diffeomorphism, there exist unique distributions (F_N^-, F_N^+) such that

$$\psi_*(F^-) = F_N^-, \quad \psi_*(F^+) = F_N^+, \quad \mathrm{rank}\, F_N^- = \mathrm{rank}\, F_N^+ = n.$$

Observe that

$$F^\psi\big|_{F_N^+} = \mathrm{Id}, \quad F^\psi\big|_{F_N^-} = -\mathrm{Id}.$$

Hence, F^ψ is a paracomplex structure.

Furthermore, for every $X, Y \in \mathfrak{X}(N)$, we compute

$$\begin{aligned}(\psi^{-1})^* g\big(F^\psi(X), F^\psi(Y)\big) &= g\big(\psi_*^{-1}F^\psi(X), \psi_*^{-1}F^\psi(Y)\big)\\ &= g\big(F(\psi_*^{-1}X), F(\psi_*^{-1}Y)\big)\\ &= -\,g\big(\psi_*^{-1}X, \psi_*^{-1}Y\big)\\ &= -(\psi^{-1})^* g(X,Y).\end{aligned}$$

In the above, we use that F is g-compatible. Therefore, F^ψ is $(\psi^{-1})^*g$-compatible.

Action on the Levi–Civita connection. Let $\nabla^g = \nabla$. Recall that the Levi–Civita connection of a pseudo-Riemannian metric on a manifold is the unique torsion-free connection that preserves the metric.

First, for all $X, Y \in \mathfrak{X}(N)$, one has

$$T_{\nabla^\psi}(X,Y) = \psi_*\Big(T_\nabla(\psi_*^{-1}X, \psi_*^{-1}Y)\Big).$$

Thus,

$$T_\nabla = 0 \Longleftrightarrow T_{\nabla^\psi} = 0. \tag{3.2}$$

Next, for $X, Y, Z \in \mathfrak{X}(N)$, we obtain

$$(\nabla^\psi_X g_N)(Y,Z) = (\nabla_{\psi_*^{-1}X} g)\big(\psi_*^{-1}Y, \psi_*^{-1}Z\big).$$

Therefore,

$$\nabla g = 0 \Longleftrightarrow \nabla^\psi g_N = 0. \tag{3.3}$$

Moreover,

$$\begin{aligned}\nabla^\psi F^\psi &= \psi_* \circ \nabla_{\psi_*^{-1}}\Big(\psi_*^{-1} \circ \big(\psi_* \circ F \circ \psi_*^{-1}\big)\Big)\\ &= \psi_* \circ \nabla_{\psi_*^{-1}X}\big(F(\psi_*^{-1})\big)\\ &= \psi_*\big(\nabla_{\psi_*^{-1}X}F\big) \circ \psi_*^{-1}.\end{aligned}$$

Since ψ is a diffeomorphism, this implies that

$$\nabla F = 0 \Longleftrightarrow \nabla^\psi F^\psi = 0. \tag{3.4}$$

By combining (3.2), (3.3), and (3.4), it follows that

$$\nabla = \nabla^g \Longleftrightarrow \nabla^\psi = \nabla^{(\psi^*)^{-1}g}. \tag{3.5}$$

Finally, the curvature of ∇^ψ satisfies

$$R_{\nabla^\psi}(X,Y)Z = \psi_*\Big(R_\nabla(\psi_*^{-1}X, \psi_*^{-1}Y)\psi_*^{-1}Z\Big).$$

Therefore,

$$R_\nabla = 0 \Longleftrightarrow R_{\nabla^\psi} = 0. \tag{3.6}$$

Consequently, the para-Kähler structure (g, F) is affine if and only if the structure $\big((\psi^*)^{-1}g,\ F^\psi := \psi_* \circ F \circ \psi_*^{-1}\big)$ is affine. □

Now, we can easily prove Theorem 3.2.

By Lemma 3.3, $\mathcal{A}$ is well defined. The action properties of $\mathcal{A}$ follow from the push-forward action of $\mathrm{Diff}(M)$ on $\mathfrak{X}(M)$. Moreover, by (3.6), the affine condition is preserved under this action. This proves Theorem 3.2. □

Remark 3.4. Let (g, F) be a para-Kähler structure on a manifold M. If $\psi \in \mathrm{Iso}_g$, then $\mathcal{A}(\psi, (g, F)) = (g, F^\psi)$ (see (3.1)). Therefore, by Theorem 3.2 or Lemma 3.3, we conclude that $(\nabla^g)^\psi = \nabla^g$. Equivalently, ψ commutes with the Levi–Civita connection ∇^g. This fact is well known in the literature.

3.2. **Prolongation of the action $\mathcal{A}$.** This section is devoted to the construction of lifted para-Kähler structures on the tangent bundle TM and the cotangent bundle T^*M, and, more generally, on the Whitney sum $E = TM \oplus T^*M$. It will then follow that the lifted action introduced in Theorem 3.2 preserves these structures. This section therefore constitutes the core of the construction announced in the title of the present work. Before formulating the main statement precisely, we recall the following definition.

Definition 3.5. Let (g, F) be a para-Kähler structure on M, and let ω denote its associated symplectic form.

On T^*M, we define

$$N^{*t}F = \big(N^{*t}F^-, N^{*t}F^+\big), \quad N^{*t}g = (d\theta, N^{*t}F), \quad \widetilde{g}_N = (\overset{*}{\pi}{}^*\omega + d\theta, N^{*t}F).$$

On TM, for each $ab \in \{cv, ch, hv\}$, we define

$$F^{ab} = \big((F^-)^{ab}, (F^+)^{ab}\big), \quad g^{ab} = \big((\omega^*)^{-1}d\theta, F^{ab}\big).$$

Theorem 3.6. *Let (g, F) be a para-Kähler structure on M, and let ω denote the associated symplectic form. Then the following assertions hold:*

1. *The pair $(N^{*t}g, N^{*t}F)$ defines an affine para-Kähler structure on T^*M.*
2. *The pair $(\widetilde{g}_N, N^{*t}F)$ defines a para-Kähler structure on T^*M. Moreover, this structure is affine if and only if (g, F) is affine.*
3. *We suppose that (g, F) is affine.*
   (a) *For each $ab \in \{cv, hv\}$, the pair (g^{ab}, F^{ab}) defines a para-Kähler structure on TM, where the horizontal lift is taken with respect to the Levi-Civita connection ∇^g.*
   (b) *The pairs (g^a, F^a), $a \in \{h, v\}$, induce a collection of 3-webs TM.*

Proof. Let (g, F) be a para-Kähler structure on M, and let $(U, p^1, \dots, p^n, q^1, \dots, q^n)$ be a coordinate chart adapted to (F^-, F^+). Thus, locally,

$$F^+ = \left\langle \frac{\partial}{\partial p^1}, \dots, \frac{\partial}{\partial p^n} \right\rangle, \quad F^- = \left\langle \frac{\partial}{\partial q^1}, \dots, \frac{\partial}{\partial q^n} \right\rangle.$$

Let $(T^*U, p^1, \dots, p^n, q^1, \dots, q^n, \xi_1, \dots, \xi_{2n})$ denote the induced cotangent bundle chart on T^*U. With respect to these coordinates, the lifted distributions admit the following local expressions:

$$\begin{cases} N^{*t}F^+ = \left\langle \frac{\partial}{\partial p^1}, \dots, \frac{\partial}{\partial p^n}, \frac{\partial}{\partial \xi_{n+1}}, \dots, \frac{\partial}{\partial \xi_{2n}} \right\rangle, \\ N^{*t}F^- = \left\langle \frac{\partial}{\partial q^1}, \dots, \frac{\partial}{\partial q^n}, \frac{\partial}{\partial \xi_1}, \dots, \frac{\partial}{\partial \xi_n} \right\rangle, \end{cases} \tag{3.7}$$

and

$$d\theta = \sum_{i=1}^{n} d\xi_i \wedge dp^i + \sum_{i=1}^{n} d\xi_{n+i} \wedge dq^i. \tag{3.8}$$

The pair $(N^{*t}g, N^{*t}F)$ is an affine para-Kähler structure on T^*M: By combining equalities (3.7) and (3.8) with Remark 2.2, the assertion follows.
The pair $(\widetilde{g}_N, N^{*t}F)$ is a para-Kähler structure on T^*M. Observe first that

$$\widetilde{\omega} = \pi^*\omega + d\theta = \sum_{1\leq i<j\leq 2n} (\omega_{ij}\circ\pi)\, dx^i\wedge dx^j + \sum_{i=1}^{n} d\xi_i\wedge dx^i + \sum_{i=1}^{n} d\xi_{n+i}\wedge dx^{n+i}, \quad (3.9)$$

where

$$(x^1,\ldots,x^{2n}) := (p^1,\ldots,p^n,q^1,\ldots,q^n), \quad (\omega_{ij}) = (\omega).$$

By combining (3.9) with (3.7), we obtain

$$(\widetilde{\omega}) = \begin{pmatrix} (\omega) & -I_{2n} \\ I_{2n} & 0 \end{pmatrix}, \quad (N^{*t}F) = \begin{pmatrix} (F) & 0 \\ 0 & -(F) \end{pmatrix}, \quad (F) = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}. \quad (3.10)$$

Since locally $(g) = (F)^t(\omega)$, it follows from (3.10) that the matrix of $N^{*t}\tilde{g}$ is given by

$$(N^{*t}\tilde{g}) = \begin{pmatrix} (\omega)_{\text{sym-}s} & -(F) \\ -(F) & 0 \end{pmatrix} = \begin{pmatrix} \big((F)^t(g)\big)_{\text{sym-}s} & -(F) \\ -(F) & 0 \end{pmatrix}, \quad (3.11)$$

where $(\omega)_{\text{sym-}s}$ denotes the symmetric $2n\times 2n$ matrix determined by the diagonal entries of (ω) together with its entries above the diagonal.

Moreover, from (3.10), one has

$$\widetilde{\omega}(N^{*t}FX, N^{*t}FY) = -\widetilde{\omega}(X,Y), \quad X,Y\in\mathfrak{X}(T^*M).$$

Consequently,

$$\widetilde{g}_N(N^{*t}FX, N^{*t}FY) = -\widetilde{g}_N(X,Y), \quad X,Y\in\mathfrak{X}(T^*M). \quad (3.12)$$

By (3.7), the endomorphism $N^{*t}F$ is completely integrable. Combining this fact with (3.12), we conclude that $(N^{*t}\tilde{g}, N^{*t}F)$ is a para-Kähler structure on T^*M.

Finally, by (3.11), the Christoffel coefficients of the Levi-Civita connection associated with $N^{*t}\tilde{g}$ are obtained from those of g by the corresponding block construction. Hence $(N^{*t}\tilde{g}, N^{*t}F)$ is affine if and only if (g,F) is affine.
We assume that (g,F) is an affine para-Kähler structure on M.
Let $(p^i,q^i) = (p^1,\ldots,p^n,q^1,\ldots,q^n)$ be an adapted coordinates to (g,F). From Remark 2.2 and the second identity in (2.1), we get

$$F^+ = \left\langle \frac{\partial}{\partial p^1},\ldots,\frac{\partial}{\partial p^n}\right\rangle, \quad F^- = \left\langle \frac{\partial}{\partial q^1},\ldots,\frac{\partial}{\partial q^n}\right\rangle, \quad (3.13)$$

and

$$\left(\omega^\flat\right)^*(d\theta) = \sum_{i=1}^{n} dy^{n+i}\wedge dp^i - \sum_{i=1}^{n} dy^i\wedge dq^i \quad (3.14)$$

where $(p^1,\ldots,p^n,q^1,\ldots,q^n,y^1,\ldots,y^{2n})$ are the tangent bundle coordinates on TM induced by the local coordinates $(p^1,\ldots,p^n,q^1,\ldots,q^n)$.

Moreover, by combining (2.3) and (3.13), we obtain

$$\begin{aligned}
(F^+)^v &= \left\langle \tfrac{\partial}{\partial y^1}, \dots, \tfrac{\partial}{\partial y^n} \right\rangle, \qquad (F^-)^v = \left\langle \tfrac{\partial}{\partial y^{n+1}}, \dots, \tfrac{\partial}{\partial y^{2n}} \right\rangle, \\
(F^+)^h &= \left\langle \tfrac{\partial}{\partial p^1}, \dots, \tfrac{\partial}{\partial p^n} \right\rangle, \qquad (F^-)^h = \left\langle \tfrac{\partial}{\partial q^1}, \dots, \tfrac{\partial}{\partial q^n} \right\rangle, \\
(F^+)^c &= \left\langle \tfrac{\partial}{\partial p^1}, \dots, \tfrac{\partial}{\partial p^n} \right\rangle, \qquad (F^-)^c = \left\langle \tfrac{\partial}{\partial q^1}, \dots, \tfrac{\partial}{\partial q^n} \right\rangle,
\end{aligned} \tag{3.15}$$

Therefore, by (2.6), for any fixed constants α, β, μ, ν, we obtain

$$\begin{aligned}
\left(F^+_{\alpha,\beta}\right)^{v(h)} &= \left\langle \alpha \tfrac{\partial}{\partial p^1} + \beta \tfrac{\partial}{\partial y^1}, \dots, \alpha \tfrac{\partial}{\partial p^n} + \beta \tfrac{\partial}{\partial y^n} \right\rangle, \\
\left(F^-_{\mu,\nu}\right)^{v(h)} &= \left\langle \mu \tfrac{\partial}{\partial q^1} + \nu \tfrac{\partial}{\partial y^{n+1}}, \dots, \mu \tfrac{\partial}{\partial q^n} + \nu \tfrac{\partial}{\partial y^{2n}} \right\rangle.
\end{aligned} \tag{3.16}$$

For each $ab \in \{cv, hv\}$, the pair (g^{ab}, F^{ab}) defines a para-Kähler structure on TM. Observe that from (2.4), the tensors F^{ab}, with $ab \in \{cv, ch, hv\}$, are involutive. Hence it remains to prove that each F^{ab} is compatible with the corresponding metric g^{ab} defined by

$$\left(g^{ab}\right) := \left(F^{ab}\right)^t \left(\left(\omega^\flat\right)^* (d\theta)\right). \tag{3.17}$$

Combining (3.17) with (3.15) and (3.14), one obtains

$$g^{ab}(F^{ab}X, F^{ab}Y) = -g^{ab}(X, Y), \quad X, Y \in \mathfrak{X}(TM).$$

Thus, the pair (g^{ab}, F^{ab}) defines a para-Kähler structure on TM.
The pairs (g^a, F^a), $a \in \{h, v\}$, induce a collection of 3-webs TM.
Let $\alpha, \beta, \mu, \nu \in \mathbb{R}$, and let

$$\left(g^{ab}_{\alpha,\beta,\mu,\nu}\right) := \left(F^{ab}_{\alpha,\beta,\mu,\nu}\right)^t \left(\left(\omega^\flat\right)^* (d\theta)\right).$$

In a similar manner as previously, by using the fact that ∇^g is a torsion-free connection, the collection $\left\{\left(g^{ab}_{\alpha,\beta,\mu,\nu}, F^{ab}_{\alpha,\beta,\mu,\nu}\right) \mid \alpha\beta\mu\nu \neq 0\right\}$ forms a family of para-Kähler structures on TM.

Now, for fixed real numbers $\alpha, \beta, \mu, \nu \in \mathbb{R} \setminus \{0\}$, let set

$$\begin{aligned}
&\Gamma\left(\mathcal{F}_1\right) = (F^-)^v \oplus (F^+)^v, \quad \Gamma\left(\mathcal{F}_2\right) = (F^-)^h \oplus (F^+)^h, \\
&\Gamma\left(\mathcal{F}_{\alpha,\beta,\mu,\nu}\right) = \left(F^-_{\alpha,\beta}\right)^{v(h)} \oplus \left(F^+_{\mu,\nu}\right)^{v(h)}.
\end{aligned}$$

From (3.15) and (3.16) the three foliations $\mathcal{F}_i$, $i \in [3]$, are pairwise transverse. Therefore, $\mathcal{W}_{\alpha,\beta,\mu,\nu} = \{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_{\alpha,\beta,\mu,\nu}\}$ defines a 3-web on TM. That is, the collection $\{\mathcal{W}_{\alpha,\beta,\mu,\nu} \mid \alpha\beta\mu\nu \neq 0\}$ is a family of 3-webs on TM. □

Corollary 3.7. *Let (g, F) be an affine para-Kähler structure on M. By Lemma 3.3, the vector bundle isomorphism*

$$\eta^\flat : TM \longrightarrow T^*M, \quad (x, X_x) \longmapsto \left(x, \eta_x(X_x, \cdot)\right), \quad \eta \in \{g, \omega\},$$

transfers the family of 3-webs on TM given in Theorem 3.6 to an associated family of 3-webs on the T^*M.

Corollary 3.8. *Let (g, F) be a para-Kähler structure on M. We denote by (g^{π}, F^{π}) and (g^{*_π}, F^{*_π}) the corresponding lifted structures on TM and T^*M, respectively.*

As a consequence of Lemma 3.3, Theorem 3.2, and Theorem 3.6, the following assertions hold.

(1) *The pairs*

$$\left((\eta^{\sharp})^* g^{\pi}, (F^{\pi})^{\eta^{\flat}}\right) \quad \text{and} \quad \left((\eta^{\flat})^* g^{*_\pi}, \left(F^{*_\pi}\right)^{\eta^{\sharp}}\right)$$

define para-Kähler structures on T^*M and TM, respectively.

Moreover, the pairs

$$\left(g^{\pi} \oplus (\eta^{\sharp})^* g^{\pi}, F^{\pi} \oplus (F^{\pi})^{\eta^{\flat}}\right)$$

and

$$\left((\eta^{\flat})^* g^{*_\pi} \oplus g^{*_\pi}, \left(F^{*_\pi}\right)^{\eta^{\sharp}} \oplus F^{*_\pi}\right)$$

define para-Kähler structures on the Whitney sum

$$E = TM \oplus T^*M.$$

If

$$(g^{*_\pi}, F^{*_\pi}) = (N^{*t} g, N^{*t} F), \quad N^{*t} g = (d\theta, N^{*t} F),$$

then the corresponding lifted structures are affine. On the other hand, if

$$(g^{*_\pi}, F^{*_\pi}) = (\widetilde{g}_N, N^{*t} F), \quad \widetilde{g}_N = ({}^{*_\pi}\omega + d\theta, N^{*t} F),$$

then the structures

$$\left((\eta^{\flat})^* g^{*_\pi}, \left(F^{*_\pi}\right)^{\eta^{\sharp}}\right)$$

and

$$\left((\eta^{\flat})^* g^{*_\pi} \oplus g^{*_\pi}, \left(F^{*_\pi}\right)^{\eta^{\sharp}} \oplus F^{*_\pi}\right)$$

are affine if and only if (g, F) is affine.

(2) *Let $\psi \in \mathrm{Diff}(M)$. We define*

$$\widetilde{L}_1(\psi) = \widehat{\psi}, \quad \widetilde{L}_2(\psi) = \eta^{\sharp} \circ \widehat{\psi} \circ \eta^{\flat}.$$

Moreover, we set

$$\widetilde{L}_1(M, g, F) = \left(T^*M, g^{*_\pi}, F^{*_\pi}\right),$$

and

$$\widetilde{L}_2(M, g, F) = \left(TM, g^{\pi}, F^{\pi}\right).$$

A lift of the action $\mathcal{A}$ defined in (3.1) of Theorem 3.2 to T^*M is given by

$$\mathcal{A}_{T^*M} : \widetilde{L}_1(\mathrm{Diff}(M)) \times \widetilde{L}_1(\mathcal{P}k) \longrightarrow \widetilde{L}_1(\mathcal{P}k),$$

$$\mathcal{A}_{T^*M}\left(\widetilde{L}_1(\psi), (g^{*_\pi}, F^{*_\pi})\right) = \left(\left(\widetilde{L}_1(\psi)^{-1}\right)^* g^{*_\pi}, \left(F^{*_\pi}\right)^{\widetilde{L}_1(\psi)}\right).$$

Similarly, the lifted action on TM is defined by

$$\mathcal{A}_{TM} : \widetilde{L}_2(\mathrm{Diff}(M)) \times \widetilde{L}_2(\mathcal{P}k) \longrightarrow \widetilde{L}_2(\mathcal{P}k),$$

$$\mathcal{A}_{TM}\left(\widetilde{L}_2(\psi),(g^\pi,F^\pi)\right)=\left(\left(\widetilde{L}_2(\psi)^{-1}\right)^* g^\pi,(F^\pi)^{\widetilde{L}_2(\psi)}\right).$$

On the Whitney sum $E=TM\oplus T^*M$, the induced action is the componentwise action $\mathcal{A}_E=\mathcal{A}_{TM}\otimes\mathcal{A}_{T^*M}$.

For each $i\in\{1,2\}$, the action $\mathcal{A}$ and the lift $\widetilde{L}_i$ commute. More precisely,

$$\mathcal{A}_i\left(\widetilde{L}_i(\psi),\widetilde{L}_i(M,g,F)\right)=\widetilde{L}_i\left(\mathcal{A}(\psi,(M,g,F))\right),$$

for every $\psi\in\mathrm{Diff}(M)$ and every $(g,F)\in\mathcal{P}k$, $\mathcal{A}_1=\mathcal{A}_{T^*M}$, $\mathcal{A}_2=\mathcal{A}_{TM}$.

Example 3.9. Let $M=\mathbb{R}^2$, and let (x,y), (x,y,u,w), and (x,y,s,t) denote the standard coordinates on M, $TM\approx\mathbb{R}^4$, and $T^*M\approx\mathbb{R}^4$, respectively. Consider

$$\omega=dy\wedge dx,\quad g=-(dx\otimes dy+dy\otimes dx),$$

together with the paracomplex structure defined by

$$F^+=\left\langle\frac{\partial}{\partial x}\right\rangle,\quad F^-=\left\langle\frac{\partial}{\partial y}\right\rangle.$$

Observe that the pair (g,F) is para-Kähler structure on $\mathbb{R}^2$.

Let $\theta=s\,dx+t\,dy$ be the canonical Liouville one-form on T^*M. The relevant lifted forms are given by

$$d\theta=ds\wedge dx+dt\wedge dy,\quad \pi^*\omega=dy\wedge dx,\quad \left(\omega^\flat\right)^*(d\theta)=dw\wedge dx-du\wedge dy. \tag{3.18}$$

Accordingly, the lifted metrics are

$$N^{*t}g=-\left(dx\otimes ds+dy\otimes dt+ds\otimes dx+dt\otimes dy\right),$$

and

$$\widetilde{g}_N=-\left(dx\otimes dy+dy\otimes dx+dx\otimes ds+dy\otimes dt+ds\otimes dx+dt\otimes dy\right).$$

The corresponding lifted paracomplex structures are defined by

$$\begin{aligned}
N^{*t}F^+&=\left\langle\tfrac{\partial}{\partial x},\tfrac{\partial}{\partial t}\right\rangle, & N^{*t}F^-&=\left\langle\tfrac{\partial}{\partial y},\tfrac{\partial}{\partial s}\right\rangle,\\
(F^+)^v&=\left\langle\tfrac{\partial}{\partial u}\right\rangle, & (F^-)^v&=\left\langle\tfrac{\partial}{\partial w}\right\rangle,\\
(F^+)^h&=\left\langle\tfrac{\partial}{\partial x}\right\rangle, & (F^-)^h&=\left\langle\tfrac{\partial}{\partial y}\right\rangle,\\
(F^+)^c&=\left\langle\tfrac{\partial}{\partial x}\right\rangle, & (F^-)^c&=\left\langle\tfrac{\partial}{\partial y}\right\rangle,\\
\left(F^+_{1,6}\right)^{v(h)}&=\left\langle\tfrac{\partial}{\partial x}+6\tfrac{\partial}{\partial u}\right\rangle, & \left(F^-_{1,1}\right)^{v(h)}&=\left\langle\tfrac{\partial}{\partial y}+\tfrac{\partial}{\partial w}\right\rangle,
\end{aligned} \tag{3.19}$$

where the horizontal lift is taken with respect to the Levi-Civita connection ∇^g.

In order to obtain a simple finite-dimensional illustration, we restrict the action $\mathcal{A}$ to the special linear group

$$\mathrm{SL}_2(\mathbb{R})=\left\{\psi_A:\mathbb{R}^2\longrightarrow\mathbb{R}^2,\quad \psi_A(x,y)=A(x,y)^t,\quad A\in M_2(\mathbb{R}),\quad \det A=1\right\}.$$

Let

$$A=\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix},\quad \alpha\delta-\beta\gamma=1,\quad \psi=\psi_A.$$

Since $\det A := \alpha\delta - \beta\gamma = 1$, every element of $\mathrm{SL}_2(\mathbb{R})$ preserves the symplectic form ω, and preserves g if and only if $\alpha\gamma = 0 = \delta\beta$.

The induced action on para-Kähler structures is given by

$$g^{\psi} = (\psi^{-1})^* g, \quad \left(F^{\psi}\right)^{+} = \left\langle \alpha\frac{\partial}{\partial x} + \gamma\frac{\partial}{\partial y} \right\rangle, \quad \left(F^{\psi}\right)^{-} = \left\langle \beta\frac{\partial}{\partial x} + \delta\frac{\partial}{\partial y} \right\rangle. \quad (3.20)$$

Observe that,

$$\psi^{-1}(x,y) = A^{-1}(x,y)^t = (\delta x - \beta y,\ -\gamma x + \alpha y).$$

Then the cotangent lift of ψ is

$$\widehat{\psi}(x,y,s,t) = (\alpha x + \beta y,\ \gamma x + \delta y,\ \delta s - \gamma t,\ -\beta s + \alpha t)\,. \quad (3.21)$$

Its differential is represented by the block matrix

$$\widehat{\psi}_* = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}.$$

Since $\widehat{\psi}$ is the cotangent lift of ψ, it preserves the canonical one-form θ, and hence it preserves the canonical symplectic form $d\theta$. Moreover, the cotangent lift acts naturally on the base and dual variables, so it also preserves the canonical neutral metric $N^{*t}g$. Therefore,

$$\widehat{\psi}^* d\theta = d\theta, \quad \widehat{\psi}^* N^{*t} g = N^{*t} g.$$

Combining (3.19), (3.18), (3.20), and (3.21), the lifted action $\mathcal{A}|_{\mathrm{SL}_2}$ may be described in the different equivalent ways appearing in Corollary 3.8. For instance,

$$\widehat{\psi}^* N^{*t} g = N^{*t} g,$$

and

$$\left(\left(N^{*t}F\right)^{\widehat{\psi}}\right)^{+} = \left\langle \widehat{\psi}_*\frac{\partial}{\partial x}, \widehat{\psi}_*\frac{\partial}{\partial t} \right\rangle, \quad \left(\left(N^{*t}F\right)^{\widehat{\psi}}\right)^{-} = \left\langle \widehat{\psi}_*\frac{\partial}{\partial y}, \widehat{\psi}_*\frac{\partial}{\partial s} \right\rangle.$$

Equivalently,

$$\left(\left(N^{*t}F\right)^{\widehat{\psi}}\right)^{+} = \left\langle \alpha\frac{\partial}{\partial x} + \gamma\frac{\partial}{\partial y}, -\gamma\frac{\partial}{\partial s} + \alpha\frac{\partial}{\partial t} \right\rangle,$$

and

$$\left(\left(N^{*t}F\right)^{\widehat{\psi}}\right)^{-} = \left\langle \beta\frac{\partial}{\partial x} + \delta\frac{\partial}{\partial y}, \delta\frac{\partial}{\partial s} - \beta\frac{\partial}{\partial t} \right\rangle.$$

These two-dimensional examples illustrate the lifting procedure in the simplest nontrivial setting and provide useful models for understanding the corresponding constructions in higher dimensions and in more general geometric contexts.

Acknowledgement. The authors thank the editor and the reviewers for the time and effort devoted to improving this manuscript..

References

1. Dombrowski, P. (1962). On the geometry of the tangent bundle. Journal für die reine und angewandte Mathematik, 210, 73–88. https://doi.org/10.1515/crll.1962.210.73
2. Hess, H. (1980). Connections on symplectic manifolds and geometric quantization. In P. L. García, A. Pérez-Rendón, & J. M. Souriau (Eds.), Differential Geometric Methods in Mathematical Physics (pp. 153–166). Springer-Verlag. https://doi.org/10.1007/BFb0092295
3. Tangue Ndawa, B. (2022). Infinite lifting of an action of symplectomorphism group on the set of bi-Lagrangian structures. Journal of Geometry and Mechanics, 14(3), 409–426. https://doi.org/10.3934/jgm.2022014
4. Tangue Ndawa, B. (2025). Dynamics on bi-Lagrangian structures and Cherry maps. arXiv preprint arXiv:2508.12350. https://arxiv.org/abs/2508.12350
5. Tangue Ndawa, B., Ngakeu, F., Saipele Nansidi, N.,& Diangitukulu NdimbaLifting S. Bi-Lagrangian and transverse Dirac structures induced on the Whitney sum. arXiv preprint arXiv:2607.11804. https://arxiv.org/abs/2607.11804
6. Yano, K., & Kobayashi, S. (1966). Prolongations of tensor fields and connections to tangent bundles I. Journal of the Mathematical Society of Japan, 18(2), 195–210. https://doi.org/10.2969/jmsj/01820195
7. Yano, K., & Kobayashi, S. (1966). Prolongations of tensor fields and connections to tangent bundles II. Journal of the Mathematical Society of Japan, 18(3), 236–246. https://doi.org/10.2969/jmsj/01830236

[1] Department of Computer Engineering, University of Ngaoundere, IUT, Cameroon.
Email address: bertuel.tangue@imsp-uac.org

[2] Department of Mathematics and Computer Science, University of Douala, Cameroon.
Email address: fngakeu@yahoo.fr

[2] Department of Mathematics and Computer Science, University of Maroua, Cameroon.
Email address: nankamla@gmail.com

[2] Department of Computer Engineering, University of Yaounde I, ENSP-Y, Cameroon.
Email address: tbouetou@gmail.com