Gulf Journal of Mathematics
Vol 24, Issue 2 (2026) 1-17

ISSN: 2309-4966
https://doi.org/10.56947/pgscms69

MILNOR FIBRATION AND MONODROMY PARITY WITHOUT ANALYTICITY

KOUAMÉ JEAN-JAURES MONDESIR N'GUESSAN[1*], PASCAL GERSHOM GOSSAN[1], KONAN FIRMIN N'GOHISSÉ[2] and GOZO YORO[1]

Abstract. We study the monodromy of vanishing cycles at an isolated critical point of a real map from an even-dimensional space to the plane which is strictly differentiable with respect to the system of compact sets. In finite dimension, this regularity is exactly continuous differentiability, without holomorphy or analyticity. A Lipschitz perturbation lemma yields a uniform implicit function theorem. Under two explicit geometric conditions, we construct a Milnor fibration whose tube is contractible and whose monodromy is trivial near the boundary. We deduce a parity criterion for the determinant of the monodromy. Explicit non-analytic examples illustrate the results.

Keywords. Monodromy, vanishing cycle, variation operator, Milnor fibration, compact differentiability, Seifert form, parity criterion, non-holomorphic maps
2020 Mathematics Subject Classification. Primary 32S55, 32S40; Secondary 57R45, 58C15

1. Introduction

Picard–Lefschetz theory describes the monodromy of the vanishing cycles of an isolated critical point of a holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ [12, 2]. It rests on two pillars: Milnor's fibration theorem, which provides a fibre F and a monodromy operator h, and the algebraic structure of the variation operator Var, which relates h to the intersection form of F.

The programme to which this work belongs aims at extending this theory to $\mathscr{C}$-differentiable functions, where $\mathscr{C}$ is the system of compact subsets [3, 13]. Let us specify at once the scope of this setting in the present paper. We work in finite dimension, on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$; bounded subsets are relatively compact there, and a map which is strictly $\mathscr{C}$-differentiable at every point of an open set is exactly a map of class C^1. The $\mathscr{C}$-differentiable programme therefore reduces here to the study of C^1 maps from $\mathbb{R}^{2n}$ to $\mathbb{R}^2$, without holomorphy. $\mathscr{C}$-differentiability only acquires a distinct meaning in infinite dimension, where compact and bounded sets differ.

Date: Received: Sep 28, 2026; Revised: Oct 26, 2026; Accepted: Nov 1, 2026.
* Corresponding author
© The Author(s) 20xx. This article is licensed under a Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License. To view a copy of the licence, visit https://creativecommons.org/licenses/by-nc-nd/4.0/.

This setting is nonetheless demanding: for a map which is only C^1, the very existence of a Milnor fibration is not guaranteed, and the classical proofs use analyticity or at least C^2 regularity. We circumvent these two obstacles. Flows of merely continuous vector fields are replaced by a tubular projection built by means of the implicit function theorem, itself derived from the Lipschitz perturbation lemma; analyticity is replaced by a construction of smooth vector fields under inequality conditions.

Let $f : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^2, 0)$, $n \geq 2$, be strictly $\mathscr{C}$-differentiable near 0, with 0 an isolated critical point, and set $g = |f|^2$. We introduce two geometric conditions. Condition (T_0) requires that, for $x \in f^{-1}(0)$ with $0 < |x| \leq \varepsilon_0$, the vector x does not belong to $\mathrm{Span}(\nabla f_1(x), \nabla f_2(x))$. Condition (R) requires that, for $0 < |x| \leq \varepsilon_0$ with $f(x) \neq 0$, the vector x is not of the form $-\lambda \nabla g(x)$ with $\lambda \geq 0$.

Our first contribution is a Milnor fibration in C^1 regularity: under (T_0) and (R), f admits a Milnor fibration on the tube, the Milnor tube is contractible, and the geometric monodromy can be chosen equal to the identity near the boundary of the fibre. Our second contribution is a parity criterion which does not require the fibre F to have the homotopy type of a bouquet of spheres:

$$\det\left(h_* \,|\, H_{n-1}(F;\mathbb{Q})\right) = (-1)^{n\, b_{n-1}(F)}$$

Our third contribution concerns stability: the hypotheses persist under small C^1 perturbations of a radially homogeneous map, with explicit constants, and in dimension $n = 3$ we obtain a map of class C^1, neither $C^{1,1}$ nor analytic, whose monodromy has determinant -1.

The closest work is that of Cisneros-Molina and Menegon [4], who establish the existence of Milnor fibrations on the tube for maps of class C^ℓ, $\ell \geq 2$, using a C^ℓ version of Ehresmann's theorem due to Wolf [18]. For a map with an isolated critical point, condition (T_0) implies their transversality property, so that the fibration statement of Theorem 5.2 is covered by [4] in C^2 regularity. With respect to [4], we obtain C^1 regularity for all the conclusions, the contractibility of the tube, the triviality of the monodromy near the boundary, and the parity criterion. These authors observe [4] that the analytic argument according to which two gradients cannot point in opposite directions relies on the curve selection lemma and does not extend to the non-analytic case; our condition (R) is precisely the explicit hypothesis replacing this argument. Ehresmann's theorem in class C^1 is itself known [18, 5]; we give a self-contained proof of it adapted to manifolds with boundary. The algebraic mechanism of Theorem 6.4, namely the expression $h = \pm G^{-1}G^t$ in terms of the Seifert form, is that of the theory of fibred knots [7, 10, 11]; our contribution is to establish it in the C^1 setting and without the bouquet assumption. The holomorphic case $\det h = (-1)^{n\mu}$ is classical. Mixed and polar weighted homogeneous maps are studied in [14, 15, 17, 16, 9], Milnor fibrations of real analytic singularities in [12] and [6], and open book structures associated with real maps in [1].

Section 2 derives a uniform implicit function theorem from strict $\mathscr{C}$-differentiability; this is the only place where the $\mathscr{C}$ structure intervenes. Section 3 fixes the hypotheses and checks them in the holomorphic and radially homogeneous cases. Section 4 proves the Ehresmann lemma in C^1 regularity,

which is used in Section 5 to establish the Milnor fibration. Section 6 deduces the parity criterion, and Section 7 establishes stability under perturbation and provides non-analytic examples.

2. From strict $\mathscr{C}$-differentiability to uniform implicit functions

In C^1 regularity, horizontal lifts cannot be integrated. We show here that strict $\mathscr{C}$-differentiability yields a uniform implicit function theorem, which replaces this integration in Sections 4 and 5. Let $\mathbb{E}$, $\mathbb{H}$ be normed vector spaces, $U \subset \mathbb{E}$ an open set and $\mathscr{C}$ the system of compact subsets of $\mathbb{E}$.

Definition 2.1. The map $F : U \to \mathbb{H}$ is *strictly $\mathscr{C}$-differentiable* at $\bar{x} \in U$ if there exists a continuous linear operator $\mathscr{C}F'(\bar{x}) : \mathbb{E} \to \mathbb{H}$ such that, for every $\epsilon > 0$ and every compact set $S \subset \mathbb{E}$, there exists $\delta > 0$ with

$$\|x - \bar{x}\| < \delta,\ |\tau| < \delta,\ v \in S,\ x + \tau v \in U \implies \big\|F(x + \tau v) - F(x) - \tau\, \mathscr{C}F'(\bar{x})v\big\| \leq \epsilon|\tau|.$$

Proposition 2.2. *If $\mathbb{E}$ is finite-dimensional, F is strictly $\mathscr{C}$-differentiable at every point of U if and only if $F \in C^1(U, \mathbb{H})$.*

Proof. [3, 13]. In finite dimension bounded sets are relatively compact, so that Definition 2.1 is strict Fréchet differentiability, and a map which is strictly Fréchet differentiable at every point of an open set is of class C^1. □

Lemma 2.3. *Let $\mathbb{E}$, $\mathbb{H}$ be Banach spaces, $A : \mathbb{E} \to \mathbb{H}$ a continuous linear isomorphism and $\Phi : \mathbb{E} \to \mathbb{H}$ a Lipschitz map with constant $\eta/\|A^{-1}\|$, where $0 < \eta < 1$. Then $A + \Phi$ is bijective and $(A + \Phi)^{-1}$ is Lipschitz with constant $\|A^{-1}\|/(1 - \eta)$.*

Proof. The equation $Ax + \Phi(x) = y$ is equivalent to $x = A^{-1}(y - \Phi(x))$; the right-hand side is a contraction with ratio η and $\mathbb{E}$ is complete.
If $Ax_i + \Phi(x_i) = y_i$, then $\|x_1 - x_2\| \leq \|A^{-1}\|\,\|y_1 - y_2\| + \eta\|x_1 - x_2\|$. □

Corollary 2.4. *Let Y be a compact metric space, $k \geq 1$, $r > 0$ and $G : Y \times \overline{B}_r \to \mathbb{R}^k$ continuous, such that $s \mapsto G(y, s)$ is strictly $\mathscr{C}$-differentiable, $(y, s) \mapsto \partial_s G(y, s)$ is continuous, $G(y, 0) = 0$ and $\partial_s G(y, 0)$ is invertible for every $y \in Y$. Then there exist $0 < r' \leq r$ and $\beta > 0$ such that, for every $y \in Y$ and every $c \in \mathbb{R}^k$ with $|c| < \beta$, the equation $G(y, s) = c$ has a unique solution $s = \sigma(y, c)$ in $B_{r'}$. The map σ is continuous; if moreover Y is a manifold of class C^1 and G is of class C^1, then σ is of class C^1.*

Proof. Set $A_y = \partial_s G(y, 0)$; by compactness, $\|A_y^{-1}\| \leq M$ for every y. By uniform continuity of $\partial_s G$ on the compact set $Y \times \overline{B}_r$, there exists $r' \leq r$ such that $\|\partial_s G(y, s) - A_y\| \leq 1/(2M)$ for $|s| \leq r'$. The map $\Phi_y = G(y, \cdot) - A_y$ is therefore Lipschitz with constant $1/(2M)$ on $\overline{B}_{r'}$; we extend it to $\mathbb{R}^k$ with the same constant. Lemma 2.3, with $\eta \leq 1/2$, shows that $s \mapsto A_y s + \Phi_y(s)$ is a bijection of $\mathbb{R}^k$ with $2M$-Lipschitz inverse; since it coincides with $G(y, \cdot)$ on $B_{r'}$ and maps 0 to 0, the equation $G(y, s) = c$ has a unique solution in $B_{r'}$ as soon as $|c| < \beta := r'/(2M)$. The continuity of σ follows from uniqueness and compactness; the C^1 regularity follows from the classical implicit function theorem, applied locally. □

3. Setting, hypotheses and first examples

Let $n \geq 2$, $U \subset \mathbb{R}^{2n}$ an open neighbourhood of 0 and $f = (f_1, f_2) : U \to \mathbb{R}^2$ with $f(0) = 0$. We identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ and $\mathbb{R}^2$ with $\mathbb{C}$ when convenient. We write $\rho(x) = |x|^2$, $g = |f|^2$, B_ε and S_ε for the ball and the sphere of radius ε, $D_\delta \subset \mathbb{R}^2$ for the disc of radius δ and $D_\delta^* = D_\delta \setminus \{0\}$.

Definition 3.1. We say that f satisfies (H1) if it is strictly $\mathscr{C}$-differentiable at every point of U, that is, of class C^1; that it satisfies (H2) if 0 is an isolated critical point, that is, $\mathrm{D}f(x)$ has rank 2 for $0 < |x| \leq \varepsilon_0$; that it satisfies ($\mathrm{T}_0$) if $x \notin \mathrm{Span}(\nabla f_1(x), \nabla f_2(x))$ for every $x \in f^{-1}(0)$ with $0 < |x| \leq \varepsilon_0$; and that it satisfies (R) if there is no $\lambda \geq 0$ such that $x = -\lambda \nabla g(x)$, for every x with $0 < |x| \leq \varepsilon_0$ and $f(x) \neq 0$.

Remark 3.2. Under (H2), the orthogonal complement of the kernel of $\mathrm{D}f(x)$ is the plane spanned by $\nabla f_1(x)$ and $\nabla f_2(x)$. Condition (T_0) at x is therefore equivalent to the surjectivity of $\mathrm{D}f(x)$ restricted to the tangent hyperplane $x^\perp$ to the sphere through x: (T_0) expresses the transversality of $f^{-1}(0) \setminus \{0\}$ to the spheres S_r. Since $\nabla g = 2\,{}^t\mathrm{D}f \cdot f$, condition (R) says that, at points where x and ∇g are collinear, the function $|f|$ increases along the ray.

Example 3.3. If $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is holomorphic with an isolated critical point, (H1) and (H2) are obvious, and (T_0) and (R) are established by Milnor by means of the curve selection lemma [12].

Definition 3.4. The map f is *radially homogeneous of degree* $d \geq 1$ if $f(tx) = t^d f(x)$ for all $t > 0$.

Lemma 3.5. *If f is radially homogeneous of degree d, of class C^1, and satisfies (H2), then it satisfies (T_0) and (R) for every $\varepsilon_0 > 0$.*

Proof. Euler's identity gives $\mathrm{D}f(x)\,x = d\,f(x)$. If $f(x) = 0$ and $x = {}^t\mathrm{D}f(x)\,a$ with $a \in \mathbb{R}^2$, then $\mathrm{D}f(x)\,{}^t\mathrm{D}f(x)\,a = \mathrm{D}f(x)\,x = 0$; the matrix $\mathrm{D}f\,{}^t\mathrm{D}f$ being invertible by (H2), we get $a = 0$ and then $x = 0$: this is (T_0).
If $f(x) \neq 0$, then $\langle x, \nabla g(x)\rangle = 2\langle \mathrm{D}f(x)x, f(x)\rangle = 2d|f(x)|^2 > 0$, whereas $x = -\lambda\nabla g$ with $\lambda \geq 0$ would give $\langle x, \nabla g\rangle = -\lambda|\nabla g|^2 \leq 0$; hence (R). □

4. An Ehresmann lemma in C^1 regularity

Ehresmann's theorem is known in regularity C^ℓ for every $\ell \geq 1$: a version due to Wolf [18] is stated in this setting in [5]. Its usual proof relies on the integration of horizontal lifts, which, for a submersion that is only C^1, are in general merely continuous. For the reader's convenience, we give a self-contained proof adapted to the manifolds with boundary that we use; it approximates the gradients by smooth vector fields, builds a tubular projection, and corrects the resulting displacement by the uniform implicit function theorem. To the best of our knowledge, this argument has not been written in this form in the literature.

Lemma 4.1. *Let $W \subset \mathbb{R}^N$ be an open set, $\psi : W \to \mathbb{R}$ a smooth function without critical points, $\varphi : W \to \mathbb{R}^k$ a map of class C^1 and $V \subset \mathbb{R}^k$ an open set. Let M be either W or $\{x \in W : \psi(x) \leq a\}$ for some real number a. For*

$x \in W$, let π_x denote the orthogonal projection onto $\nabla\psi(x)^\perp$. Assume that (i) $\varphi : M \cap \varphi^{-1}(V) \to V$ is proper, (ii) $\mathrm{D}\varphi(x)$ is surjective for every $x \in M \cap \varphi^{-1}(V)$, and (iii) if $M \neq W$, the map $\mathrm{D}\varphi(x) \circ \pi_x$ is surjective for every $x \in \varphi^{-1}(V)$ with $\psi(x) = a$. Then $\varphi : M \cap \varphi^{-1}(V) \to V$ is a locally trivial fibration, and the local trivialisations can be chosen to be C^1-diffeomorphisms of manifolds with boundary.

In the applications, $\psi(x) = |x|^2$ on an open set W not containing the origin, or $\psi(x, s) = |x|^2$ on an open subset of $(\mathbb{R}^{2n} \setminus \{0\}) \times \mathbb{R}$; in both cases ψ has no critical point, and π_x is well defined.

Proof. Fix $c_0 \in V$ and set $F_0 = M \cap \varphi^{-1}(c_0)$, which is compact by (i). By (ii), (iii) and the submersion theorem in class C^1, F_0 is a submanifold of class C^1 of dimension $N - k$; when $M \neq W$, its boundary is $\partial F_0 = F_0 \cap \{\psi = a\}$, and when $M = W$ it has no boundary.

When $M \neq W$, hypothesis (iii), the continuity of $\mathrm{D}\varphi$ and the compactness of $F_0 \cap \{\psi = a\}$ provide $\theta > 0$ such that $\mathrm{D}\varphi(x) \circ \pi_x$ is surjective for every $x \in \varphi^{-1}(c_0)$ close to F_0 with $|\psi(x) - a| \leq \theta$. When $M = W$, this condition is void, only hypothesis (ii) is used, and we set $\chi \equiv 0$ below.

Let $u_1, \dots, u_k$ be smooth vector fields uniformly approximating $\nabla\varphi_1, \dots, \nabla\varphi_k$ near the compact set F_0 (by convolution), and $\chi = \chi_0 \circ \psi$, where χ_0 is smooth and

$$\begin{cases} \chi_0(t) = 1, & t \in \left[a - \dfrac{\theta}{3},\, a + \dfrac{\theta}{3}\right], \\ \chi_0(t) = 0, & t \notin \left]a - \dfrac{2\theta}{3},\, a + \dfrac{2\theta}{3}\right[. \end{cases}$$

We set

$$\nu_i(x) = \chi(x)\, \pi_x u_i(x) + (1 - \chi(x))\, u_i(x), \qquad \text{and} \qquad A(x) = \big(\mathrm{D}\varphi_i(x)\, \nu_j(x)\big)_{1 \leq i,j \leq k}.$$

The two reference matrices

$$\mathrm{D}\varphi(x)\, {}^t\mathrm{D}\varphi(x) \qquad \text{and} \qquad (\mathrm{D}\varphi(x)\pi_x)\, {}^t(\mathrm{D}\varphi(x)\pi_x)$$

are symmetric positive definite in the regions where they are used, by (ii) and by the choice of θ respectively. By continuity and compactness, their smallest eigenvalues are bounded below by a constant $2\lambda > 0$ near F_0. Since $\mathrm{D}\varphi_i\, u_j = \langle \nabla\varphi_i, u_j \rangle$ and $\mathrm{D}\varphi_i\, \pi_x u_j = \langle \pi_x \nabla\varphi_i, \pi_x u_j \rangle$, a fine enough approximation makes $A(x)$ close, up to λ in norm, to the convex combination $\chi\, (\mathrm{D}\varphi\pi_x)\, {}^t(\mathrm{D}\varphi\pi_x) + (1 - \chi)\, \mathrm{D}\varphi\, {}^t\mathrm{D}\varphi$. We thus obtain an open neighbourhood O of F_0 on which

$$\mathrm{Sym}\, A(x) = \tfrac{1}{2}\big(A(x) + {}^t A(x)\big) \;\geq\; \lambda\, I_k;$$

in particular $A(x)$ is invertible on O. Finally, in the region where $|\psi - a| \leq \theta/3$, we have $\chi = 1$ and $\nu_i(x) = \pi_x u_i(x) \in \nabla\psi(x)^\perp$: the vector fields ν_i are tangent to the level sets of ψ near $\{\psi = a\}$.

Let Γ_t^i denote the (unique) flows of the smooth vector fields ν_i, and set

$$\Phi(y, s) = \Gamma_{s_k}^k \circ \cdots \circ \Gamma_{s_1}^1(y), \qquad y \in F_0,\ s = (s_1, \dots, s_k) \in \overline{B}_r \subset \mathbb{R}^k.$$

The map Φ is of class C^1, $\Phi(y,0)=y$ and

$$\mathrm{D}\Phi(y,0)(v,s)=v+\sum_{j=1}^{k}s_j\,\nu_j(y),\qquad v\in T_yF_0.$$

Since $T_yF_0=\ker \mathrm{D}\varphi(y)$ and $A(y)$ is invertible, the vectors $\nu_1(y),\dots,\nu_k(y)$ are linearly independent modulo T_yF_0; hence $T_yF_0\oplus \mathrm{Span}(\nu_1(y),\dots,\nu_k(y))=\mathbb{R}^N$ and $\mathrm{D}\Phi(y,0)$ is an isomorphism.

In the region where $|\psi-a|\le\theta/3$, each flow Γ_t^i preserves ψ; a trajectory starting in this region therefore stays there, and ψ is constant along compositions of flows. In particular $\Phi(\partial F_0\times B_r)\subset\{\psi=a\}$. In the region where $\psi(y)<a-\theta/3$, a displacement of length at most $r\sup_O|\nu|$ does not meet $\{\psi=a\}$ if r is small enough. Hence $\Phi(F_0\times B_r)\subset M$.

By the inverse function theorem for manifolds with boundary (applied to an extension of Φ beyond $\{\psi=a\}$, which preserves ψ near this level), Φ is a local C^1-diffeomorphism from $F_0\times B_r$ onto an open subset of M near each point $(y,0)$.

There exists $r_1\le r$ such that Φ is injective on $F_0\times B_{r_1}$. Otherwise, there would be pairs $(y_k,s_k)\neq(y_k',s_k')$ with $s_k,s_k'\to 0$ and $\Phi(y_k,s_k)=\Phi(y_k',s_k')$. By compactness of F_0, after passing to a subsequence, $y_k\to y$ and $y_k'\to y'$; since $\Phi(y,0)=y$ and $\Phi(y',0)=y'$, we get $y=y'$. Both sequences of pairs then converge to $(y,0)$, which contradicts the local injectivity of Φ near $(y,0)$. Thus Φ is a C^1-diffeomorphism from $F_0\times B_{r_1}$ onto an open neighbourhood $\mathcal{N}$ of F_0 in M.

The function $G(y,s)=\varphi(\Phi(y,s))-c_0$ is of class C^1 on $F_0\times\overline{B}_{r_1}$, with $G(y,0)=0$ and

$$\partial_s G(y,0)=\big(\mathrm{D}\varphi_i(y)\,\nu_j(y)\big)_{i,j}=A(y),$$

which is invertible for every $y\in F_0$. The uniform implicit function theorem, applied on the compact set F_0, provides $r'\le r_1$, $\beta>0$ and a map σ of class C^1 such that, for every $y\in F_0$ and every c with $|c-c_0|<\beta$, the equation $G(y,s)=c-c_0$ has a unique solution $s=\sigma(y,c)$ in $B_{r'}$. In other words, $\varphi\big(\Phi(y,\sigma(y,c))\big)=c$. This is where the Lipschitz perturbation lemma comes in: the local version of the implicit function theorem would guarantee neither a radius r' common to all $y\in F_0$ nor uniqueness within this radius.

By (i), there exists a neighbourhood $V_0\subset c_0+B_\beta$ of c_0 such that $M\cap\varphi^{-1}(V_0)\subset\Phi(F_0\times B_{r'})$. By the injectivity established above, every $x\in M\cap\varphi^{-1}(V_0)$ can be written uniquely as $x=\Phi(y,s)$ with $|s|<r'$; by the uniqueness established above, the relation $\varphi(x)=c$ then forces $s=\sigma(y,c)$. The map

$$\Theta: V_0\times F_0\to M\cap\varphi^{-1}(V_0),\qquad \Theta(c,y)=\Phi\big(y,\sigma(y,c)\big),$$

is therefore a bijection of class C^1, with inverse $x\mapsto\big(\varphi(x),\mathrm{pr}_1\Phi^{-1}(x)\big)$ of class C^1, and $\varphi\circ\Theta(c,y)=c$: it is a local trivialisation of class C^1 of φ over V_0. $\square$

5. Milnor fibration, contractible tube and monodromy trivial near the boundary

Lemma 5.1. *Assume (H1), (H2) and (R). Let $0 < \varepsilon \le \varepsilon_0$, $\delta > 0$ and $r > 0$ be such that $\overline{B}_r \subset E = f^{-1}(\overline{D}_\delta) \cap \overline{B}_\varepsilon$, and set $A = E \setminus B_r$. There exists a smooth vector field w, defined on an open neighbourhood of A, such that*

$$\begin{aligned} &\langle w(x), x\rangle < 0 \quad \text{for every } x \in A, \\ &\langle w(x), \nabla g(x)\rangle < 0 \quad \text{for every } x \in A \text{ with } g(x) = \delta^2. \end{aligned} \tag{5.1}$$

Proof. Let $\Gamma = \{x \in A : g(x) = \delta^2\}$, which is closed in A; the local construction is illustrated in Figure 1(b)–(c). For each $p \in A$, we construct an open ball U_p centred at p and a constant vector w_p. If $p \notin \Gamma$, we set $w_p = -p$; since $\langle -p, x\rangle \to -|p|^2 < 0$ as $x \to p$, there is a ball U_p not meeting Γ on which $\langle w_p, x\rangle < 0$. If $p \in \Gamma$, then $f(p) \neq 0$, hence $\nabla g(p) = 2\,{}^t\mathrm{D}f(p)f(p) \neq 0$ since ${}^t\mathrm{D}f(p)$ is injective by (H2); moreover $p \neq 0$ since $|p| \ge r$. By (R), the unit vectors $a = p/|p|$ and $b = \nabla g(p)/|\nabla g(p)|$ satisfy $a \neq -b$. We set $w_p = -(a+b) \neq 0$. Then $\langle w_p, p\rangle = -|p|(1 + \langle a, b\rangle) < 0$ and $\langle w_p, \nabla g(p)\rangle = -|\nabla g(p)|(1 + \langle a, b\rangle) < 0$, since $\langle a, b\rangle > -1$; by continuity, there is a ball U_p on which $\langle w_p, x\rangle < 0$ and $\langle w_p, \nabla g(x)\rangle < 0$.

By compactness, A is covered by finitely many balls $U_{p_1}, \dots, U_{p_N}$. Let (θ_j) be a smooth partition of unity subordinate to this cover, with $\sum_j \theta_j = 1$ on an open neighbourhood O of A, and $w = \sum_j \theta_j\, w_{p_j}$ on O. Let $x \in A$ and $J(x) = \{j : \theta_j(x) > 0\}$, which is nonempty. For $j \in J(x)$ we have $x \in U_{p_j}$, hence $\langle w_{p_j}, x\rangle < 0$, so that $\langle w(x), x\rangle < 0$. If moreover $x \in \Gamma$, no ball centred outside Γ contains x; therefore $\langle w_{p_j}, \nabla g(x)\rangle < 0$ for every $j \in J(x)$, whence $\langle w(x), \nabla g(x)\rangle < 0$. □

Theorem 5.2. *Assume (H1), (H2), (T_0) and (R). There exists $\varepsilon_1 > 0$ such that for every $0 < \varepsilon \le \varepsilon_1$ there exists $\delta_\varepsilon > 0$ with the following properties for $0 < \delta < \delta_\varepsilon$ and $E = f^{-1}(\overline{D}_\delta) \cap \overline{B}_\varepsilon$. The map $f : E^* = E \cap f^{-1}(\overline{D}^*_\delta) \to \overline{D}^*_\delta$ is a locally trivial fibration, whose fibre is a compact oriented manifold F of class C^1 and dimension $2n-2$, with boundary $\partial F \cong K = f^{-1}(0) \cap S_\varepsilon$. There exists a geometric monodromy $h : F \to F$ which is an orientation-preserving homeomorphism equal to the identity on a collar of ∂F. Finally, E is a compact topological manifold with boundary which is contractible; in particular ∂E is an integral homology sphere of dimension $2n-1$.*

The proof of Theorem 5.2 occupies the rest of this section. Since $\mathrm{D}f(x)\,{}^t\mathrm{D}f(x)$ is invertible for $0 < |x| \le \varepsilon_0$ by (H2), the matrix

$$P(x) = {}^t\mathrm{D}f(x)\big(\mathrm{D}f(x)\,{}^t\mathrm{D}f(x)\big)^{-1}\mathrm{D}f(x)$$

is the orthogonal projection onto the plane $\mathcal{V}(x) = \mathrm{Span}(\nabla f_1(x), \nabla f_2(x))$, and $x \mapsto P(x)$ is continuous. We set

$$\alpha(x) = \operatorname{dist}\Big(\frac{x}{|x|},\ \mathcal{V}(x)\Big) = \Big|\big(\mathrm{id} - P(x)\big)\frac{x}{|x|}\Big|, \qquad 0 < |x| \le \varepsilon_0, \tag{5.2}$$

which is a continuous function.

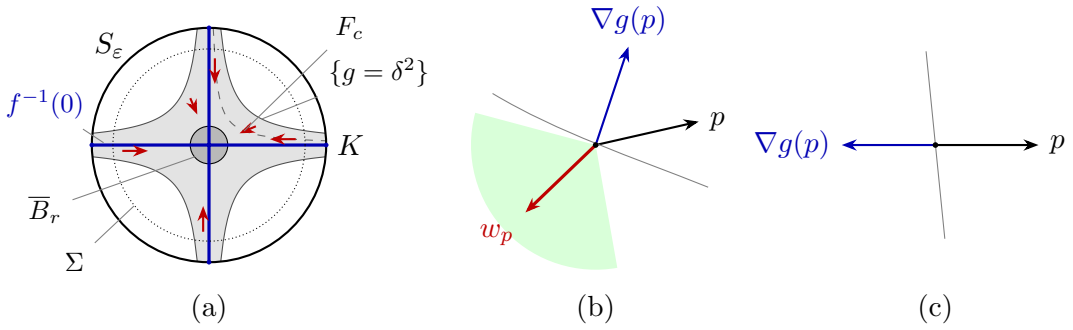


Fig. 1. (a) Schematic picture of the Milnor tube $E = f^{-1}(\overline{D}_\delta) \cap \overline{B}_\varepsilon$ (in grey) in a 2-dimensional slice. The set $f^{-1}(0)$ is a cone over K; the boundary of E consists of pieces of S_ε and of the hypersurface $\{g = \delta^2\}$. A fibre F_c is drawn dashed, the collar Σ is bounded by the dotted circle, and the arrows represent the vector field w of Lemma 5.1, whose flow retracts E onto $\overline{B}_r$. (b) At a point p with $g(p) = \delta^2$ satisfying (R), the vectors w with $\langle w, p\rangle < 0$ and $\langle w, \nabla g(p)\rangle < 0$ form a nonempty open cone (in green) containing w_p. (c) If $p = -\lambda \nabla g(p)$ with $\lambda > 0$, no such w exists: this is the configuration ruled out by (R).

Lemma 5.3. *Assume (H1), (H2) and (T$_0$), and let $\varepsilon_1 < \varepsilon_0$. For $0 < \varepsilon \le \varepsilon_1$, fix $0 < \kappa < \varepsilon/2$ with $\varepsilon + \kappa < \varepsilon_0$ and set $\Sigma' = \{\varepsilon - \kappa \le |x| \le \varepsilon + \kappa\}$. There exists $\delta_\varepsilon > 0$ such that $\alpha > 0$ on $\Sigma' \cap f^{-1}(\overline{D}_{\delta_\varepsilon})$. On this set, $\mathrm{D}f(x)$ restricted to $x^\perp$ is surjective, and (f, ρ) is a submersion.*

Proof. By (T$_0$), $\alpha > 0$ on the compact set $\Sigma' \cap f^{-1}(0)$; by continuity and compactness, $\alpha > 0$ on $\Sigma' \cap f^{-1}(\overline{D}_{\delta_\varepsilon})$ for some $\delta_\varepsilon > 0$. There, $x \notin \mathcal{V}(x) = (\ker \mathrm{D}f(x))^\perp$, hence $\mathrm{D}f(x)$ restricted to $x^\perp$ is surjective. □

In what follows, $\varepsilon \le \varepsilon_1$, κ and δ_ε are as in Lemma 5.3, we fix $0 < \delta < \delta' < \delta_\varepsilon$, and we write $\Sigma = \{\varepsilon - \kappa \le |x| \le \varepsilon\}$.

Lemma 5.4. *The tube $E = \{\rho \le \varepsilon^2,\ g \le \delta^2\}$ is a submanifold with corners of class C^1 and dimension $2n$; in particular, it is a compact topological manifold with boundary.*

Proof. We have $\nabla\rho(x) = 2x$ and $\nabla g(x) = 2\,{}^t\mathrm{D}f(x)\,f(x)$. If $g(p) = \delta^2$, then $f(p) \neq 0$ and $\nabla g(p) \neq 0$, since ${}^t\mathrm{D}f(p)$ is injective by (H2). If $\rho(p) = \varepsilon^2$, then $\nabla\rho(p) = 2p \neq 0$. If both equalities hold, then $\nabla g(p) \in \mathcal{V}(p)$ whereas $p \notin \mathcal{V}(p)$ by Lemma 5.3, so that $\nabla\rho(p)$ and $\nabla g(p)$ are linearly independent. In each case, the submersion theorem in class C^1 provides near p a chart of class C^1 in which E coincides with a half-space $\{u_1 \le 0\}$ or with the quadrant $\{u_1 \le 0,\ u_2 \le 0\} \times \mathbb{R}^{2n-2}$. Since the quadrant is homeomorphic to a half-space, the result follows. □

Lemma 5.5. *There is a homeomorphism*

$$P :\ K \times \overline{D}_\delta \times [(\varepsilon - \kappa)^2, \varepsilon^2] \ \xrightarrow{\sim}\ f^{-1}(\overline{D}_\delta) \cap \Sigma,$$

compatible with (f,ρ). In particular, every fibre F_c, $c \in \overline{D}^*_\delta$, contains the collar $F_c \cap \Sigma \cong K \times [(\varepsilon-\kappa)^2, \varepsilon^2]$.

Proof. We apply Lemma 4.1, with $M = W$, to $\varphi = (f,\rho)$ on the open set $\{(\varepsilon-\kappa)^2 < \rho < (\varepsilon+\kappa)^2\}$, over $V = D_{\delta'} \times](\varepsilon-\kappa)^2, (\varepsilon+\kappa)^2[$; hypotheses (i) and (ii) follow from Lemma 5.3. The base being contractible, the resulting locally trivial bundle is trivial [8], and its fibre is $K = f^{-1}(0) \cap S_\varepsilon$. □

Proposition 5.6. *Assume (H1), (H2), (T_0) and (R). The tube E is contractible, and ∂E is an integral homology sphere of dimension $2n-1$.*

Proof. Let L be a Lipschitz constant of f on $\overline{B}_\varepsilon$ and $0 < r < \min(\varepsilon, \delta/L)$; then $|f(x)| \le L|x| < \delta$ on $\overline{B}_r$, hence $\overline{B}_r \subset E$. Set $A = E \setminus B_r$, which is compact, and let w be the smooth vector field given by Lemma 5.1, defined on an open neighbourhood O of A. By compactness and by (5.1), there exists $c > 0$ such that $\langle w(x), x\rangle \le -c$ for $x \in A$. Let ϕ_t be the flow of w, which is unique since w is smooth.

Let $x \in A$ and $t^* = \sup\{t \ge 0 : \phi_s(x) \in E \text{ for } s \in [0,t]\}$. If $\phi_{t^*}(x) \in A$ and the trajectory left E right after t^*, then at the point $y = \phi_{t^*}(x) \in \partial E$ we would have either $\rho(y) = \varepsilon^2$ with $\frac{d}{dt}\rho(\phi_t(x))|_{t^*} = 2\langle w(y), y\rangle < 0$, or $g(y) = \delta^2$ with $\frac{d}{dt} g(\phi_t(x))|_{t^*} = \langle \nabla g(y), w(y)\rangle < 0$ by (5.1). In both cases the corresponding function is strictly decreasing at t^*, and the trajectory cannot cross the corresponding level. Hence the trajectory stays in E as long as it stays in A, and then

$$\frac{d}{dt}\rho(\phi_t(x)) = 2\langle w(\phi_t(x)), \phi_t(x)\rangle \le -2c, \qquad \text{so that} \qquad \rho(\phi_t(x)) \le \varepsilon^2 - 2ct.$$

The trajectory therefore reaches S_r at a time $\tau(x) \le (\varepsilon^2 - r^2)/(2c)$, and $\tau(x) = 0$ if and only if $|x| = r$. Since $t \mapsto \rho(\phi_t(x))$ is strictly decreasing on $[0, \tau(x)]$, $\tau(x)$ is the unique solution of $R(x,t) := \rho(\phi_t(x)) - r^2 = 0$. The map R is of class C^1 and $\partial_t R(x, \tau(x)) \le -2c < 0$, so that τ is continuous by the implicit function theorem and uniqueness.

Define $H : E \times [0,1] \to E$ by $H(x,s) = \phi_{s\tau(x)}(x)$ if $x \in A$ and $H(x,s) = x$ if $x \in \overline{B}_r$. The two definitions agree on $S_r \times [0,1]$, where $\tau = 0$, so H is continuous by the pasting lemma. We have $H(\cdot, 0) = \mathrm{id}_E$, $H(x,1) \in S_r$ for $x \in A$, and $H(x,s) = x$ for $x \in \overline{B}_r$: H is a deformation retraction of E onto $\overline{B}_r$, and E is contractible.

By Lemma 5.4, E is a compact orientable topological manifold of dimension $2n$ with nonempty boundary. Lefschetz duality gives $H_q(E, \partial E; \mathbb{Z}) \cong H^{2n-q}(E; \mathbb{Z})$, which is $\mathbb{Z}$ for $q = 2n$ and 0 otherwise. In the exact sequence

$$H_{q+1}(E, \partial E) \to H_q(\partial E) \to H_q(E) \to H_q(E, \partial E),$$

we have $H_q(E) = 0$ for $q \ge 1$. Hence $H_q(\partial E) = 0$ for $1 \le q \le 2n-2$, $H_{2n-1}(\partial E) \cong H_{2n}(E, \partial E) \cong \mathbb{Z}$, and $H_0(\partial E) \cong \mathbb{Z}$. Thus ∂E is an integral homology sphere of dimension $2n-1$. □

Proof of Theorem 5.2. We apply Lemma 4.1 to $W = B_{\varepsilon_0} \setminus \{0\}$, $\varphi = f$, $M = \{\rho \le \varepsilon^2\} \cap W$ and $V = D^*_{\delta'}$. Properness follows from the fact that $\overline{B}_\varepsilon \cap f^{-1}(Q)$ is compact and avoids 0 for every compact $Q \subset V$; hypothesis (ii) is (H2), and (iii)

is Lemma 5.3. We obtain a locally trivial fibration over $D^*_{\delta'}$, which we restrict to $\overline{D}^*_\delta$. The fibres are compact submanifolds of class C^1 and dimension $2n-2$, oriented by the orientations of $\mathbb{R}^{2n}$ and of the base, and $\partial F \cong K$ by Lemma 5.5.

We now construct the monodromy. Let $F_c^{\mathrm{in}} = F_c \cap \overline{B}_{\varepsilon-\kappa}$ and $\gamma(\theta) = \delta e^{i\theta}$, $\theta \in [0, 2\pi]$. Lemma 4.1, applied with $a = (\varepsilon-\kappa)^2$, shows that the sets F_c^{in} form a locally trivial bundle over $\overline{D}^*_\delta$. Over the interval $[0, 2\pi]$, it admits a trivialisation $\tau^{\mathrm{in}} : F^{\mathrm{in}}_{\gamma(0)} \times [0, 2\pi] \to \bigcup_\theta F^{\mathrm{in}}_{\gamma(\theta)}$, which we normalise by $\tau_0^{\mathrm{in}} = \mathrm{id}$. On the inner boundary $K_{\mathrm{in}} = \partial F^{\mathrm{in}} \cong K$, the trivialisations τ^{in} and P of Lemma 5.5 differ by a continuous path of homeomorphisms g_θ of K, with $g_0 = \mathrm{id}$. Let $\lambda : [(\varepsilon-\kappa)^2, \varepsilon^2] \to [0, 1]$ be continuous, equal to 1 at $(\varepsilon-\kappa)^2$ and vanishing near ε^2, and define on the collar

$$P'(k, t, \theta) = P\big(g_{\lambda(t)\theta}(k),\, \gamma(\theta),\, t\big).$$

Then $\tau = \tau^{\mathrm{in}} \cup P'$ is a continuous trivialisation of the bundle $\bigcup_\theta F_{\gamma(\theta)}$ over $[0, 2\pi]$, the two pieces agreeing on K_{in}. The monodromy $h = \tau_0^{-1} \circ \tau_{2\pi}$ is a homeomorphism of $F = F_{\gamma(0)}$. On the region where $\lambda = 0$, we have $\tau_{2\pi} = P(\cdot, \gamma(2\pi), \cdot) = P(\cdot, \gamma(0), \cdot) = \tau_0$, hence $h = \mathrm{id}$ on a collar of ∂F. Finally, h preserves the orientation, since the fibres are coherently oriented along γ. The remaining assertions are Lemma 5.4 and Proposition 5.6. □

Remark 5.7. The proof only uses the C^1 regularity of f and never integrates a merely continuous vector field: the flows used in Lemma 4.1 and in Proposition 5.6 are those of smooth vector fields, and the monodromy is built by gluing trivialisations. The specific contribution with respect to the analytic case lies in Proposition 5.6, where condition (R) replaces the curve selection lemma.

6. Monodromy, variation and parity criterion

We place ourselves under the conclusions of Theorem 5.2. Set $m = n - 1 \geq 1$; the fibre F is a compact oriented manifold of dimension $2m$, and $h = \mathrm{id}$ near ∂F.

Definition 6.1. For a relative class $a \in H_q(F, \partial F)$ represented by a relative cycle α, the chain $h(\alpha) - \alpha$ is an absolute cycle, since $h = \mathrm{id}$ near ∂F. We obtain a well-defined homomorphism

$$\mathrm{Var} : H_q(F, \partial F) \to H_q(F), \qquad \mathrm{Var}(a) = [h(\alpha) - \alpha].$$

If $j : H_q(F) \to H_q(F, \partial F)$ is induced by the inclusion, then $h_* = \mathrm{id} + \mathrm{Var} \circ j$.

We write $\langle \cdot, \cdot \rangle$ for intersection numbers. The pairing

$$H_q(F; \mathbb{Q}) \times H_{2m-q}(F, \partial F; \mathbb{Q}) \to \mathbb{Q}$$

is perfect, and $\langle a, b \rangle = (-1)^{q(2m-q)} \langle b, a \rangle$. We write $S(x, y) = \langle x, jy \rangle$ for the intersection form on $H_m(F; \mathbb{Q})$.

Lemma 6.2. *For $a, b \in H_m(F, \partial F)$, we have $\langle \mathrm{Var}\, a, \mathrm{Var}\, b \rangle + \langle a, \mathrm{Var}\, b \rangle + \langle \mathrm{Var}\, a, b \rangle = 0$.*

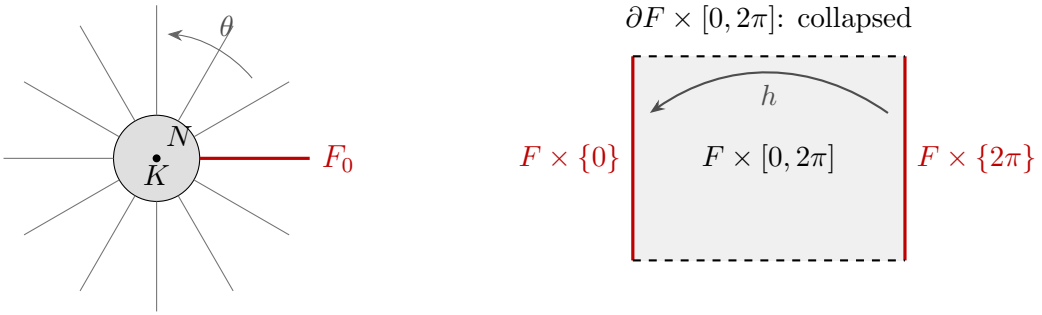
Proof. We represent a and b by relative cycles whose boundaries, shifted inside the collar where $h = \mathrm{id}$, are disjoint; their intersection number $\langle a, b \rangle_*$ is then defined and preserved by h, an orientation-preserving homeomorphism equal to

the identity on the collar. Expanding $\langle ha-a, hb-b\rangle$ we obtain $\langle ha, hb\rangle_* - \langle a, b\rangle_* = 0$. This is the argument of [2]; it is homological and does not depend on any holomorphic structure. □

Lemma 6.3. *Under the conclusions of Theorem 5.2, let $F_0 \subset \partial E$ be the fibre over $\gamma(0) = \delta$. There is an exact sequence, with coefficients in $\mathbb{Z}$ or $\mathbb{Q}$,*

$$\cdots \to H_{q+1}(\partial E) \to H_q(F, \partial F) \xrightarrow{\pm \mathrm{Var}} H_q(F) \xrightarrow{i_*} H_q(\partial E) \to H_{q-1}(F, \partial F) \to \cdots$$

where $i : F \cong F_0 \hookrightarrow \partial E$ is the inclusion. Consequently, $\mathrm{Var} : H_q(F, \partial F; \mathbb{Q}) \to H_q(F; \mathbb{Q})$ *is an isomorphism for $1 \le q \le 2m-1$.*



(a) $\partial E = T \cup N$: pages F_θ around the binding

(b) the pair (X', A'') of the proof: $H_q \cong H_{q-1}(F, \partial F)$

Fig. 2. The open book ∂E of Lemma 6.3. (a) Picture transverse to the binding K: the pages F_θ, fibres of f over $\delta e^{i\theta}$, wind around the neighbourhood $N \cong K \times \overline{D}_\delta$. (b) After cutting along the page F_0 and collapsing N, one obtains $F \times [0, 2\pi]$, whose two faces are glued by the monodromy h and whose lateral face $\partial F \times [0, 2\pi]$ is collapsed; the boundary of a chain $\alpha \times [0, 2\pi]$ is then $\pm\big(h(\alpha) - \alpha\big) = \pm\,\mathrm{Var}(\alpha)$.

Proof. See Figure 2. We have $\partial E = T \cup N$ with $T = f^{-1}(\partial D_\delta) \cap \overline{B}_\varepsilon$ and $N = f^{-1}(\overline{D}_\delta) \cap S_\varepsilon$. The trivialisation τ constructed in the proof of Theorem 5.2 identifies T with the mapping torus $F \times [0, 2\pi]/(x, 2\pi) \sim (h(x), 0)$, and the homeomorphism P of Lemma 5.5 (for $t = \varepsilon^2$) identifies N with $K \times \overline{D}_\delta$. On $T \cap N = K \times \partial D_\delta$ the two identifications agree, since $\tau = P$ where $\lambda = 0$. Thus ∂E is homeomorphic to the quotient $q : X \to \partial E$, where

$$X = \big(F \times [0, 2\pi]\big) \cup_{\partial F \times [0, 2\pi]} \big(K \times \overline{D}_\delta\big),$$

where the gluing sends $(k, \theta) \in \partial F \times [0, 2\pi]$ to $(k, \delta e^{i\theta})$, and the quotient map q identifies $(x, 2\pi)$ with $(h(x), 0)$ for $x \in F$; this is compatible at the boundary since $h = \mathrm{id}$ near ∂F.

Let $A' = q^{-1}(F_0) = F \times \{0\} \cup F \times \{2\pi\}$. The map q induces a homeomorphism from $X \setminus A'$ onto $\partial E \setminus F_0$ and a homeomorphism $X/A' \cong \partial E/F_0$; since the pairs are

compact and the subspaces are deformation retracts of neighbourhoods (thanks to the collars), we get $H_*(\partial E, F_0) \cong H_*(X, A')$.

On the other hand, let X' be the quotient of X obtained by collapsing each disc $\{k\} \times \overline{D}_\delta$ to a point: $X' = \big(F \times [0, 2\pi]\big)/\sim$, where $(k, \theta) \sim (k, \theta')$ for $k \in \partial F$. Since $K \times \overline{D}_\delta \to K$ is a homotopy equivalence and $K \times \partial D_\delta \hookrightarrow K \times \overline{D}_\delta$ is a cofibration, the gluing lemma for homotopy equivalences shows that $X \to X'$ is a homotopy equivalence; it induces a homeomorphism from A' onto its image A''. Hence $H_*(X, A') \cong H_*(X', A'')$.

In $F \times [0, 2\pi]$, the collapsed subspace $\partial F \times [0, 2\pi]$ is contained in $B = F \times \{0, 2\pi\} \cup \partial F \times [0, 2\pi]$, whose image in X' is A''. For a good pair, collapsing a closed subset of the subspace does not change relative homology; hence

$$\begin{aligned} H_q(X', A'') &\cong H_q\big(F \times [0, 2\pi],\, B\big) \\ &= H_q\big((F, \partial F) \times ([0, 2\pi], \{0, 2\pi\})\big) \cong H_{q-1}(F, \partial F), \end{aligned}$$

by the relative Künneth formula, the isomorphism sending the class of a relative cycle α of $(F, \partial F)$ to the class of $\alpha \times [0, 2\pi]$. In the exact sequence of the pair $(\partial E, F_0)$,

$$\cdots \to H_q(F_0) \xrightarrow{i_*} H_q(\partial E) \to H_q(\partial E, F_0) \cong H_{q-1}(F, \partial F) \xrightarrow{\partial} H_{q-1}(F_0) \to \cdots,$$

the boundary of the chain $\alpha \times [0, 2\pi]$ is $\pm\big(\alpha \times \{2\pi\} - \alpha \times \{0\}\big)$ plus the chain $\partial\alpha \times [0, 2\pi]$, which is degenerate in X'; its image in F_0 is $\pm\big(h(\alpha) - \alpha\big)$. Hence $\partial = \pm\,\mathrm{Var}$, which gives the stated sequence (after an index shift).

By Proposition 5.6, $\tilde{H}_j(\partial E; \mathbb{Q}) = 0$ for $j \neq 2m + 1$. For $1 \leq q \leq 2m - 1$, the terms $H_{q+1}(\partial E)$ and $H_q(\partial E)$ surrounding Var vanish, hence Var is an isomorphism. □

Theorem 6.4. *Under hypotheses (H1), (H2), (T_0) and (R), with $n \geq 2$, we have*

$$\det\big(h_* \,\big|\, H_{n-1}(F; \mathbb{Q})\big) = (-1)^{n\, b_{n-1}(F)}.$$

Proof. By Lemma 6.3 with $q = m$ (indeed $1 \leq m \leq 2m - 1$), Var is an isomorphism in degree m. For $x, y \in H_m(F; \mathbb{Q})$, set

$$\Lambda(x, y) = \langle x, \mathrm{Var}^{-1}\, y \rangle,$$

a nondegenerate bilinear form. Writing $x = \mathrm{Var}\, p$, $y = \mathrm{Var}\, r$ in Lemma 6.2, we get $S(x, y) + \langle \mathrm{Var}^{-1}\, x, y\rangle + \Lambda(x, y) = 0$, and since $\langle \mathrm{Var}^{-1}\, x, y\rangle = (-1)^m \Lambda(y, x)$,

$$S(x, y) = -\Lambda(x, y) - (-1)^m \Lambda(y, x). \tag{6.1}$$

Since $h_* y = y + \mathrm{Var}(jy)$, we have $\mathrm{Var}^{-1}(h_* y) = \mathrm{Var}^{-1}\, y + jy$, hence, by (6.1),

$$\Lambda(z, h_* y) = \Lambda(z, y) + S(z, y) = (-1)^{m+1} \Lambda(y, z).$$

Let G be the Gram matrix of Λ and H the matrix of h_* in a basis of $H_m(F; \mathbb{Q})$, of dimension $b = b_m(F)$. The previous relation reads $GH = (-1)^n\, {}^t G$, whence

$$H = (-1)^n\, G^{-1}\, {}^t G, \qquad \det H = (-1)^{nb}\, \frac{\det {}^t G}{\det G} = (-1)^{nb}. \qquad \square$$

Remark 6.5. In particular $\det h_* = 1$ if n is even and $\det h_* = (-1)^{b_{n-1}(F)}$ if n is odd. As for fibred knots [7], the relation $H = (-1)^n G^{-1}\,{}^tG$ also implies the reciprocity $\Delta(t) = \pm t^b \Delta(1/t)$ of the characteristic polynomial Δ of h_* and the equality $\Delta(1) = \pm \det S / \det G$.

Corollary 6.6. *If $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is holomorphic with an isolated critical point of Milnor number μ, then $\det h_* = (-1)^{n\mu}$.*

Proof. By Example 3.3, Theorem 6.4 applies, and $b_{n-1}(F) = \mu$ [12]. □

Proposition 6.7. *Assume $n = 1$. Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be strictly $\mathscr{C}$-differentiable, satisfying (H2) and (R), with 0 isolated in $f^{-1}(0)$. For ε small and $\delta < \min_{S_\varepsilon} |f|$, the fibre F consists of $\mu+1$ points and $\det\big(h_* \,|\, \tilde{H}_0(F;\mathbb{Q})\big) = (-1)^\mu$; this is the formula of Theorem 6.4 for $n = 1$, with $b = \operatorname{rank} \tilde{H}_0(F) = \mu$.*

Proof. Condition (T_0) is void and $K = \emptyset$; Lemmas 5.3 and 5.4, Proposition 5.6 and the construction of the fibration in the proof of Theorem 5.2 apply without change. The tube E is a compact contractible surface, hence a closed disc, and $E \cap S_\varepsilon = \emptyset$ by the choice of δ. Thus $\partial E = f^{-1}(\partial D_\delta) \cap \overline{B}_\varepsilon$ is a circle, and $f : \partial E \to \partial D_\delta$ is a covering of degree $\pm(\mu+1)$. The monodromy therefore permutes the $\mu+1$ points of F cyclically; on $\tilde{H}_0(F;\mathbb{Q})$, the quotient of the permutation representation by the trivial line, its determinant is the signature of a cycle of length $\mu+1$, namely $(-1)^\mu$. □

7. Stability and non-analytic examples

Lemma 7.1. *Let $A, B : \mathbb{R}^N \to \mathbb{R}^2$ be linear, with A of smallest singular value $\sigma_{\min}(A) \geq \sigma > 0$ and $\|B\| \leq \eta \leq \sigma/2$. Write $\mathcal{V}_A = \operatorname{Im}{}^tA$ and $\mathcal{V}_{A+B} = \operatorname{Im}{}^t(A+B)$. For every unit vector $u \in \mathbb{R}^N$,*

$$\operatorname{dist}(u, \mathcal{V}_{A+B}) \;\geq\; \operatorname{dist}(u, \mathcal{V}_A) - \frac{2\eta}{\sigma}.$$

Proof. We have $\sigma_{\min}(A+B) \geq \sigma - \eta \geq \sigma/2$. Let v be the orthogonal projection of u onto $\mathcal{V}_{A+B}$; then $|v| \leq 1$ and $v = {}^t(A+B)a$ with $|v| \geq \frac{\sigma}{2}|a|$, hence $|a| \leq 2/\sigma$. The vector ${}^tA\,a$ belongs to $\mathcal{V}_A$ and $|v - {}^tA\,a| = |{}^tB\,a| \leq \eta|a| \leq 2\eta/\sigma$. Hence $\operatorname{dist}(u, \mathcal{V}_A) \leq |u - v| + 2\eta/\sigma = \operatorname{dist}(u, \mathcal{V}_{A+B}) + 2\eta/\sigma$. □

Let $f : \mathbb{R}^{2n} \to \mathbb{R}^2$ be of class C^1, radially homogeneous of degree $d \geq 2$ and satisfying (H2), and let $\mathbb{S}$ be the unit sphere. We introduce the positive constants $\sigma_0 = \min_{u\in\mathbb{S}} \sigma_{\min}(\mathrm{D}f(u))$, positive by (H2) and compactness, and $\alpha_0 = \frac{1}{2}\min\{\alpha(u) : u \in \mathbb{S},\ f(u) = 0\}$, positive by Lemma 3.5, where α is the function (5.2). By continuity of α and compactness, there exists $\kappa > 0$ such that $\alpha(u) \geq \alpha_0$ for every $u \in \mathbb{S}$ with $|f(u)| \leq \kappa$. Finally, we set $\eta_0 = \min\big\{\sigma_0/2,\ \kappa/2,\ \sigma_0\alpha_0/4\big\}$.

Proposition 7.2. *Let e be a map of class C^1 on B_{ε_0} such that, for $|x| \leq \varepsilon_0$,*

$$\begin{cases} |e(x)| \leq \eta|x|^d, \\ \|\mathrm{D}e(x)\| \leq \eta|x|^{d-1}, \end{cases} \qquad 0 \leq \eta \leq \eta_0.$$

Then the map $\tilde f = f + e$ satisfies (H1), (H2), (T_0) and (R) on B_{ε_0}. If moreover $\eta \le \frac{4}{5}\eta_0$, then, for ε, δ suitable for both f and $\tilde f$, there exists a homeomorphism $\Psi : F \to \tilde F$ between their Milnor fibres such that $\Psi \circ h$ and $\tilde h \circ \Psi$ are isotopic. In particular $b_{n-1}(\tilde F) = b_{n-1}(F)$ and $\tilde h_* = \Psi_*\, h_*\, \Psi_*^{-1}$ in homology.

Proof. We first check the hypotheses for $\tilde f$. For $0 < |x| \le \varepsilon_0$, set $u = x/|x|$. By homogeneity, $\mathrm{D}f(x) = |x|^{d-1}\mathrm{D}f(u)$, hence

$$\begin{cases} \mathrm{D}\tilde f(x) = |x|^{d-1}\big(\mathrm{D}f(u) + B_x\big), \\ B_x = |x|^{1-d}\mathrm{D}e(x), \\ \|B_x\| \le \eta. \end{cases}$$

By (H2), $\sigma_{\min}(\mathrm{D}\tilde f(x)) \ge |x|^{d-1}(\sigma_0 - \eta) \ge |x|^{d-1}\sigma_0/2 > 0$, and condition (H1) is clear. Define the conical region $\mathcal{K} = \{0 < |x| \le \varepsilon_0,\ |f(x)| \le \kappa|x|^d\}$. If $x \in \mathcal{K}$, then $|f(u)| \le \kappa$, hence $\alpha(u) \ge \alpha_0$. The planes $\mathcal{V}_f(x)$ and $\mathcal{V}_{\tilde f}(x)$ do not change when the differentials are multiplied by $|x|^{1-d}$; Lemma 7.1 applied to $A = \mathrm{D}f(u)$, $B = B_x$, $\sigma = \sigma_0$ gives

$$\operatorname{dist}\big(u, \mathcal{V}_{\tilde f}(x)\big) \ \ge\ \alpha_0 - \frac{2\eta}{\sigma_0} \ \ge\ \frac{\alpha_0}{2} > 0,$$

since $\eta \le \sigma_0\alpha_0/4$. Hence $x \notin \mathcal{V}_{\tilde f}(x)$ for every $x \in \mathcal{K}$. If $\tilde f(x) = 0$, then $|f(x)| = |e(x)| \le \eta|x|^d \le \kappa|x|^d$, hence $x \in \mathcal{K}$ and $x \notin \mathcal{V}_{\tilde f}(x)$.

The vector $\nabla|\tilde f|^2 = 2\,{}^t\mathrm{D}\tilde f \cdot \tilde f$ belongs to $\mathcal{V}_{\tilde f}(x)$; since $x \notin \mathcal{V}_{\tilde f}(x)$, x is not of the form $-\lambda\nabla|\tilde f|^2(x)$.

Then $|f(x)| > \kappa|x|^d$, and Euler's identity $\mathrm{D}f(x)x = d\,f(x)$ together with the bounds on e give

$$\tfrac12\langle x, \nabla|\tilde f|^2(x)\rangle = \langle d\,f + \mathrm{D}e(x)x,\ f + e\rangle \ge d|f|^2 - d\eta|x|^d|f| - \eta|x|^d|f| - \eta^2|x|^{2d}.$$

Since $|x|^d < |f|/\kappa$ and $\eta \le \kappa/2$,

$$\begin{aligned} \tfrac12\langle x, \nabla|\tilde f|^2(x)\rangle &\ge |f|^2\Big(d - \frac{(d+1)\eta}{\kappa} - \frac{\eta^2}{\kappa^2}\Big) \\ &\ge |f|^2\Big(d - \frac{d+1}{2} - \frac14\Big) = \frac{2d-3}{4}\,|f|^2 > 0, \end{aligned}$$

because $d \ge 2$. But $x = -\lambda\nabla|\tilde f|^2(x)$ with $\lambda \ge 0$ would give $\langle x, \nabla|\tilde f|^2\rangle = -\lambda|\nabla|\tilde f|^2|^2 \le 0$.

We now compare the Milnor fibrations of f and $\tilde f$. For $s \in J =]-\frac18, \frac98[$, set $f_s = f + s\,e$; since $|s|\eta \le \frac98 \cdot \frac45\eta_0 < \eta_0$, the first part of the proof applies to each f_s, with the same constants. The set $\{(x,s) : x \in \Sigma',\ s \in \overline{J},\ f_s(x) = 0\}$ is compact and, denoting by α_s the function (5.2) associated with f_s, the map $(x,s) \mapsto \alpha_s(x)$ is continuous and positive on it; we can therefore choose ε and then $\delta < \delta' < \delta_\varepsilon$ as in Lemma 5.3, uniformly in $s \in \overline{J}$. We then apply Lemma 4.1 to

$$\varphi(x,s) = (f_s(x), s), \qquad W = (B_{\varepsilon_0} \setminus \{0\}) \times J,$$
$$\psi(x,s) = |x|^2, \quad a = \varepsilon^2, \qquad V = D^*_{\delta'} \times J.$$

Properness is clear; hypothesis (ii) follows from $\mathrm{D}\varphi = \begin{pmatrix} \mathrm{D}f_s & e \\ 0 & 1 \end{pmatrix}$ and the surjectivity of $\mathrm{D}f_s$; (iii) follows from the surjectivity of $\mathrm{D}f_s$ restricted to $x^\perp$ on $\{|x| = \varepsilon,\ |f_s(x)| < \delta'\}$. We obtain a locally trivial bundle over V, which we restrict to $\partial D_\delta \times [0,1]$. By homotopy invariance of bundles [8], this bundle is isomorphic, over $\partial D_\delta \times [0,1]$, to the product with $[0,1]$ of its restriction to $\partial D_\delta \times \{0\}$; restricting to $s = 1$, we obtain an isomorphism of topological bundles over the circle ∂D_δ between the Milnor fibrations of f and of $\tilde f$. Two isomorphic bundles over a circle have monodromies conjugate up to isotopy, whence Ψ. □

Example 7.3. On $\mathbb{C}^3 = \mathbb{R}^6$, we consider

$$f(z) = z_1^3\bar z_1 + z_2^3\bar z_2 + z_3\bar z_3^3,$$

which is radially homogeneous of degree 4. The term $z^3\bar z$ has real Jacobian $|3z^2\bar z|^2 - |z^3|^2 = 8|z|^6$, and the term $z\bar z^3$ has Jacobian $|\bar z^3|^2 - |3z\bar z^2|^2 = -8|z|^6$. The matrix $\mathrm{D}f$ consists of three 2×2 blocks, each invertible as soon as the corresponding variable is nonzero; hence $\mathrm{D}f(z)$ has rank 2 for $z \neq 0$, (H2) holds, and f satisfies (T_0) and (R) by Lemma 3.5. Each term is polar of degree ± 2: for $\theta \in \mathbb{R}$, $f(e^{i\theta/2}z_1, e^{i\theta/2}z_2, e^{-i\theta/2}z_3) = e^{i\theta}f(z)$, so that the geometric monodromy is $h(z) = (-z_1, -z_2, -z_3)$. By Oka's join theorem for polar weighted homogeneous mixed polynomials [14], the fibre F has the homotopy type of the join $S^0 * S^0 * S^0 \cong S^2$, each S^0 being the two-point fibre of one term, and h exchanges the two points of each factor. This exchange acts by -1 on $\tilde H_0(S^0)$, hence h_* acts on $H_2(F) \cong \tilde H_0(S^0)^{\otimes 3}$ by $(-1)^3 = -1$, in agreement with Theorem 6.4, which predicts $(-1)^{n\,b_2} = (-1)^3 = -1$.

Now let, with $x_1 = \operatorname{Re} z_1$,

$$e(x) = \eta\,\big(x_1|x_1|^{1/2}\,|x|^{5/2},\ 0\big).$$

We have $|e(x)| \le \eta|x|^{3/2}|x|^{5/2} = \eta|x|^4$. Away from the origin, the first component e_1 has gradient

$$\nabla e_1(x) = \eta\Big(\tfrac{3}{2}|x_1|^{1/2}|x|^{5/2}\,\mathbf{e}_1 + \tfrac{5}{2}\,x_1|x_1|^{1/2}|x|^{1/2}\,x\Big),$$

whence $\|\mathrm{D}e(x)\| \le \eta\big(\frac{3}{2} + \frac{5}{2}\big)|x|^3 = 4\eta|x|^3$; moreover $\mathrm{D}e(x) \to 0$ as $x \to 0$, so e is of class C^1. On the other hand, near a point $p \neq 0$ of the hyperplane $\{x_1 = 0\}$, the derivative $\partial_{x_1} e_1 \sim \frac{3}{2}\eta|x_1|^{1/2}|p|^{5/2}$ is not Lipschitz in x_1: e is not of class $C^{1,1}$. Proposition 7.2, applied with the constant 4η, applies as soon as $\eta \le \eta_0/5$: then $\tilde f = f + e$ satisfies all the hypotheses, its fibre is homeomorphic to that of f, hence homotopy equivalent to S^2, and

$$\det\big(\tilde h_* \,|\, H_2(\tilde F; \mathbb{Q})\big) = -1.$$

The map $\tilde f$ is of class C^1 but not $C^{1,1}$; it is neither holomorphic, nor a mixed polynomial, nor real analytic. The standard arguments based on analyticity, in particular on the curve selection lemma, do not apply directly to this example.

Remark 7.4. The same reasoning applies to any sum $\sum_j z_j^{a_j}\bar z_j^{b_j}$ with $a_j \neq b_j$ and $a_j + b_j = d$ independent of j: the fibre has the homotopy type of the join of $|a_j - b_j|$ points [14], and Theorem 6.4 gives $\det h_* = (-1)^{n\prod_j(|a_j - b_j| - 1)}$.

8. Conclusion

We have shown that the theory of the monodromy of vanishing cycles of an isolated critical point extends, in its essentials, to strictly $\mathscr{C}$-differentiable maps $f : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^2, 0)$, that is, maps of class C^1, under the two explicit geometric conditions (T_0) and (R). Three tools make this extension possible without analyticity: the uniform implicit function theorem derived from the Lipschitz perturbation lemma, which replaces the integration of merely continuous vector fields; the construction of smooth vector fields under inequality conditions, which replaces the curve selection lemma; and the description of the boundary of the tube as an open book, which transfers the mechanism of the Seifert form to this setting. The outcome is a parity criterion valid without the bouquet assumption, and a class of examples of class C^1, not $C^{1,1}$, which is stable under perturbation. The restriction to maps with values in $\mathbb{R}^2$ and to isolated critical points is that of the classical Picard–Lefschetz setting; maps to $\mathbb{R}^k$ and non-isolated singularities in low regularity are natural directions for further work.

References

1. Araújo dos Santos, R. N., Chen, Y., & Tibăr, M. (2013). Singular open book structures from real mappings. *Central European Journal of Mathematics, 11*(5), 817–828. https://doi.org/10.2478/s11533-013-0212-1
2. Arnold, V. I., Gusein-Zade, S. M., & Varchenko, A. N. (1988). *Singularities of differentiable maps: Vol. II. Monodromy and asymptotics of integrals* (Monographs in Mathematics, Vol. 83). Birkhäuser. https://doi.org/10.1007/978-1-4612-3940-6
3. Averbukh, V. I., & Smolyanov, O. G. (1967). The theory of differentiation in linear topological spaces. *Russian Mathematical Surveys, 22*(6), 201–258. https://doi.org/10.1070/RM1967v022n06ABEH003761
4. Cisneros-Molina, J. L., & Menegon, A. (2024). Milnor fibration theorem for differentiable maps. *Research in the Mathematical Sciences, 11*(2), Article 19. https://doi.org/10.1007/s40687-024-00431-4
5. Cisneros-Molina, J. L., Menegon, A., Seade, J., & Snoussi, J. (2024). Fibration theorems à la Milnor for analytic maps with non-isolated singularities. *São Paulo Journal of Mathematical Sciences, 18*(2), 1178–1199. https://doi.org/10.1007/s40863-023-00370-y
6. Cisneros-Molina, J. L., Seade, J., & Snoussi, J. (2010). Milnor fibrations and d-regularity for real analytic singularities. *International Journal of Mathematics, 21*(4), 419–434. https://doi.org/10.1142/S0129167X10006124
7. Durfee, A. H. (1974). Fibered knots and algebraic singularities. *Topology, 13*(1), 47–59. https://doi.org/10.1016/0040-9383(74)90037-8
8. Husemoller, D. (1994). *Fibre bundles* (3rd ed., Graduate Texts in Mathematics, Vol. 20). Springer. https://doi.org/10.1007/978-1-4757-2261-1
9. Inaba, K. (2016). On deformations of isolated singularities of polar weighted homogeneous mixed polynomials. *Osaka Journal of Mathematics, 53*(3), 813–842. https://doi.org/10.18910/58887
10. Kauffman, L. H. (1974). Branched coverings, open books and knot periodicity. *Topology, 13*(2), 143–160. https://doi.org/10.1016/0040-9383(74)90005-6
11. Kauffman, L. H., & Neumann, W. D. (1977). Products of knots, branched fibrations and sums of singularities. *Topology, 16*(4), 369–393. https://doi.org/10.1016/0040-9383(77)90042-8
12. Milnor, J. (1968). *Singular points of complex hypersurfaces* (Annals of Mathematics Studies, No. 61). Princeton University Press. https://doi.org/10.1515/9781400881819

13. Nijenhuis, A. (1974). Strong derivatives and inverse mappings. *The American Mathematical Monthly, 81*(9), 969–980. https://doi.org/10.1080/00029890.1974.11993706
14. Oka, M. (2008). Topology of polar weighted homogeneous hypersurfaces. *Kodai Mathematical Journal, 31*(2), 163–182. https://doi.org/10.2996/kmj/1214442793
15. Oka, M. (2010). Non-degenerate mixed functions. *Kodai Mathematical Journal, 33*(1), 1–62. https://doi.org/10.2996/kmj/1270559157
16. Pichon, A., & Seade, J. (2008). Fibred multilinks and singularities $f\bar{g}$. *Mathematische Annalen, 342*(3), 487–514. https://doi.org/10.1007/s00208-008-0234-3
17. Ruas, M. A. S., Seade, J., & Verjovsky, A. (2002). On real singularities with a Milnor fibration. In A. Libgober & M. Tibăr (Eds.), *Trends in singularities* (pp. 191–213). Birkhäuser. https://doi.org/10.1007/978-3-0348-8161-6_9
18. Wolf, J. A. (1964). Differentiable fibre spaces and mappings compatible with Riemannian metrics. *Michigan Mathematical Journal, 11*(1), 65–70. https://doi.org/10.1307/mmj/1028999036

[1] Department of Mathematics and Computer Science, UFR-SFA, Université Nangui Abrogoua, Abidjan, Côte d'Ivoire.
Email address: jeanjaureskouame052@gmail.com
Email address: gershom2024@yahoo.com

[2] Department of MPC, UFR-Sciences Biologiques, Université Peleforo Gbon Coulibaly, Korhogo, Côte d'Ivoire.