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COMPACT DIFFERENCE SCHEMES FOR QUASILINEAR PARABOLIC EQUATIONS

PIOTR MATUS[1], BAKHADIR UTEBAEV[2*], JOANNA KAPUSTA[3] and BERDAKH MNAZHOV[2]

Abstract. A compact difference scheme with fourth-order spatial and first-order temporal accuracy is developed for a one-dimensional quasilinear heat equation. The proposed scheme uses minimal stencils and does not require an iterative process for its implementation. Computational efficiency is achieved by parallelizing the Thomas algorithm over even and odd grid nodes. Conditions ensuring the monotonicity of the difference scheme are established. Two-sided and a priori estimates in the uniform norm are derived. Numerical experiments confirm the method's effectiveness and theoretical convergence order. The possibility of constructing compact difference scheme for a two-dimensional quasilinear equation is also demonstrated.

Keywords. Compact difference scheme, quasilinear parabolic equation, approximation error, monotonicity, two-sided estimates
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1. Introduction

One of the pressing problems of modern computational mathematics is the development of high-precision algorithms that allow mathematical modeling of multidimensional problems with a sufficiently large number of computational points to be performed on modern computers [8]. Some methods for constructing high-order approximation algorithms can be conditionally classified as follows: the Richardson extrapolation method [21]; the use of multipoint stencils and the application of differential consequences of the differential equations [19]; exact [8, 11], compact [18, 16, 25] and bicompact [6] approximations. Among the numerous methods, compact difference schemes have gained great popularity, which allow for a significant increase in the accuracy of algorithms without increasing the standard stencil of the scheme approximating a specific equation of mathematical physics. Some of the most important studies in this area are fundamental works [18, 16, 25, 6, 1, 2, 22, 4, 7], in which equations with constant and variable coefficients are considered.

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* Corresponding author
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Compact difference schemes for the non-stationary diffusion equation with a linear operator $Lu = (k(x)u')'$ were obtained in [22]. The scheme is constructed using a discrete functional a_i, which approximates the $k(x)$ at three consecutive grid nodes $x_{i-1}, x_{i-0.5}, x_i$ such that $(k(x)u')' - (au_{\bar{x}})_x = O(h^4)$, where

$$a_i = 6(1/k(x_{i-1}) + 4/k(x_{i-0.5}) + 1/k(x_i))^{-1},\ v_{\bar{x}} = (v_i - v_{i-1})/h,\ v_x = (v_{i+1} - v_i)/h.$$

A compact scheme for the two-dimensional stationary convection-diffusion equation with variable coefficients, defined on a 9-point uniform grid stencil, monotone (satisfying the maximum principle) and possessing the fourth order of approximation, was constructed in [20]. In [26], compact difference schemes for the diffusion equation with variable coefficients with divergent and non-divergent convective terms were constructed and investigated. Compact difference schemes for parabolic equations with variable coefficients were also investigated in [3, 5].

Numerous modern publications in the field of computational mathematics are devoted to the development of numerical methods for solving quasilinear equations of mathematical physics, including multidimensional problems. In [27], compact difference schemes for the one-dimensional and multidimensional quasilinear Burgers equation are constructed and studied. Three-layer compact schemes for the quasilinear equation are proposed in [14].

Until recently, it was considered impossible to generalize high-order compact schemes to arbitrary quasilinear equations with the operator $Lu = (k(x,u)u')'$ due to the need to determine the stencil functionals a_i at non-calculated points $x_{i-0.5}$ [22]. However, in [13], compact difference schemes were constructed on a three-point stencil for the stationary quasilinear diffusion equation $(k(x,u)u')' - q(x)u = -f(x)$ using the central difference derivative at even and odd nodes of a uniform grid. The use of the central derivative is associated with the desire to avoid calculating the value at an intermediate (non-integer) point of the grid. Thus, it was proposed to apply the discrete functional only to whole nodes, so that

$$a_i = 6(1/k(x_{i-1}, y_{i-1}) + 4/k(x_i, y_i) + 1/k(x_{i+1}, y_{i+1}))^{-1}.$$

Regarding the construction of compact schemes for quasilinear parabolic equations, when constructing them, the property of conservatism is lost, which is simply necessary when modeling applied problems with generalized solutions [17].

In [9], a compact 4+2 scheme (fourth order in space and second order in time) was constructed for a quasilinear parabolic fast diffusion equation. Using the results obtained by V.K. Polevikov in [20], compact difference schemes of approximation order 4+1 and 4+2 for the one-dimensional and multidimensional Fisher equations with nonlinear convection were constructed and investigated in [10].

In this paper, we propose and investigate a compact difference scheme of approximation order 4+1 on minimal stencils for a one-dimensional non-stationary quasilinear heat conduction equation, which does not require an iterative process for their implementation. The computational effect is achieved as a result of parallelization of the Thomas algorithm over even and odd nodes [12]. Monotonicity conditions are obtained, two-sided estimates of the difference solution, and a priori estimates in the uniform norm are derived. Newton interpolation polynomials are used to approximate the unknown grid functions at the boundary nodes. A

compact scheme of order 4+2 is briefly introduced, but its detailed study is omitted due to the extremely strict conditions necessary to ensure the monotonicity of the scheme (or, equivalently, the solvability condition for the difference problem). Computational results are also presented to illustrate the effectiveness of the proposed methods and their convergence with the corresponding order. The possibility of constructing compact finite difference schemes for a two-dimensional quasilinear parabolic equation is also demonstrated.

2. Compact finite difference scheme of fourth-order approximation for a one-dimensional equation

In the rectangular $\bar{Q}_T = \bar{\Omega} \times [0,T]$, $\bar{\Omega} = \Omega \cup \partial\Omega$, $\bar{\Omega} = \{x : 0 \le x \le l\}$, where $\partial\Omega$ is the boundary of the domain, we consider the initial-boundary value problem for the quasilinear parabolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(x,t,u)\frac{\partial u}{\partial x}\right) + f(x,t), \quad 0 < x < l, \quad 0 < t \le T, \tag{2.1}$$

$$u(x,0) = u_0(x), \quad u(0,t) = \mu_1(t), \quad u(l,t) = \mu_2(t). \tag{2.2}$$

We assume that the coefficient $k(x,t,u)$ and right-hand side $f(x,t)$ are sufficiently smooth in such a way that the solution $u(x,t) \in C^{6,2}(Q_T)$. Moreover, we suppose that there exist constants k_1 and k_2 such that

$$0 < k_1 \le k(x,t,u) \le k_2 \quad \text{for all} \quad x \in [0,l], \quad u \in \mathbb{R}.$$

We define uniform spatial and temporal grids with constant mesh sizes h and τ, respectively

$$\bar{\omega} = \bar{\omega}_h \times \bar{\omega}_\tau, \quad \bar{\omega}_h = \left\{x_i = ih,\ i = \overline{0,N},\ hN = l\right\}, \quad \bar{\omega}_h = \omega_h \cup \gamma_h,$$

$$\bar{\omega}_\tau = \left\{t_n = n\tau,\ n = \overline{0,N_0},\ \tau N_0 = T\right\}, \quad \bar{\omega}_\tau = \omega_\tau \cup \{t_0 = 0\}, \quad \omega = \omega_h \times \omega_\tau.$$

We denote the set of boundary nodes by $\gamma_h = \{x_i = ih, i = 0, N\}$, where N is even.

On a uniform grid of nodes, using the central difference derivative, we approximate the differential problem (2.1), (2.2) by a compact difference scheme

$$y_{t,i}^n = \left(a^n y^{n+1}{}_{\overset{\circ}{x}}\right)_{\overset{\circ}{x},i} - \frac{h^2}{3}\left(a^n (p^n y_t^n)_{\overset{\circ}{x}}\right)_{\overset{\circ}{x},i} + \varphi_i^{n+1}, \quad i = \overline{2, N-2}, \tag{2.3}$$

$$y_i^0 = u_0(x_i),\ i = \overline{0,N},\ y_0^{n+1} = \mu_1(t_{n+1}),\ y_N^{n+1} = \mu_2(t_{n+1}),\ n = \overline{0, N_0 - 1}, \tag{2.4}$$

where

$$a_i^n = 6(p_{i-1}^n + 4p_i^n + p_{i+1}^n)^{-1}, \quad p_i^n = 1/k_i^n, \quad k_i^n = k(x_i, t_n, y_i^n),$$

$$\varphi_i^{n+1} = f_i^{n+1} + \frac{h^2}{3}(a^n (p^n f^{n+1})_{\overset{\circ}{x}})_{\overset{\circ}{x},i}.$$

Here, we have used the standard notation of the theory of difference schemes [23, 24]:

$$v = v^n = v(x_i, t_n), \quad \hat{v} = v^{n+1}, \quad v_{t,i}^n = \frac{v_i^{n+1} - v_i^n}{\tau}, \quad v_{\overset{\circ}{x},i} = \frac{v_{i+1} - v_{i-1}}{2h}.$$

The idea of implementing the proposed scheme (2.3), (2.4) is described in our work [12] and is based on the splitting of the algorithm into even and odd nodes with subsequent application of the scalar Thomas algorithm [23]. In the case of even nodes $i = 2, 4, ..., N-2$, the difference scheme (2.3), (2.4) is reduced to the canonical form of three-point difference equations on one time layer [23]

$$A_i^n y_{i-2}^{n+1} - C_i^n y_i^{n+1} + B_i^n y_{i+2}^{n+1} = -F_i^n, \quad i = 2, 4, ..., N-2, \tag{2.5}$$

$$y_0^{n+1} = \mu_1(t^{n+1}), \quad y_N^{n+1} = \mu_2(t^{n+1}) \tag{2.6}$$

with coefficients

$$A_i^n = \frac{\tau}{4h^2} a_{i-1}^n \left(1 - \frac{h^2}{3\tau} p_{i-2}^n\right), \quad B_i^n = \frac{\tau}{4h^2} a_{i+1}^n \left(1 - \frac{h^2}{3\tau} p_{i+2}^n\right),$$

$$C_i^n = 1 + \frac{\tau}{4h^2}(a_{i+1}^n + a_{i-1}^n)\left(1 - \frac{h^2}{3\tau} p_i^n\right),$$

$$F_i^n = \left(1 - \frac{p_i^n}{12}\left(a_{i+1}^n + a_{i-1}^n\right)\right) y_i^n + \frac{1}{12}\left(a_{i+1}^n p_{i+2}^n y_{i+2}^n + a_{i-1}^n p_{i-2}^n y_{i-2}^n\right) + \tau \varphi_i^{n+1}.$$

Note that the values of the function y_i^{n+1} at even grid nodes can be computed independently of those at odd grid nodes.

For odd i, we obtain a similar system of equations

$$A_i^n y_{i-2}^{n+1} - C_i^n y_i^{n+1} + B_i^n y_{i+2}^{n+1} = -F_i^n, \quad i = 3, 5, ..., N-3, \tag{2.7}$$

whose boundary conditions y_1^{n+1} and y_{N-1}^{n+1} are found using Newton interpolation polynomials

$$y_1^{n+1} = y_0^{n+1} + \frac{\Delta y_0^{n+1}}{1!} q + \frac{\Delta^2 y_0^{n+1}}{2!} q(q-1) + \frac{\Delta^3 y_0^{n+1}}{3!} q(q-1)(q-2), \tag{2.8}$$

$$y_{N-1}^{n+1} = y_N^{n+1} + \frac{\Delta y_{N-2}^{n+1}}{1!} \tilde{q} + \frac{\Delta^2 y_{N-4}^{n+1}}{2!} \tilde{q}(\tilde{q}+1) + \frac{\Delta^3 y_{N-6}^{n+1}}{3!} \tilde{q}(\tilde{q}+1)(\tilde{q}+2), \tag{2.9}$$

where $q = 0.5$, $\tilde{q} = -0.5$, $\quad \Delta y_i^{n+1} = y_{i+2}^{n+1} - y_i^{n+1}$.

2.1. **Approximation error.** The truncation error associated with scheme (2.3) is expressed as follows:

$$\psi = \left(a^n u^{n+1}{}_{\overset{\circ}{x}}\right)_{\overset{\circ}{x},i} - \frac{h^2}{3}\left(a^n (p^n u_t^n)_{\overset{\circ}{x}}\right)_{\overset{\circ}{x},i} - u_{t,i}^n + \varphi_i^{n+1}, \quad i = \overline{2, N-2}.$$

Using Taylor's expansion, one can easily derive representations

$$\left(a u_{\overset{\circ}{x}}\right)_{i+1} = \frac{1}{p}\frac{\partial u}{\partial x} + hLu + \frac{2h^2}{3}\frac{1}{\sqrt{p}}\frac{\partial}{\partial x}\left(\sqrt{p} Lu\right) + \frac{h^3}{3} L\left(pLu\right) + O\left(h^4\right), \tag{2.10}$$

$$\left(a u_{\overset{\circ}{x}}\right)_{i-1} = \frac{1}{p}\frac{\partial u}{\partial x} - hLu + \frac{2h^2}{3}\frac{1}{\sqrt{p}}\frac{\partial}{\partial x}\left(\sqrt{p} Lu\right) - \frac{h^3}{3} L\left(pLu\right) + O\left(h^4\right), \tag{2.11}$$

from which the following relation can be derived:

$$\left(a u_{\overset{\circ}{x}}\right)_{\overset{\circ}{x}} = Lu + \frac{h^2}{3} L\left(pLu\right) + O\left(h^4\right).$$

Taking into account

$$L\left(pLu\right)=L\left(p\frac{\partial u}{\partial t}-pf\right),\hat{u}=u+\tau\dot{u}+O\left(\tau\right),u_t=\dot{u}+O\left(\tau\right),\ \dot{u}=\frac{\partial u}{\partial t},$$

we get

$$\begin{aligned}\psi&=\left(Lu-\dot{u}+\varphi_i^{n+1}\right)+\frac{h^2}{3}L\left(p\frac{\partial u}{\partial t}-pf\right)-\frac{h^2}{3}\left(a(pu_{t,n})_{\overset{\circ}{x}}\right)_{\overset{\circ}{x}}+O\left(h^4+\tau\right)\\&=\varphi_i^{n+1}-\left(f+\frac{h^2}{3}L(pf)\right)+O\left(h^4+\tau\right)=O\left(h^4+\tau\right).\end{aligned}$$

Let us now find the approximation error at the boundary nodes. The residual terms of the polynomials (2.8), (2.9) have the form respectively

$$R_1\left(q\right)=h^4\frac{q\left(q-1\right)\left(q-2\right)\left(q-3\right)}{4!}\frac{\partial^4u}{\partial x^4}\left(\xi,t\right),\quad\xi\in\left[x_0,x_6\right],$$

$$R_{N-1}\left(\tilde{q}\right)=h^4\frac{\tilde{q}\left(\tilde{q}+1\right)\left(\tilde{q}+2\right)\left(\tilde{q}+3\right)}{4!}\frac{\partial^4u}{\partial x^4}\left(\xi,t\right),\quad\xi\in\left[x_{N-6},x_N\right],$$

from which it is clear that they are of the fourth order in space, i.e.

$$\max_{x\in[x_0,x_6]}\left|R_1\left(q\right)\right|\le M_1h^4,\quad\max_{x\in[x_{N-6},x_N]}\left|R_{N-1}\left(\tilde{q}\right)\right|\le M_2h^4,\quad M_1,M_2=const>0.$$

Consequently, the scheme (2.3), (2.4) approximates problem (2.1), (2.2) with an accuracy of order $O(h^4+\tau)$ for the solution $u(x,t)$.

Thus, the scheme (2.3), (2.4) approximates the differential problem (2.1), (2.2) with a fourth-order precision in space and a first-order precision in time for the solution $u(x,t)$. For the grid function ψ, the following estimate is valid:

$$\left\|\psi\right\|_C\le M_3\left(h^4+\tau\right),\quad M_3=const,$$

where, as usual, $\left\|\cdot\right\|_C=\max\limits_{x\in\omega_h}\left|\cdot\right|,\ \left\|\cdot\right\|_{\bar{C}}=\max\limits_{x\in\bar{\omega}_h}\left|\cdot\right|$.

2.2. **Two-sided and a priori estimates in the norm C.** According to [23], the difference scheme (2.3), (2.4) will be monotone if the conditions of the maximum principle are met:

$$A_i^n>0,\ B_i^n>0,\ D_i^n=C_i^n-A_i^n-B_i^n>0,$$

i.e. when

$$\tau>\frac{h^2}{3k_1}.\tag{2.12}$$

In what follows, we will need the following statement.

Lemma 2.1. *Let the condition for the positivity of the coefficients (2.12) be satisfied. Then, for the solution of problem (2.5), (2.6) at even nodes $i=2,4,...,N-2$, the two-sided estimate holds*

$$\begin{aligned}&\min\left\{\min_{x\in\gamma_h}\mu\left(x,t_{n+1}\right),\min_{i=2,4,...,N-2}\left(y_i^n+\tau f_i^{n+1}\right)\right\}\le y_i^{n+1}\\&\le\max\left\{\max_{x\in\gamma_h}\mu\left(x,t_{n+1}\right),\max_{i=2,4,...,N-2}\left(y_i^n+\tau f_i^{n+1}\right)\right\}.\end{aligned}$$

The proof of the lemma is similar to the statements in [13, 15].

Using induction on n, we obtain a two-sided estimate of the form

$$m_1^{n+1} \le y_i^{n+1} \le m_2^{n+1},\ \ i = 0, 2, ..., N-2, N,$$

where

$$m_1^{n+1} = \min\left\{ \min_{n=0,1,...,N_0-1} \left\{\mu_1(t_{n+1}), \mu_2(t_{n+1})\right\}, \min_{i=0,2,...,N} u_0(x_i) \right\} + t_{n+1} \min\left\{ 0, \min_{n=0,1,...,N_0-1} \min_{i=0,2,...,N} f(x_i, t_{n+1}) \right\},$$

$$m_2^{n+1} = \max\left\{ \max_{n=0,1,...,N_0-1} \left\{\mu_1(t_{n+1}), \mu_2(t_{n+1})\right\}, \max_{i=0,2,...,N} u_0(x_i) \right\} + t_{n+1} \max\left\{ 0, \max_{n=0,1,...,N_0-1} \max_{i=0,2,...,N} f(x_i, t_{n+1}) \right\}.$$

Let us now consider problem (2.7)–(2.9) for odd i.

By virtue of the Lemma 1, if condition (2.12) is satisfied, we obtain a two-sided estimate

$$\min\left\{ \min\left\{y_1^{n+1}, y_{N-1}^{n+1}\right\}, \min_{i=3,5,...,N-3} \left(y_i^n + \tau f_i^{n+1}\right) \right\} \le y_i^{n+1} \le \max\left\{ \max\left\{y_1^{n+1}, y_{N-1}^{n+1}\right\}, \max_{i=3,5,...,N-3} \left(y_i^n + \tau f_i^{n+1}\right) \right\}. \tag{2.13}$$

Since

$$c \cdot \min_{i=0,2,4,6} y_i^{n+1} \le y_1^{n+1} \le c \cdot \max_{i=0,2,4,6} y_i^{n+1},$$

$$c \cdot \min_{i=N-6,N-4,N-2,N} y_i^{n+1} \le y_{N-1}^{n+1} \le c \cdot \max_{i=N-6,N-4,N-2,N} y_i^{n+1},\quad c = \frac{13}{8},$$

and using induction on n, from (2.13) we obtain a two-sided estimate without the assumption of sign-definiteness of the input data for odd nodes

$$m_3^{n+1} \le y_i^{n+1} \le m_4^{n+1},\ \ i = 1, 3, ..., N-3, N-1,$$

where

$$m_3^{n+1} = \min\left\{ c \cdot \min_{n=0,1,...,N_0-1} \left\{ \min_{i=0,2,4,6} y_i^{n+1}, \min_{i=N-6,N-4,N-2,N} y_i^{n+1} \right\}, \min_{i=1,3,...,N-1} u_0(x_i) \right\} + t_{n+1} \min\left\{ 0, \min_{n=0,1,...,N_0-1} \min_{i=1,3,...,N-1} f(x_i, t_{n+1}) \right\},$$

$$m_4^{n+1} = \max\left\{ c \cdot \max_{n=0,1,...,N_0-1} \left\{ \max_{i=0,2,4,6} y_i^{n+1}, \max_{i=N-6,N-4,N-2,N} y_i^{n+1} \right\}, \max_{i=1,3,...,N-1} u_0(x_i) \right\} + t_{n+1} \max\left\{ 0, \max_{n=0,1,...,N_0-1} \max_{i=1,3,...,N-1} f(x_i, t_{n+1}) \right\}.$$

Theorem 2.2. *Under the assumptions of the lemma, the following estimate holds for the grid solution of scheme (2.3), (2.4) in the discrete C norm:*

$$\|y(t_n)\|_{\bar{C}} \le c \cdot \max\left\{\max_{t\in[0,t_n]}\left\{|\mu_1(t)|, |\mu_2(t)|\right\}, \|u_0\|_C\right\} + t_n \max_{t\in[0,t_n]} \|f(t)\|_C.$$

Remark 2.3. We limited ourselves to studying the compact 4+1 scheme. Using (2.10), (2.11), it is easy to show that for equation (2.1) the compact 4+2 scheme has the form

$$y_{t,n} = \frac{1}{2}\left(a^{n+1}y^{n+1}_{\overset{\circ}{x}}\right)_{\overset{\circ}{x},i} + \frac{1}{2}\left(a^n y^n_{\overset{\circ}{x}}\right)_{\overset{\circ}{x},i} - \frac{h^2}{3}\left(a^{n+\frac12}\left(p^{n+\frac12}y_{t,n}\right)_{\overset{\circ}{x}}\right)_{\overset{\circ}{x},i} + \varphi_i^{n+\frac12},$$

$$\varphi_i^{n+\frac12} = \frac{1}{2}\left(f_i^{n+1} + \frac{h^2}{3}(a^{n+1}(p^{n+1}f^{n+1})_{\overset{\circ}{x}})_{\overset{\circ}{x},i}\right) + \frac{1}{2}\left(f_i^{n} + \frac{h^2}{3}(a^{n}(p^{n}f^{n})_{\overset{\circ}{x}})_{\overset{\circ}{x},i}\right),$$

$$i = \overline{2, N-2}.$$

3. Compact finite difference scheme of fourth-order approximation for the two-dimensional equation

In this section, a compact finite difference scheme of order 4+1 for a two-dimensional quasilinear parabolic equation is presented as a theoretical generalization of the results obtained for the one-dimensional case.

Let it be required to find a function $u(x,t)$ in the domain

$$\bar{Q}_T = \left\{(x,t) : x \in \bar{\Omega},\ 0 \le t \le T\right\}, \quad x = (x_1, x_2),$$

$$\bar{\Omega} = \{0 \le x_\alpha \le l_\alpha,\ \alpha = 1,2\}, \quad \bar{\Omega} = \Omega \cup \partial\Omega,$$

satisfying the initial-boundary value problem for the two-dimensional quasilinear parabolic equation

$$\frac{\partial u}{\partial t} = Lu + f(x,t), \quad x \in \Omega, \quad t \in (0,T], \tag{3.1}$$

$$Lu = \sum_{\alpha=1}^{2} L_\alpha u, \quad L_\alpha u = \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x,t,u)\frac{\partial u}{\partial x_\alpha}\right),$$

$$u(x,0) = u_0(x), \quad x \in \bar{\Omega}, \tag{3.2}$$

$$u(x,t) = \mu(x,t), \quad (x,t) \in \partial\Omega \times (0,T]. \tag{3.3}$$

In the domain $\bar{Q}_T$ we consider a uniform grid

$$\bar{\omega} = \bar{\omega}_h \times \bar{\omega}_\tau, \quad \bar{\omega}_h = \omega_h \cup \gamma_h,$$

where the set of interior spatial grid nodes is defined as follows:

$$\bar{\omega}_h = \left\{x = \left(x_1^{i_1}, x_2^{i_2}\right),\ x_\alpha^{i_\alpha} = i_\alpha h_\alpha,\ i_\alpha = \overline{1, N_\alpha - 1},\ h_\alpha N_\alpha = l_\alpha,\ \alpha = 1,2\right\},$$

γ_h is the set of its boundary nodes, N_1, N_2 are even integers.

On a uniform grid $\bar{\omega}$ we approximate the differential problem (3.1)–(3.3) by a compact difference scheme of the fourth order of approximation

$$y_t = \Lambda\hat{y} - \frac{1}{3}\sum_{\alpha=1}^{2} h_\alpha^2 \Lambda_\alpha\left(p_\alpha y_t\right) + \frac{1}{3}\sum_{\beta\ne\alpha}^{1\div2} h_\alpha^2 \Lambda_\alpha\left(p_\alpha \Lambda_\beta y\right) + \varphi,$$

$$y(x,t)=\mu(x,t),\quad x\in\gamma_h,\quad t\in\omega_\tau,$$
$$y(x,0)=u_0(x),\quad x\in\bar{\omega}_h,$$
$$\Lambda y=\sum_{\alpha=1}^{2}\Lambda_\alpha y,\ \Lambda_\alpha y=\left(a_\alpha y_{\overset{\circ}{x_\alpha}}\right)_{\overset{\circ}{x_\alpha}},\quad y_{\overset{\circ}{x_\alpha}}=\frac{y_{i_\alpha+1}-y_{i_\alpha-1}}{2h_\alpha},\quad i_\alpha=\overline{2,N_\alpha-2},$$
$$a_1=6(p_{1i_1-1i_2}+4p_{1i_1i_2}+p_{1i_1+1i_2})^{-1},\quad a_2=6(p_{2i_1i_2-1}+4p_{2i_1i_2}+p_{2i_1i_2+1})^{-1},$$
$$p_{\alpha i_1i_2}=(k_{\alpha i_1i_2})^{-1},\quad \varphi=\left(f+\frac{1}{3}\sum_{\alpha=1}^{2}h_\alpha^2\Lambda_\alpha\left(p_\alpha f\right)\right).$$

The implementation of a similar difference scheme was discussed in our previous study [13]. A detailed investigation of the present scheme will be presented in a separate paper.

4. Numerical results

In this section, numerical simulations are performed to verify the theoretical analysis of the proposed methods. Consider the function

$$u(x,t)=e^{x+t}(t+1),\tag{4.1}$$

which is the exact solution of the problem (2.1), (2.2). The input data for $k(u)=u^2$ are determined from the exact solution (4.1).

The convergence rates with respect to the temporal and spatial variables in the C and L_2 norms are estimated using Runge's rule. To evaluate the numerical results, we introduce the following error and convergence rate functions:

$$\|e(h,\tau)\|=\|y(h,\tau)-u(h,\tau)\|,$$
$$order^h=\log_2\frac{\|e(h,\tau)\|}{\|e(h/2,\tau/2^4)\|},\quad order^\tau=\log_2\frac{\|e(h,\tau)\|}{\|e(h,\tau/2)\|}.$$

Table 1. The C and L_2 norm errors of the numerical solution computed by the proposed scheme (2.3), (2.4) with $l=0.5$ at $t=0.5$

$h=0.05$	$\tau=0.05$	$\|e\|_C$	$order_C^h$	$\|e\|_{L_2}$	$order_{L_2}^h$
h	τ	0.0547	–	0.0217	–
$h/2$	$\tau/2^4$	0.0031	4.1412	0.0011	4.3021
$h/4$	$\tau/4^4$	$1.91e-04$	4.0206	$7.28e-05$	3.9174
$h/8$	$\tau/8^4$	$1.18e-05$	4.0167	$4.63e-06$	3.9748

Tables 1 and 2 present the errors in the C and L_2 norms, along with the observed convergence rates. The computational results confirm that the compact scheme (2.3), (2.4) exhibits fourth-order accuracy in space and first-order accuracy in time.

TABLE 2. The C and L_2 norm errors of the numerical solution computed by the proposed scheme (2.3), (2.4) with $h = 0.05, l = 0.5$ at $t = 0.5$

$\tau = 0.05$	$\|e\|_C$	$order_C^\tau$	$\|e\|_{L_2}$	$order_{L_2}^\tau$
τ	0.0547	–	0.0217	–
$\tau/2$	0.0259	1.0785	0.0103	1.0750
$\tau/4$	0.0126	1.0395	0.0050	1.0426
$\tau/8$	0.0062	1.0230	0.0024	1.0588

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[1] National Academy of Sciences of Belarus, Institute of Mathematics, Minsk, Belarus.
Email address: piotr.p.matus@gmail.com

[2] Karakalpak State University named after Berdakh, Department of Applied Mathematics and Informatics, Nukus, Uzbekistan.
Email address: bakhadir1992@gmail.com
Email address: mberdax@mail.ru

[3] The John Paul II Catholic University of Lublin, Institute of Mathematics, Informatics and Landscape Architecture, Lublin, Poland.
Email address: jkapusta@kul.pl