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# DEFECT-FREE LIMITS IN UNDAMPED $p$-LAPLACIAN WAVE EQUATIONS

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Abstract. This paper studies conditional solvability and regularity of an undamped quasilinear $p$-Laplacian wave equation. We formulate an energy framework that distinguishes weak convergence from identification of the nonlinear flux. A defect-free Minty criterion yields strong gradient convergence and supports a conditional local existence result. Conditional uniqueness is obtained for sufficiently regular solutions satisfying a relative-energy remainder estimate. Under an energy bound and a supercritical Sobolev exponent, spatial Hölder estimates and time continuity in weaker Hölder spaces follow. The defect-free and stability hypotheses are not verified for arbitrary initial data; hence general well-posedness is not claimed.

*Keywords.* quasilinear wave equation, p-Laplacian, relative energy, defect-free convergence, Hölder regularity.
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## 1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $T > 0$. We consider the initial–boundary value problem

$$\begin{cases} u_{tt} - \Delta_p u = f(x,u) + \operatorname{div} g(x,t), & (x,t) \in \Omega \times (0,T), \\ u = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(0) = u_0, \qquad u_t(0) = u_1, & x \in \Omega, \end{cases} \tag{1.1}$$

where

$$\Delta_p u := \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big), \qquad p > n \geq 2.$$

The principal part of (1.1) is quasilinear and degenerate when $p > 2$, since the ellipticity of the mapping $A(\xi) = |\xi|^{p-2}\xi$ degenerates at $\xi = 0$. Consequently, the analysis of (1.1) differs substantially from that of the classical semilinear wave equation. In particular, the construction of weak solutions requires careful treatment of the nonlinear flux, while uniqueness cannot in general be obtained by

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the standard quadratic energy argument. The basic functional-analytic framework for monotone operators and nonlinear evolution equations may be found, for instance, in [16, 14].

An early analysis directly devoted to a wave equation governed by the $p$-Laplacian was given by Dreher [4], who studied the Cauchy problem and obtained local existence results together with estimates for the lifespan of solutions. This work demonstrated that the nonlinear elastic operator $\Delta_p$ produces analytical features that are absent from linear wave equations. Nevertheless, the solution concept, the Cauchy setting, and the regularity framework considered there do not directly resolve the initial–boundary value problem (1.1) in the natural energy class.

A large part of the subsequent well-posedness theory for nonlinear wave equations has been developed in the presence of dissipative mechanisms. Bociu and Lasiecka [1] established local Hadamard well-posedness for nonlinear wave equations with supercritical sources and damping. For viscoelastic hyperbolic equations, Guo et al. [7] proved local existence, uniqueness, continuous dependence, and global continuation under an appropriate balance between the source and damping terms. The complementary blow-up regime was later investigated by Guo, Rammaha, and Sakuntasathien [6]. These studies emphasize the competition between energy-producing source terms and dissipative mechanisms.

For wave equations involving the $p$-Laplacian, dissipation also plays a central role. Pei, Rammaha, and Toundykov [13] considered a $p$-Laplacian wave equation with Kelvin–Voigt damping and supercritical sources, proving the existence of local weak solutions, global continuation under suitable assumptions, and finite-time blow-up for negative initial energy. Kass and Rammaha [10] subsequently established local and global existence for a strongly damped wave equation of $p$-Laplacian type, while their later work [11] treated supercritical interior and boundary nonlinearities in a related quasilinear framework.

More recent developments further illustrate the importance of damping in the theory of $p$-Laplacian wave equations. Yang and Han [19] studied a damped $p$-Laplacian-type wave equation with a logarithmic source and obtained finite-time blow-up criteria together with lifespan estimates. Boumaza and Gheraibia [3] investigated a model containing weak and $p$-Laplacian damping, nonlinear boundary delay, and source terms, and derived global existence, decay, and blow-up results. Irkıl [8] considered a strongly damped $p$-Laplacian equation with logarithmic nonlinearity and combined Galerkin approximation, compactness, and potential-well arguments to obtain global existence, decay, and blow-up properties.

In the same direction, Wu, Yang, and Cheng [18] established global well-posedness and analyzed energy decay and blow-up for a $p$-Laplacian wave equation containing both weak and strong damping. Zu, Sun, and Wu [20] investigated global existence and blow-up for a strongly damped wave equation of $p$-Laplacian type and obtained estimates for the lifespan of solutions. Irkıl [9] more recently established local existence and uniqueness for a $p$-Laplacian equation with logarithmic nonlinearity and nonlinear damping.

The recent literature continues to develop along this dissipative direction. Fu and Zhou [5] considered a $p$-Laplacian wave equation with logarithmic nonlinearity and singular dissipation and used cut-off arguments together with the Faedo–Galerkin method to establish local weak solutions before analyzing global existence, energy decay, and blow-up. Sabitbek [15] developed a potential-well framework for a strongly damped $p$-Laplacian wave equation and obtained threshold criteria separating global weak solutions from finite-time blow-up. Boulaaras [2] studied strongly damped $p$-Laplacian wave equations with logarithmic source terms and derived nonexistence results for global weak solutions together with lifespan estimates. These recent results show that dissipative terms continue to provide an essential analytical mechanism in the existence and long-time theory of quasilinear wave equations.

In contrast to the models discussed above, problem (1.1) contains no strong damping, nonlinear damping, memory effects, or singular dissipation. This distinction is not merely formal. In strongly damped models, a term such as $-\Delta u_t$ provides spatial control of the velocity and contributes additional compactness. For the undamped problem (1.1), the natural energy estimates yield only

$$u \in L^\infty(0,T;W_0^{1,p}(\Omega)), \qquad u_t \in L^\infty(0,T;L^2(\Omega)),$$

and do not provide direct control of $\nabla u_t$. The absence of this additional regularity is responsible for two fundamental difficulties in the existence and uniqueness analysis.

The first difficulty arises in the identification of the nonlinear flux. A Galerkin approximation naturally yields, up to subsequences,

$$\nabla u_m \rightharpoonup \nabla u \quad \text{in } L^p(Q_T),$$

and

$$|\nabla u_m|^{p-2}\nabla u_m \rightharpoonup \chi \quad \text{in } L^{p'}(Q_T;\mathbb{R}^n).$$

Because the mapping $\xi \mapsto |\xi|^{p-2}\xi$ is nonlinear, these weak convergences alone do not imply

$$\chi = |\nabla u|^{p-2}\nabla u.$$

The Minty argument identifies the nonlinear limit only when an appropriate lim sup inequality is available. In particular, weak lower semicontinuity, which produces a lim inf inequality, does not provide the required information. Thus, the possible occurrence of an energy defect must be treated explicitly rather than hidden inside the compactness argument.

The second difficulty concerns uniqueness. If $u$ and $v$ are two energy solutions and $z = u - v$, the natural difference estimate would formally test the equation with $z_t$. However, the energy class provides only

$$z_t \in L^\infty(0,T;L^2(\Omega)),$$

whereas the nonlinear $p$-Laplacian term acts naturally on $W_0^{1,p}(\Omega)$. Therefore, $z_t$ need not be an admissible test function. Moreover, even when additional regularity permits this test, the term

$$\int_\Omega \big(A(\nabla u) - A(\nabla v)\big)\cdot \nabla z_t \, dx$$

does not have the direct sign structure required by the usual energy method. This motivates the use of a relative-energy formulation together with an explicit stability condition controlling the nonlinear Taylor remainder.

A third aspect of the present problem is spatial regularity. Since $p > n$, Morrey's embedding gives

$$W_0^{1,p}(\Omega) \hookrightarrow C^{0,\gamma}(\overline{\Omega}), \qquad \gamma = 1 - \frac{n}{p}.$$

Hence an energy bound immediately produces spatial Hölder bounds. This regularity should, however, be distinguished from a genuine smoothing effect of the hyperbolic equation: it is a consequence of the choice $p > n$ and the embedding properties of the energy space. Combining this spatial estimate with temporal continuity in $L^2(\Omega)$ also yields continuity in weaker Hölder spaces $C^{0,\beta}(\overline{\Omega})$ for $0 \le \beta < \gamma$.

The contribution of this article is the explicit separation of what the undamped energy estimate proves from what requires additional nonlinear compactness or stability. The gradient-oscillation and relative-energy counterexamples delineate these obstructions; no unconditional existence or uniqueness theorem is claimed. Against this background, we develop an analytical framework for problem (1.1). We first formulate weak and energy solutions in a way that avoids inadmissible formal testing and identifies the correct modified energy associated with the divergence forcing. We then establish a precise Minty criterion showing that a defect-free lim sup condition identifies the nonlinear flux and indeed yields strong convergence of the gradients. Based on this criterion, a conditional local existence theorem is obtained for Galerkin approximations satisfying the required defect-free property. We further prove conditional uniqueness in a relative-energy stable class, where the additional time regularity and the nonlinear Taylor-remainder estimate needed for the argument are stated explicitly. Finally, local energy estimates and Hölder regularity are derived for solutions satisfying the corresponding energy bounds.

The qualifications in these results are essential. The existence theorem depends on the additional defect-free compactness condition, whereas the uniqueness theorem is established in a more regular relative-energy stable class. Consequently, existence, uniqueness, and continuous dependence are not established simultaneously in the entire natural energy class. We therefore do not claim local Hadamard well-posedness for arbitrary energy solutions. Rather, the results identify precisely which analytical obstructions arise when the dissipative mechanisms used in much of the recent $p$-Laplacian wave literature are removed, and which additional conditions are sufficient to overcome those obstructions.

The remainder of the paper is organized as follows. Section 2 introduces the functional setting and the preliminary estimates, including the Morrey–Sobolev embedding and the basic properties of the $p$-Laplace operator. Section 3 states the assumptions, solution concepts, and the Minty criterion required for flux identification. Section 4 contains the main results on conditional local existence, conditional uniqueness, local energy estimates, and Hölder regularity. The final

section summarizes the limitations of the present framework and discusses possible directions for removing the additional compactness and stability assumptions.

## 2. Notation and preliminary results

Throughout this paper, $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary, $n \geq 2$, and

$$p > n, \qquad p' = \frac{p}{p-1}, \qquad Q_T = \Omega \times (0,T).$$

We use the Gelfand triple

$$V := W_0^{1,p}(\Omega) \hookrightarrow H := L^2(\Omega) \cong H^* \hookrightarrow V^* := W^{-1,p'}(\Omega).$$

The norm on $V$ is taken as $\|v\|_V = \|\nabla v\|_{L^p(\Omega)}$. The duality pairing between $V^*$ and $V$ is denoted by $\langle \cdot, \cdot \rangle_{V^*,V}$, while $(\cdot,\cdot)_2$ denotes the inner product in $L^2(\Omega)$.

For $0 < \alpha \leq 1$, set

$$[v]_{C^{0,\alpha}(\overline{\Omega})} := \sup_{\substack{x,y \in \overline{\Omega} \\ x \neq y}} \frac{|v(x) - v(y)|}{|x-y|^\alpha}.$$

The letter $C$ denotes a positive constant that may vary from line to line; its dependence will be specified whenever necessary.

**Lemma 2.1.** *If $p > n$ and $\gamma = 1 - n/p$, then*

$$W_0^{1,p}(\Omega) \hookrightarrow C^{0,\gamma}(\overline{\Omega})$$

*continuously. There exists $C_M = C(n,p,\Omega) > 0$ such that*

$$\|v\|_{L^\infty(\Omega)} + [v]_{C^{0,\gamma}(\overline{\Omega})} \leq C_M \|\nabla v\|_{L^p(\Omega)} \quad \text{for every } v \in W_0^{1,p}(\Omega).$$

*Moreover, the embedding into $C^{0,\beta}(\overline{\Omega})$ is compact for every $0 \leq \beta < \gamma$.*

*Proof.* This is the classical Morrey–Sobolev embedding, followed by the compact inclusion of a bounded set in $C^{0,\gamma}$ into $C^{0,\beta}$ for $\beta < \gamma$; see [14]. □

Define $A : \mathbb{R}^n \to \mathbb{R}^n$ by

$$A(\xi) = |\xi|^{p-2}\xi.$$

**Lemma 2.2.** *For $p \geq 2$ and $\xi, \eta \in \mathbb{R}^n$, one has*

$$|A(\xi)| = |\xi|^{p-1}, \tag{2.1}$$

$$A(\xi) \cdot \xi = |\xi|^p, \tag{2.2}$$

$$(A(\xi) - A(\eta)) \cdot (\xi - \eta) \geq 2^{2-p}|\xi - \eta|^p, \tag{2.3}$$

$$|A(\xi) - A(\eta)| \leq (p-1)(|\xi| + |\eta|)^{p-2}|\xi - \eta|. \tag{2.4}$$

*The operator $\mathcal{A} : V \to V^*$ defined by*

$$\langle \mathcal{A}v, \varphi \rangle_{V^*,V} = \int_\Omega A(\nabla v) \cdot \nabla \varphi \, dx$$

*is bounded on bounded subsets of $V$, hemicontinuous, coercive, and strictly monotone.*

*Proof.* The identities (2.1)–(2.2) are immediate. The inequalities (2.3) and (2.4) follow by applying the mean value theorem to the gradient of the convex function $\xi \mapsto |\xi|^p/p$; see [16, 14]. Moreover,

$$|\langle \mathcal{A}v, \varphi\rangle_{V^*,V}| \le \|\nabla v\|_p^{p-1} \|\nabla\varphi\|_p, \qquad \langle \mathcal{A}v, v\rangle_{V^*,V} = \|\nabla v\|_p^p.$$

Monotonicity and hemicontinuity follow by integrating the corresponding pointwise properties of $A$. □

**Lemma 2.3.** *Let $0 < \gamma \le 1$ and $0 \le \beta < \gamma$. There exists a constant $C = C(\Omega, n, \beta, \gamma)$ such that, for every $v \in C^{0,\gamma}(\overline{\Omega})$,*

$$\|v\|_{L^\infty} \le C\,[v]_{C^{0,\gamma}}^{\frac{n}{n+2\gamma}} \|v\|_{L^2}^{\frac{2\gamma}{n+2\gamma}} + C\,\|v\|_{L^2},$$
$$\|v\|_{C^{0,\beta}} \le C\big(1 + \|v\|_{C^{0,\gamma}}\big) \max\left\{ \|v\|_{L^2}, \|v\|_{L^2}^{\frac{2(\gamma-\beta)}{n+2\gamma}} \right\}.$$

*Proof.* The first estimate follows by averaging $v$ over balls intersected with $\Omega$, using $|B_r(x) \cap \Omega| \ge cr^n$, and optimizing the radius. The second follows from the standard interpolation bound $[v]_{C^{0,\beta}} \le C[v]_{C^{0,\gamma}}^{\beta/\gamma} \|v\|_\infty^{1-\beta/\gamma}$ and the first estimate; compare [14]. □

**Lemma 2.4.** *Assume that $f$ satisfies* (H4) *below. Let $z_m, z \in L^2(Q_S)$, suppose that $z_m \to z$ strongly in $L^2(Q_S)$, and assume that there exists $M > 0$ such that $|z_m|, |z| \le M$ almost everywhere. Then*

$$f(\cdot, z_m) \longrightarrow f(\cdot, z) \quad \text{strongly in } L^2(Q_S).$$

*Proof.* For almost every $(x,t)$, local Lipschitz continuity on $[-M, M]$ gives

$$|f(x, z_m(x,t)) - f(x, z(x,t))| \le L_M |z_m(x,t) - z(x,t)|.$$

Squaring, integrating over $Q_S$, and taking the square root, we obtain

$$\|f(\cdot, z_m) - f(\cdot, z)\|_{L^2(Q_S)} \le L_M \|z_m - z\|_{L^2(Q_S)} \longrightarrow 0.$$

This elementary lemma makes explicit why strong convergence of $u_m$, rather than merely weak convergence, is needed when passing to the limit in the source term. □

## 3. The equation, assumptions, and solution concepts

We impose the following assumptions.

**(H1):** $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary, $n \ge 2$, and $p > n$.
**(H2):** $u_0 \in W_0^{1,p}(\Omega)$ and $u_1 \in L^2(\Omega)$.
**(H3):** $g \in W^{1,p'}(0,T; L^{p'}(\Omega; \mathbb{R}^n))$.
**(H4):** $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $f(\cdot, 0) \in L^2(\Omega)$, and for every $M > 0$ there exists $L_M > 0$ such that

$$|f(x,s) - f(x,r)| \le L_M |s - r|$$

for almost every $x \in \Omega$ and all $|s|, |r| \le M$.

*Remark* 3.1. A $C^{1,1}$ boundary is not required for the results of this paper; a Lipschitz boundary is sufficient for the Poincaré inequality, Sobolev extension, and the Morrey embedding. Hölder assumptions with respect to the spatial variable $x$ on $f$ and $g$ are also omitted because they are not used. By contrast, the assumption $f(\cdot,0)\in L^2$ is stated explicitly in order to ensure that $f(\cdot,u)\in L^2$. The time regularity $g\in W^{1,p'}$ is required when differentiating the modified energy.

The local Lipschitz condition (H4) includes polynomial and smooth exponential nonlinearities on bounded ranges. Similarly, separable forcing $g(x,t)=h(t)G(x)$ satisfies (H3) whenever $h\in W^{1,p'}(0,T)$ and $G\in L^{p'}(\Omega;\mathbb{R}^n)$. In contrast, $s\log|s|$ (continuously extended by zero at the origin) is not locally Lipschitz at zero and is not covered by (H4).

To formulate the existence result rigorously, one additional approximation hypothesis is required. Let $V_m\subset V$ be an increasing family of finite-dimensional subspaces such that $\bigcup_m V_m$ is dense in $V$. Let $P_m$ denote the orthogonal projection from $H$ onto $V_m$. With this choice, $(z_m,\varphi)_2=(z_m,P_m\varphi)_2$ for every $z_m\in V_m$; this is the identity used when $u_{m,tt}$ is viewed as an element of $V^*$.

**(HC):** The Galerkin scheme can be chosen so that

(a) $P_m$ is uniformly stable on $V$, namely $\|P_m\varphi\|_V\le C\,\|\varphi\|_V$;

(b) the initial data can be approximated strongly: $u_{0m}\to u_0$ in $V$ and $u_{1m}\to u_1$ in $H$;

(c) for every sequence of Galerkin solutions satisfying the uniform energy bound and the convergences

$$u_m\rightharpoonup u \text{ in } L^p(0,S;V),\qquad A(\nabla u_m)\rightharpoonup \chi \text{ in } L^{p'}(Q_S),$$

the following defect-free property holds:

$$\limsup_{m\to\infty}\int_{Q_S}A(\nabla u_m)\cdot\nabla u_m\le\int_{Q_S}\chi\cdot\nabla u. \tag{3.1}$$

*Remark* 3.2. Condition (HC)(c) supplies additional nonlinear compactness beyond the energy estimate. This paper does not prove (HC)(c) for general initial data; in particular, the construction of a nontrivial data class satisfying it remains open here. A sufficient *a posteriori* criterion is strong convergence of the Galerkin gradients in $L^p(Q_S)$, but that criterion is not obtained from the standard undamped energy bounds. The existence theorem is therefore deliberately conditional and cannot be interpreted as general Hadamard well-posedness.

**Definition 3.3.** A function $u$ is called a weak solution of (1.1) on $[0,S]$, $S\le T$, if

$$u\in L^\infty(0,S;V)\cap W^{1,\infty}(0,S;H),$$
$$u_{tt}\in L^{p'}(0,S;V^*),$$

and, for every $\varphi\in V$, the identity

$$\langle u_{tt},\varphi\rangle_{V^*,V}+\int_\Omega A(\nabla u)\cdot\nabla\varphi\,dx=\int_\Omega f(x,u)\varphi\,dx-\int_\Omega g\cdot\nabla\varphi\,dx \tag{3.2}$$

holds in $\mathcal{D}'(0,S)$. The initial conditions are understood as $u(0)=u_0$ in $H$ and $u_t(0)=u_1$ weakly in $H$.

*Remark* 3.4. From $u \in W^{1,\infty}(0,S;H)$, one obtains $u \in C^{0,1}([0,S];H)$. In addition, $u_t \in L^\infty(0,S;H)$ and $u_{tt} \in L^{p'}(0,S;V^*)$ yield a representative $u_t \in C_w([0,S];H)$; see [12, 14]. We do not claim $u_{tt} \in W^{-1,2}$, since the $p$-Laplace operator naturally takes values in $W^{-1,p'}$.

**Definition 3.5.** A weak solution $u$ is called an energy solution if the function

$$\mathcal{E}(t) := \frac{1}{2}\,\|u_t(t)\|_2^2 + \frac{1}{p}\,\|\nabla u(t)\|_p^p$$

admits a representative satisfying, for every $t \in [0,S]$,

$$\mathcal{E}(t) + \int_\Omega g(t)\cdot\nabla u(t)\,dx = \mathcal{E}(0) + \int_\Omega g(0)\cdot\nabla u_0\,dx + \int_0^t\int_\Omega f(x,u)u_t\,dx\,ds + \int_0^t\int_\Omega g_t\cdot\nabla u\,dx\,ds. \quad (3.3)$$

*Remark* 3.6. The identity (3.3) does not follow automatically from Definition 3.3, because $u_t$ need not belong to $V$ and therefore need not be an admissible test function. The energy identity must be obtained from an approximation scheme that preserves energy, from additional regularity, or else included as part of the definition of an energy solution. This distinction prevents the use of an inadmissible formal test.

For smooth solutions, the sign in (3.3) can be checked directly. Testing (3.2) with $u_t$ gives

$$\frac{d}{dt}\mathcal{E}(t) = \int_\Omega f(x,u)u_t\,dx - \int_\Omega g\cdot\nabla u_t\,dx.$$

Since

$$\frac{d}{dt}\int_\Omega g\cdot\nabla u\,dx = \int_\Omega g_t\cdot\nabla u\,dx + \int_\Omega g\cdot\nabla u_t\,dx,$$

the correct modified energy is

$$\mathcal{E}(t) + \int_\Omega g(t)\cdot\nabla u(t)\,dx,$$

not the energy with the cross term subtracted.

The energy estimate yields only weak convergence of the gradients and the fluxes. Since the mapping $\xi \mapsto |\xi|^{p-2}\xi$ is nonlinear, these two limits need not be related in the expected way. The following theorem states the additional information needed in the existence proof.

**Theorem 3.7.** *Suppose that* $u_m \rightharpoonup u$ *in* $L^p(0,S;V)$ *and*

$$A(\nabla u_m) \rightharpoonup \chi \quad \text{in } L^{p'}(Q_S;\mathbb{R}^n).$$

*If the limsup condition*

$$\limsup_{m\to\infty}\int_{Q_S} A(\nabla u_m)\cdot\nabla u_m\,dx\,dt \le \int_{Q_S}\chi\cdot\nabla u\,dx\,dt \quad (3.4)$$

*holds, then*

$$\chi = A(\nabla u) \quad \text{a.e. in } Q_S.$$

*Moreover,* $\nabla u_m \to \nabla u$ *strongly in* $L^p(Q_S;\mathbb{R}^n)$.

*Proof.* For arbitrary $v \in L^p(0,S;V)$, the monotonicity of $A$ yields

$$\begin{aligned} 0 &\le \int_{Q_S} \big(A(\nabla u_m) - A(\nabla v)\big)\cdot\big(\nabla u_m - \nabla v\big) \\ &= \int_{Q_S} A(\nabla u_m)\cdot\nabla u_m - \int_{Q_S} A(\nabla u_m)\cdot\nabla v \\ &\quad - \int_{Q_S} A(\nabla v)\cdot\nabla u_m + \int_{Q_S} A(\nabla v)\cdot\nabla v. \end{aligned}$$

Using (3.4) for the first term and the corresponding weak convergences in the two mixed terms, we may pass to the limit as $m\to\infty$ and obtain

$$0 \le \int_{Q_S} \big(\chi - A(\nabla v)\big)\cdot(\nabla u - \nabla v).$$

We now choose $v = u - \varepsilon w$, where $w \in L^p(0,S;V)$ and $\varepsilon > 0$. Dividing by $\varepsilon$ gives

$$0 \le \int_{Q_S} \big[\chi - A(\nabla u - \varepsilon\nabla w)\big]\cdot\nabla w.$$

By the growth condition (2.1), the hemicontinuity of $A$, and dominated convergence, letting $\varepsilon \downarrow 0$ yields

$$\int_{Q_S} \big(\chi - A(\nabla u)\big)\cdot\nabla w \ge 0.$$

Replacing $w$ by $-w$ gives the reverse inequality; thus

$$\int_{Q_S} \big(\chi - A(\nabla u)\big)\cdot\nabla w = 0 \qquad \text{for every } w \in L^p(0,S;V). \tag{3.5}$$

Crucially, (3.5) first identifies the two fluxes only *as functionals on gradients*; it does not alone imply their pointwise equality as vector fields. Taking $w = u$ in (3.5), expanding the monotonicity expression, and using (3.4), we obtain

$$\limsup_{m\to\infty} \int_{Q_S} \big(A(\nabla u_m) - A(\nabla u)\big)\cdot\big(\nabla u_m - \nabla u\big) \le 0.$$

The integrand is nonnegative and (2.3) gives

$$2^{2-p}\limsup_{m\to\infty} \|\nabla u_m - \nabla u\|^p_{L^p(Q_S)} \le 0.$$

Consequently, $\nabla u_m \to \nabla u$ strongly in $L^p(Q_S;\mathbb{R}^n)$. The continuity estimate (2.4) now yields $A(\nabla u_m) \to A(\nabla u)$ strongly in $L^{p'}(Q_S;\mathbb{R}^n)$. Comparing this strong limit with the given weak flux limit finally proves $\chi = A(\nabla u)$ almost everywhere. □

*Remark* 3.8. Weak lower semicontinuity yields an inequality of lim inf type, whereas Theorem 3.7 requires a lim sup estimate. Since these inequalities point in opposite directions, weak lower semicontinuity cannot be used to establish (3.4). Example 3.9 below shows that this obstruction is genuine and is not merely a technical gap in the proof.

**Example 3.9.** Let $\Omega = (0,1)^n$ and write $x = (x_1, x')$. Choose $0 \le \psi \in C_c^\infty((0,1)^{n-1})$, $\psi \not\equiv 0$. Define the one-periodic function

$$q(s) = \begin{cases} 1, & 0 \le s < 2/3, \\ -2, & 2/3 \le s < 1, \end{cases} \qquad q(s+1) = q(s).$$

The mean of $q$ is zero. Hence the primitive $Q(s) = \int_0^s q(r)\,dr$ extends to a bounded periodic function. For every integer $m \ge 1$, set

$$u_m(x) = \frac{1}{m} Q(mx_1)\psi(x').$$

Since $Q(0) = Q(m) = 0$ and $\psi$ has compact support, we have $u_m \in W_0^{1,p}(\Omega)$. Moreover,

$$\partial_{x_1} u_m = q(mx_1)\psi(x'), \qquad \nabla_{x'} u_m = \frac{1}{m} Q(mx_1)\nabla_{x'}\psi(x').$$

Thus $u_m \to 0$ uniformly and $\nabla u_m \rightharpoonup 0$ in $L^p$. However, the first component of the flux has the weak limit

$$\begin{aligned} |\partial_{x_1} u_m|^{p-2}\partial_{x_1} u_m &\rightharpoonup \psi(x')^{p-1} \int_0^1 |q(s)|^{p-2} q(s)\,ds \\ &= \frac{2 - 2^{p-1}}{3}\,\psi(x')^{p-1}, \end{aligned}$$

which is nonzero for $p > 2$. The transverse components of the flux converge to zero and do not affect this limit. Hence

$$A(\nabla u_m) \rightharpoonup \chi \neq 0 = A(\nabla 0).$$

Furthermore,

$$\lim_{m\to\infty} \int_\Omega |\nabla u_m|^p dx = \frac{2 + 2^p}{3} \int_{(0,1)^{n-1}} \psi(x')^p dx' > 0.$$

Thus the Minty limsup condition fails. This sequence is not claimed to be a sequence of PDE solutions; its purpose is to refute the incorrect assertion that weak convergence of the gradients automatically yields weak convergence of the nonlinear flux to the expected limit.

## 4. Main results

4.1. **Conditional local existence.** The route to existence is as follows. The source is first truncated so that the local nonlinearity becomes globally Lipschitz. The resulting finite-dimensional problem admits an ODE solution. The energy identity then yields a common lifespan and uniform bounds. These bounds provide compactness for $u_m$ and $u_{m,t}$, but only weak convergence for the gradients and the fluxes. Hypothesis (HC)(c), together with Minty's criterion, closes this

sole nonlinear gap. Once the flux limit is identified, the truncation is removed by a Morrey bootstrap.

For $M > 0$, define

$$T_M(s) := \max\{-M, \min\{s, M\}\}, \qquad f_M(x, s) := f(x, T_M(s)).$$

Then $f_M$ is globally Lipschitz with respect to $s$, with Lipschitz constant $L_M$.

**Theorem 4.1.** *Assume that* (H1)–(H4) *and* (HC) *hold. Then there exist* $S_0 \in (0, T]$ *and at least one weak solution* $u$ *of* (1.1) *on* $[0, S_0]$ *such that*

$$u \in L^\infty(0, S_0; V) \cap W^{1,\infty}(0, S_0; H), \qquad u_{tt} \in L^{p'}(0, S_0; V^*).$$

*Moreover,*

$$\operatorname*{ess\,sup}_{0<t<S_0} \left( \|u_t(t)\|_2^2 + \|\nabla u(t)\|_p^p \right) \le C_0, \tag{4.1}$$

*where* $C_0$ *depends on the data,* $f$, $g$, *and* $S_0$. *If, in addition, for every* $t$ *the time-slice convergences*

$$u_m(t) \rightharpoonup u(t) \text{ in } V, \qquad u_{m,t}(t) \rightharpoonup u_t(t) \text{ in } H,$$

*hold and the Galerkin energies converge to the limiting energy, then* $u$ *is an energy solution in the sense of Definition 3.5.*

*Proof.* Choose a basis $\{w_1, \dots, w_m\}$ of $V_m$ and seek

$$u_m(t) = \sum_{j=1}^{m} d_{mj}(t) w_j$$

satisfying, for $k = 1, \dots, m$,

$$\begin{aligned} (u_{m,tt}, w_k)_2 + \int_\Omega A(\nabla u_m) \cdot \nabla w_k \, dx \\ = \int_\Omega f_M(x, u_m) w_k \, dx - \int_\Omega g \cdot \nabla w_k \, dx. \end{aligned} \tag{4.2}$$

In finite dimensions, the right-hand side of the coefficient system is measurable in $t$ and continuous, in fact locally Lipschitz, in the coefficient vector. Carathéodory's ODE theorem therefore yields a maximal solution on $[0, t_m)$.

Multiply the $k$th equation by $d'_{mk}$, sum over $k$, and integrate in time. Since $u_{m,t} \in V_m$, this test is admissible and gives

$$\begin{aligned} \mathcal{E}_m(t) + \int_\Omega g(t) \cdot \nabla u_m(t) \, dx = \mathcal{E}_m(0) + \int_\Omega g(0) \cdot \nabla u_{0m} \, dx \\ + \int_0^t \int_\Omega f_M(x, u_m) u_{m,t} \, dx \, ds \\ + \int_0^t \int_\Omega g_t \cdot \nabla u_m \, dx \, ds, \end{aligned} \tag{4.3}$$

where

$$\mathcal{E}_m(t) = \frac{1}{2} \|u_{m,t}(t)\|_2^2 + \frac{1}{p} \|\nabla u_m(t)\|_p^p .$$

Notice the positive sign in front of the cross term $g \cdot \nabla u_m$.

Young's inequality gives

$$\left|\int_\Omega g\cdot\nabla u_m\right| \le \frac{1}{2p}\|\nabla u_m\|_p^p + C\,\|g\|_{p'}^{p'}.$$

Since $T_M$ is a contraction, $|T_M(s)|\le|s|$, and $f_M$ is globally Lipschitz,

$$\|f_M(\cdot,u_m)\|_2 \le \|f(\cdot,0)\|_2 + L_M\,\|u_m\|_2 \le C_M\big(1+\|\nabla u_m\|_p\big).$$

Consequently,

$$\begin{aligned}\left|\int_\Omega f_M(x,u_m)u_{m,t}\,dx\right| &\le C_M(1+\|\nabla u_m\|_p)\,\|u_{m,t}\|_2\\ &\le C_M(1+\mathcal{E}_m),\end{aligned}$$

because $p>2$ and $\|\nabla u_m\|_p\,\|u_{m,t}\|_2 \le C\mathcal{E}_m^{1/p+1/2}\le C(1+\mathcal{E}_m)$. Likewise,

$$\left|\int_\Omega g_t\cdot\nabla u_m\,dx\right| \le \frac{1}{2p}\|\nabla u_m\|_p^p + C\,\|g_t\|_{p'}^{p'}.$$

Define

$$Y_m(t) := 1+\mathcal{E}_m(t)+C_g\,\|g(t)\|_{p'}^{p'},$$

with $C_g$ sufficiently large to absorb the cross term. From (4.3) and the preceding estimates, we obtain

$$Y_m(t)\le C\left(1+\mathcal{E}_m(0)+\|g(0)\|_{p'}^{p'}\right)+C_M\int_0^t Y_m(s)\,ds + C\int_0^t\|g_t(s)\|_{p'}^{p'}\,ds.$$

Gronwall's lemma yields

$$Y_m(t)\le Ce^{C_Mt}\left[1+\mathcal{E}_m(0)+\|g(0)\|_{p'}^{p'}+\int_0^t\|g_t(s)\|_{p'}^{p'}\,ds\right]. \tag{4.4}$$

The strong convergence of the initial data makes the right-hand side uniform in $m$. Hence there exists $S_0>0$, independent of $m$, such that

$$\sup_m \operatorname*{ess\,sup}_{0<t<S_0}\left(\|u_{m,t}(t)\|_2^2+\|\nabla u_m(t)\|_p^p\right)\le C_0. \tag{4.5}$$

Morrey's inequality gives $\|u_m(t)\|_\infty\le C\,\|\nabla u_m(t)\|_p$. Choose $R$ above the data-dependent threshold required to absorb the modified-energy cross term, and then choose $M>C_M(pR)^{1/p}$. Since the right-hand side of (4.4) depends continuously on the upper time limit, $S_0$ may be reduced so that the bootstrap condition $\mathcal{E}_m(t)<R$ remains closed. On this interval, $T_M(u_m)=u_m$; hence the truncation does not alter the original equation.

Next, for $\varphi\in V$, use $P_m\varphi$ in the Galerkin equation. By the projection stability in (HC)(a), the growth of the operator, and (4.5),

$$|\,\langle u_{m,tt},\varphi\rangle_{V^*,V}\,|\le C\big(\|\nabla u_m\|_p^{p-1}+\|f(\cdot,u_m)\|_2+\|g\|_{p'}\big)\,\|\varphi\|_V.$$

Thus $u_{m,tt}$ is bounded in $L^{p'}(0,S_0;V^*)$.

The above bounds allow us to apply Banach–Alaoglu together with the compact embedding $V \Subset H$. Therefore, after passing to a subsequence, not relabeled,

$$\begin{aligned} u_m &\rightharpoonup^* u && \text{in } L^\infty(0,S_0;V),\\ u_{m,t} &\rightharpoonup^* u_t && \text{in } L^\infty(0,S_0;H),\\ u_{m,tt} &\rightharpoonup u_{tt} && \text{in } L^{p'}(0,S_0;V^*),\\ A(\nabla u_m) &\rightharpoonup \chi && \text{in } L^{p'}(Q_{S_0};\mathbb{R}^n). \end{aligned}$$

The Aubin–Lions–Simon compactness criterion [17] gives

$$u_m \to u \qquad \text{strongly in } C([0,S_0];H). \tag{4.6}$$

Indeed, the bound on $u_{m,t}$ yields

$$\|u_m(t)-u_m(s)\|_2 \le \int_s^t \|u_{m,t}(r)\|_2\, dr \le C|t-s|.$$

Thus $\{u_m\}$ is equicontinuous with values in $H$. At each time, $\{u_m(t)\}$ is bounded in $V$ and relatively compact in $H$ because $V \Subset H$. The Banach-valued Arzelà–Ascoli theorem then yields (4.6). Combining this convergence with the uniform bound in $C^{0,\gamma}$ and Lemma 2.3, we also obtain

$$u_m \to u \quad \text{strongly in } C([0,S_0];C^{0,\beta}(\overline{\Omega})) \quad \text{for every } 0 \le \beta < \gamma.$$

Since the solutions remain in the same $L^\infty$ ball, Lemma 2.4 and (4.6) give

$$f(x,u_m) \to f(x,u) \qquad \text{strongly in } L^2(Q_{S_0}).$$

To identify the flux limit, (HC)(c) provides the limsup condition (3.1). By Theorem 3.7, proved before the main results,

$$\chi = A(\nabla u), \qquad \nabla u_m \to \nabla u \text{ strongly in } L^p(Q_{S_0}).$$

Once the flux limit has been identified, every term in (4.2) can be passed to the limit, and therefore (3.2) holds. The strong convergence (4.6), the convergence of the initial data, and the weak continuity of $u_t$ yield $u(0)=u_0$ and $u_t(0)=u_1$ in the sense of Definition 3.3. To justify the initial velocity condition in detail, take $\varphi \in V$ and integrate the scalar equation for $(u_{m,t},P_m\varphi)_2$ from 0 to $t$. All integral terms pass to the limit and $(u_{1m},P_m\varphi)_2 \to (u_1,\varphi)_2$. The density of $V$ in $H$ then yields $u_t(0)=u_1$ weakly in $H$. Weak lower semicontinuity gives (4.1). If the time-slice and energy convergences stated in the theorem also hold, the cross term involving $g(t)\cdot\nabla u_m(t)$ converges by duality, the source term converges because one factor converges strongly and the other weakly, and the $g_t$ term converges through the $L^{p'}$–$L^p$ duality. Hence the limit of (4.3) is precisely (3.3); in this case, $u$ is an energy solution. □

*Remark* 4.2. Theorem 4.1 is a *conditional* local existence result. The difficult part is not the Galerkin ODE or the energy estimate, but the verification of (HC)(c). In the undamped equation, this condition cannot be replaced by weak lower semicontinuity because the direction of the resulting inequality is incorrect.

*Remark* 4.3. Eigenfunctions of the $p$-Laplacian should not be described as an $L^2$-orthogonal basis, since the $p$-Laplace operator is nonlinear. One possible choice is a smooth system that is total in $V$ and orthonormal in $H$, provided that stability of the associated projections in $V$ is verified. Alternatively, one may use a regularization or a time discretization based on monotone operators.

4.2. **Conditional uniqueness.** Existence only addresses whether at least one solution is available. To compare two solutions, subtracting the equations removes the forcing $g$ but produces a nonlinear flux that has no simple quadratic energy. We therefore use a relative energy rather than the ordinary norm of the difference. The price is stronger time regularity and an explicit control of the Taylor remainder.

*Remark* 4.4. In the general energy class, if $z = u - v$, then one only knows $z_t \in L^\infty(0, S; L^2(\Omega))$, whereas a test function for the $p$-Laplace term must belong to $W_0^{1,p}(\Omega)$. Hence testing the difference equation with $z_t$ is not justified without additional regularity. Even formally, the flux term takes the form $\int_\Omega (A(\nabla u) - A(\nabla v)) \cdot \nabla z_t \, dx$, so monotonicity does not directly provide a sign.

**Example 4.5.** In $V = H_0^1(0,\pi) \Subset H = L^2(0,\pi)$, let $e_k(x) = \sqrt{2/\pi}\sin(kx)$ and $w(t,x) = \sum_{k\geq 1} k^{-2}\sin(kt)e_k(x)$. Then $w \in L^\infty(0,S;V) \cap W^{1,\infty}(0,S;H)$, but $w_t \notin L^2(0,S;V)$, since the squared gradient norm would have a divergent sum $\sum_{k\geq 1}\cos^2(kt)$ after integration in time. Thus energy-class time derivatives need not be admissible test functions.

Define the stored energy $W(\xi) = |\xi|^p/p$ and the relative energy density

$$W(\xi \mid \eta) := W(\xi) - W(\eta) - A(\eta)\cdot(\xi - \eta) \geq 0. \tag{4.7}$$

**Theorem 4.6.** *Let $u$ and $v$ be two solutions of (1.1) with the same initial data. Assume that both solutions are sufficiently regular to permit the use of $u_t - v_t$ as a test function and to justify the chain rule in (4.7); in particular, it is sufficient to assume*

$$u, v \in W^{1,\infty}(0,S;V), \qquad u_{tt}, v_{tt} \in L^1(0,S;H).$$

*Assume further that there exists $k \in L^1(0,S)$ such that, with $z = u - v$, for almost every $t$,*

$$\left|\int_\Omega \big[A(\nabla u) - A(\nabla v) - DA(\nabla v)\nabla z\big]\cdot\nabla v_t \, dx\right| \leq k(t)\int_\Omega W(\nabla u \mid \nabla v)\, dx. \tag{4.8}$$

*Then $u = v$ on $[0,S]$.*

*Proof.* Since the forcing $g$ is the same for both solutions, subtracting the two equations gives

$$z_{tt} - \operatorname{div}\big(A(\nabla u) - A(\nabla v)\big) = f(x,u) - f(x,v).$$

Define

$$\mathscr{R}(t) := \frac{1}{2}\|z_t(t)\|_2^2 + \frac{1}{2}\|z(t)\|_2^2 + \int_\Omega W(\nabla u(t) \mid \nabla v(t))\, dx.$$

The convexity of $W$ implies $\mathscr{R}(t) \geq 0$.

Under the regularity assumptions of the theorem, testing the difference equation with $z_t$ is legitimate. By the chain rule,

$$\frac{d}{dt}\int_\Omega W(\nabla u \mid \nabla v)\,dx = \int_\Omega (A(\nabla u) - A(\nabla v)) \cdot \nabla z_t\,dx + \int_\Omega \big[A(\nabla u) - A(\nabla v) - DA(\nabla v)\nabla z\big] \cdot \nabla v_t\,dx.$$

The first term cancels exactly with the flux term arising from the difference equation. Therefore,

$$\mathscr{R}'(t) = \int_\Omega z\, z_t\,dx + \int_\Omega (f(x,u) - f(x,v))z_t\,dx + \int_\Omega \big[A(\nabla u) - A(\nabla v) - DA(\nabla v)\nabla z\big] \cdot \nabla v_t\,dx. \tag{4.9}$$

By Morrey's embedding, both solutions are bounded in $L^\infty(Q_S)$. Choose $M$ containing the ranges of both solutions. Then (H4), the Cauchy–Schwarz inequality, and Young's inequality yield

$$\left|\int_\Omega z z_t\right| + \left|\int_\Omega (f(x,u) - f(x,v))z_t\right| \leq C(1 + L_M)\mathscr{R}(t).$$

Using (4.8) in (4.9), we obtain

$$\mathscr{R}'(t) \leq [C(1 + L_M) + k(t)]\mathscr{R}(t).$$

Since the initial data coincide, $\mathscr{R}(0) = 0$. Gronwall's lemma then gives $\mathscr{R}(t) = 0$ for every $t \in [0, S]$. Hence $z = z_t = 0$ and therefore $u = v$. □

*Remark* 4.7. The estimate (4.8) is an additional stability hypothesis, not a consequence of (H1)–(H4) or the natural energy bound. It controls the Taylor remainder of the nonlinear flux along the comparison solution. It is automatic in the linear case $p = 2$, because $DA$ is constant and the remainder vanishes. For $p > 2$, especially near points where $\nabla v = 0$, this condition does not follow from the standard energy bound; this is precisely where the degeneracy of the operator enters.

**Example 4.8.** At $\eta = 0$, the relative density is $W(\xi \mid 0) = |\xi|^p/p$. For $p > 2$, the ratio $W(\xi \mid 0)/|\xi|^2 = |\xi|^{p-2}/p$ tends to zero as $\xi \to 0$. Thus relative energy does not uniformly control a quadratic gradient distance near degenerate states.

*Remark* 4.9. Theorem 4.6 does not assert uniqueness of all energy solutions. A solution in the sense of Definition 3.5 only satisfies $u_t \in L^\infty(L^2)$, so $u_t - v_t$ need not be an admissible test function in $W_0^{1,p}$. Moreover, the remainder estimate (4.8) is not available from the standard energy alone. Thus, this is a conditional uniqueness result for a more regular class of solutions, not a general uniqueness theorem in the energy class.

*Remark* 4.10. The existence and uniqueness results above are established in different classes. Theorem 4.1 still depends on the defect-free condition (HC)(c),

whereas Theorem 4.6 requires stronger time regularity and the remainder control (4.8). Continuous dependence has not been established for the entire energy class either. Therefore, these two results are not combined into a claim of *local well-posedness* in the Hadamard sense.

### 4.3. **Local energy estimate.**

**Theorem 4.11.** *Assume that* (H1)–(H4) *hold and that $u$ is an energy solution on $[0,S]$. There exists a threshold $R_* > \mathcal{E}(0)$, depending on the initial data and $g(0)$, such that for every $R > R_*$ there exists $\tau = \tau(R, u_0, u_1, f, g, p, n, \Omega) > 0$ with the following property: if $S \geq \tau$ and*

$$\mathcal{E}(t) \leq R \quad \text{on } [0,\tau]$$

*in the bootstrap argument, then*

$$\sup_{0\leq t\leq \tau} \mathcal{E}(t) \leq C_R \left(1 + \mathcal{E}(0) + \|g\|^{p'}_{L^\infty(0,\tau;L^{p'})} + \|g_t\|^{p'}_{L^{p'}(0,\tau;L^{p'})}\right), \tag{4.10}$$

*where $C_R$ is independent of the approximating solution. The time $\tau$ can be chosen sufficiently small so that the right-hand side remains below $R$, thereby closing the bootstrap argument.*

*Proof.* By Young's inequality, for every $t$,

$$\begin{aligned}\left|\int_\Omega g(t)\cdot\nabla u(t)\,dx\right| &\leq \|g(t)\|_{p'}\,\|\nabla u(t)\|_p \\ &\leq \frac{1}{2p}\,\|\nabla u(t)\|_p^p + C_p\,\|g(t)\|_{p'}^{p'}.\end{aligned}$$

Set

$$G := 1 + \|g\|^{p'}_{L^\infty(0,S;L^{p'})}$$

and choose $C_0$ sufficiently large. The function

$$\mathcal{H}(t) := \mathcal{E}(t) + \int_\Omega g(t)\cdot\nabla u(t)\,dx + C_0 G$$

satisfies the equivalence

$$c_0(1+\mathcal{E}(t)) \leq \mathcal{H}(t) \leq C_0'(1+\mathcal{E}(t)+G). \tag{4.11}$$

On the bootstrap region $\mathcal{E}(t)\leq R$, Lemma 2.1 gives

$$\|u(t)\|_\infty \leq C_M\,\|\nabla u(t)\|_p \leq C_M(pR)^{1/p} =: M_R.$$

Assumption (H4) then implies

$$\begin{aligned}\|f(\cdot,u(t))\|_2 &\leq \|f(\cdot,0)\|_2 + L_{M_R}\,\|u(t)\|_2 \\ &\leq C_R(1+\mathcal{E}(t)^{1/p}).\end{aligned}$$

Since $p>2$, one has $1/2+1/p<1$. By Cauchy–Schwarz and Young's inequalities,

$$\left|\int_\Omega f(x,u)u_t\,dx\right| \leq C_R(1+\mathcal{E}^{1/p})\mathcal{E}^{1/2} \tag{4.12}$$

$$\leq C_R(1+\mathcal{E}) \leq C_R\mathcal{H}. \tag{4.13}$$

Again by Young's inequality,

$$\left|\int_\Omega g_t \cdot \nabla u \, dx\right| \le \|g_t\|_{p'} \|\nabla u\|_p \tag{4.14}$$

$$\le \frac{1}{2p} \|\nabla u\|_p^p + C_p \|g_t\|_{p'}^{p'} \tag{4.15}$$

$$\le C\mathcal{H} + C \|g_t\|_{p'}^{p'}. \tag{4.16}$$

From the energy identity (3.3), together with (4.13) and (4.16), it follows that for almost every $t$,

$$\mathcal{H}'(t) \le C_R \mathcal{H}(t) + C \|g_t(t)\|_{p'}^{p'}.$$

Gronwall's lemma yields

$$\mathcal{H}(t) \le e^{C_R t} \left[\mathcal{H}(0) + C \int_0^t \|g_t(s)\|_{p'}^{p'} \, ds\right]. \tag{4.17}$$

Using (4.11) gives (4.10). Since the right-hand side of (4.17) depends continuously on $t$ and its value at $t = 0$ is determined by the initial data, one can choose $\tau > 0$ so that the energy does not reach the bootstrap level $R$. This closes the local argument. □

*Remark* 4.12. The preceding estimate shows that an energy solution can be continued as long as $\|u(t)\|_{W_0^{1,p}} + \|u_t(t)\|_2$ remains bounded and the local Lipschitz constant $L_M$ remains controlled on the range of the solution. This is an a priori continuation criterion, not by itself an existence or continuation theorem.

4.4. **Spatial Hölder regularity.** Once existence has been addressed and the limits of the uniqueness class have been clarified, the regularity analysis is more direct. For every solution satisfying an energy bound, the $L^p$ control of the gradient can be converted immediately into Hölder control because $p > n$. No additional testing of the equation is required.

**Theorem 4.13.** *Assume that (H1)–(H4) hold and that $u$ is an energy solution on $[0, S]$ satisfying*

$$K_S := \operatorname*{ess\,sup}_{0<t<S} \|\nabla u(t)\|_p < \infty.$$

*If $\gamma = 1 - n/p$, then*

$$u \in L^\infty(0, S; C^{0,\gamma}(\overline{\Omega}))$$

*and*

$$\operatorname*{ess\,sup}_{0<t<S} \|u(t)\|_{C^{0,\gamma}(\overline{\Omega})} \le C_M K_S.$$

*If the estimate (4.10) holds, then $K_S$ can be replaced by $(p \sup_t \mathcal{E}(t))^{1/p}$.*

*Proof.* For almost every $t$, one has $u(t) \in W_0^{1,p}(\Omega)$. By Lemma 2.1,

$$\|u(t)\|_{C^{0,\gamma}} \le C_M \|\nabla u(t)\|_p.$$

Taking the essential supremum in time gives the stated estimate. Since

$$p^{-1} \|\nabla u(t)\|_p^p \le \mathcal{E}(t),$$

the final assertion follows immediately. □

*Remark* 4.14. The additional assumption $u_0 \in C^{0,\alpha}$ is unnecessary when $u_0 \in W_0^{1,p}$ and $p > n$, since Lemma 2.1 already gives $u_0 \in C^{0,\gamma}$. Thus, Theorem 4.13 is more appropriately viewed as a Hölder bound inherited from the energy space, rather than a regularizing effect generated by the equation.

**Theorem 4.15.** *Under the assumptions of Theorem 4.13, choose* $0 \le \beta < \gamma$ *and set*

$$\theta_\beta := \frac{2(\gamma - \beta)}{n + 2\gamma} > 0.$$

*Then* $u$ *admits a representative satisfying*

$$u \in C^{0,\theta_\beta}([0, S]; C^{0,\beta}(\overline{\Omega})).$$

*More precisely, for* $s, t \in [0, S]$ *with* $|t - s| \le 1$,

$$\|u(t) - u(s)\|_{C^{0,\beta}} \le C(K_S + 1) \max\Big\{ N_S, N_S^{\theta_\beta} \Big\} |t - s|^{\theta_\beta}, \tag{4.18}$$

*where* $N_S := \|u_t\|_{L^\infty(0,S;L^2)}$.

*Proof.* Since $u \in W^{1,\infty}(0, S; L^2)$,

$$\|u(t) - u(s)\|_2 \le \int_s^t \|u_t(r)\|_2 \, dr \le \|u_t\|_{L^\infty(L^2)} |t - s|. \tag{4.19}$$

On the other hand, Theorem 4.13 yields

$$\|u(t) - u(s)\|_{C^{0,\gamma}} \le 2C_M K_S.$$

Applying Lemma 2.3 to $v = u(t) - u(s)$ and using (4.19), we find that, for $|t - s| \le 1$, the smallest exponent that appears is $2(\gamma - \beta)/(n + 2\gamma)$. This gives (4.18). Defining $u(t)$ by limits in $C^{0,\beta}$ along Lebesgue times yields a representative continuous on the whole interval $[0, S]$. □

*Remark* 4.16. This theorem does not use the false embedding $L^2(\Omega) \hookrightarrow C^0(\overline{\Omega})$. The correct direction is $C^0(\overline{\Omega}) \hookrightarrow L^2(\Omega)$. Time continuity is obtained by interpolating strong continuity in $L^2$ with the uniform bound in $C^{0,\gamma}$.

## 5. Discussion and further directions

The results of this paper distinguish three levels of analysis:

1. *A priori estimates*, which hold for energy solutions and follow directly from the energy identity.
2. *Spatial regularity*, which follows from the energy bound and the Morrey embedding when $p > n$.
3. *Existence and uniqueness*, which require additional arguments not contained in the preceding two levels.

Several directions may lead to stronger results. First, one may consider the viscous approximation

$$u_{tt} - \Delta_p u - \varepsilon \Delta u_t = f(x, u) + \operatorname{div} g$$

and then study the limit $\varepsilon \downarrow 0$, taking into account the possible formation of an energy defect. Second, Young-measure or dissipative solutions may be appropriate

if gradient oscillations cannot be excluded. Weak–strong uniqueness arguments based on relative energy in elastodynamic systems provide a relevant conceptual model, although their adaptation to the scalar equation (1.1) must still be proved and cannot be taken for granted. Third, when $p \leq n$, Hölder regularity must arise from the PDE structure rather than from the standard Sobolev embedding. This third direction may lead to a more substantive regularity theory.

*Remark* 5.1. Examples 3.9, 4.5, and 4.8 explain several common failures in formal proof arguments, but they do not prove that the PDE (1.1) possesses two distinct energy solutions. Accordingly, the appropriate conclusion is that "general uniqueness in the energy class has not been established within the present framework," rather than that "general energy uniqueness is false."

## 6. Conclusion

For the undamped $p$-Laplacian wave equation, the natural functional setting is $W_0^{1,p} \hookrightarrow L^2 \hookrightarrow W^{-1,p'}$. With an explicit energy-solution concept, the sign of the modified energy is identified correctly and a local energy estimate can be derived. The condition $p > n$ yields spatial $C^{0,1-n/p}$ bounds, while interpolation with time continuity in $L^2$ gives time continuity in $C^{0,\beta}$ for every $\beta < 1 - n/p$. By contrast, the existence of a weak solution cannot be concluded before the nonlinear flux limit is identified, and uniqueness cannot be proved by using $u_t - v_t$ as a test function in the general energy class. These restrictions are not merely expository limitations; they reflect the structure of the undamped hyperbolic problem.

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