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SCHATTEN-CLASS TRANSITIONS FOR DIAGONAL KERNELS

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Abstract. We study operators generated by diagonal positive-definite kernels on weighted sequence spaces and describe how their Schatten properties depend on the underlying weighted measure. We completely characterize changes of weight that preserve Schatten-class membership uniformly over all positive diagonal kernels, and we describe transitions between different Schatten classes in terms of boundedness or summability of the ratio of the two weight sequences. The same criteria hold for the associated reproducing kernel Hilbert space embeddings. We also obtain stability estimates under perturbations of the weight, a trace stability formula, and threshold results for power-type and regularly varying products.

Keywords. Reproducing kernel Hilbert space, Schatten class, diagonal kernel operator, weighted sequence space, weight stability, trace class.
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1. Introduction

Reproducing kernel Hilbert spaces provide a natural setting for the study of positive-definite kernels and the operators associated with them. We refer to the classical work of Aronszajn [1] and to the monographs [2, 11]. When the natural embedding of an RKHS into an L_2-space is bounded, the corresponding kernel operator factors through this embedding. This makes it possible to relate Schatten properties of kernel operators to those of RKHS embeddings. The operator-theoretic background can be found in [8, 10], while the dependence of kernel operators and RKHS embeddings on the underlying measure is discussed in [12].

We consider diagonal positive-definite kernels

$$k(m,n) = c_n \delta_{mn}, \qquad c_n > 0,$$

on $\mathbb{N}$ endowed with weighted counting measures

$$\mu_w(\{n\}) = w_n, \qquad w_n > 0.$$

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For a fixed weight sequence, the associated kernel operator is unitarily equivalent to the diagonal operator with entries $\{c_n w_n\}$. Its boundedness, compactness, and Schatten-class membership are therefore determined by the corresponding diagonal sequence. Our main interest is in what happens when the kernel is kept fixed and the weight is changed.

Let $w = \{w_n\}$ and $v = \{v_n\}$ be two positive weight sequences. We determine when

$$T_k^w \in S_p \quad \Longrightarrow \quad T_k^v \in S_q$$

holds for every positive diagonal kernel. The criterion depends only on the ratio

$$r_n = \frac{v_n}{w_n}.$$

When $p \le q$, boundedness of this ratio is necessary and sufficient. When $q < p$, the corresponding condition is a summability requirement on the ratio. The same criterion is obtained for the natural RKHS embeddings. In the case $p = q$, the condition no longer depends on the Schatten index. It follows that preservation of one Schatten class for the whole family of positive diagonal kernels implies preservation of every Schatten class.

The proof is based on a reduction of the change-of-weight problem to a classical coordinate-multiplier problem between sequence spaces. The multiplier characterization itself is standard. Here it is used to describe, in a uniform way, how a change of the weighted measure affects both the kernel operator and the corresponding RKHS embedding. This approach also gives criteria for equivalent weights and quantitative estimates for perturbations of the weight. Related multiplication operators on weighted sequence spaces were considered in [5], while related change-of-measure identities for kernel covariance embeddings were obtained in [9].

The diagonal setting also allows the general criteria to be evaluated explicitly. We apply them to power-type and regularly varying kernel–weight products. This gives exact Schatten thresholds and shows how slowly varying factors affect the critical case. In the trace-class setting, the perturbation results also lead to an explicit formula for the change of the trace.

Section 2 contains the necessary background on Schatten classes and RKHS embeddings. Sections 3 and 4 describe the diagonal kernel operator and the associated RKHS embedding. Section 5 is devoted to changes of weight and the corresponding stability results. Applications to power-type and regularly varying sequences are given in Section 6.

2. Preliminaries

We fix notation and recall only the facts used below. For trace ideals and Schatten classes, see [8, 10]; for reproducing kernel Hilbert spaces, see [1, 2]. All Hilbert spaces are complex and separable.

Let $w = \{w_n\}_{n\ge1}$ be a sequence of positive numbers and let μ_w be the weighted counting measure on $\mathbb{N}$ given by $\mu_w(\{n\}) = w_n$. We write $\ell^2(\mathbb{N}, \mu_w)$ for the associated Hilbert space, with inner product $\langle f, g\rangle = \sum_{n\ge1} f(n)\overline{g(n)}\, w_n$. If e_n denotes

the standard coordinate vector, then $u_n = e_n/\sqrt{w_n}$, $n \geq 1$, is an orthonormal basis of $\ell^2(\mathbb{N}, \mu_w)$, and

$$U_w : \ell^2(\mathbb{N}, \mu_w) \longrightarrow \ell^2(\mathbb{N}), \qquad (U_w f)_n = \sqrt{w_n}\, f(n),$$

is unitary. This identification reduces the weighted kernel operators considered below to diagonal operators on $\ell^2(\mathbb{N})$.

For a compact operator $T : H_1 \to H_2$, the singular values $s_n(T)$ are the eigenvalues of $|T| = (T^*T)^{1/2}$, counted with multiplicity and arranged in non-increasing order. For $1 \leq p < \infty$, the Schatten class $S_p(H_1, H_2)$ consists of the compact operators with $\|T\|_{S_p}^p = \sum_{n\geq 1} s_n(T)^p < \infty$ and we write $S_p(H)$ when $H_1 = H_2 = H$. In particular, S_1 and S_2 are the trace and Hilbert–Schmidt classes. If $T \in S_1(H)$, then $\mathrm{Tr}(T) = \sum_{n\geq 1} \langle Te_n, e_n\rangle_H$ for any orthonormal basis $\{e_n\}$ of H.

We shall use the following standard relation between an operator and the positive operator it generates; it is a direct consequence of the singular-value decomposition, see [10, Chapter 1]. We state it explicitly because the factorisation $T_k = JJ^*$ is used repeatedly.

Proposition 2.1. *Let $J : H_1 \to H_2$ be bounded and let $1 \leq p < \infty$. Then $JJ^* \in S_p(H_2)$ if and only if $J \in S_{2p}(H_1, H_2)$. Whenever these conditions hold,*

$$s_n(JJ^*) = s_n(J)^2, \qquad n \geq 1, \qquad \text{and} \qquad \|JJ^*\|_{S_p}^p = \|J\|_{S_{2p}}^{2p}.$$

Finally, let $k : X \times X \to \mathbb{C}$ be a positive-definite kernel. By the Moore–Aronszajn theorem [1], there is a unique reproducing kernel Hilbert space H_k with $f(x) = \langle f, k(\cdot, x)\rangle_{H_k}$ for $f \in H_k$ and $x \in X$. If (X, ν) is a measure space and the natural embedding $J : H_k \to L_2(\nu)$ is bounded, then, under the usual measurability assumptions, $(J^*f)(x) = \int_X k(x, y) f(y)\, d\nu(y)$, so that the associated kernel operator admits the factorisation $T_k = JJ^*$. The precise measurable setting is discussed in [12, Lemma 2.2]. These facts become completely explicit for diagonal kernels in the next section.

3. Diagonal Reproducing Kernels and the Induced Operator

Let $k(m, n) = c_n \delta_{mn}$, $m, n \in \mathbb{N}$, $c_n > 0$, and let μ_w be the weighted counting measure $\mu_w(\{n\}) = w_n > 0$.

Lemma 3.1. *The kernel k is positive definite, and its reproducing kernel Hilbert space is*

$$H_k = \left\{ f = \{f(n)\}_{n\geq 1} : \sum_{n=1}^{\infty} \frac{|f(n)|^2}{c_n} < \infty \right\},$$

with inner product

$$\langle f, g\rangle_{H_k} = \sum_{n=1}^{\infty} \frac{f(n)\overline{g(n)}}{c_n}.$$

Moreover,

$$\varphi_n = \sqrt{c_n}\, e_n, \qquad n \geq 1,$$

is an orthonormal basis of H_k.

Proof. For every finitely supported scalar sequence $\{a_n\}$,

$$\sum_{m,n} a_m \overline{a_n}\, k(m,n) = \sum_n c_n |a_n|^2 \geq 0,$$

so k is positive definite. Since $k(\cdot, n) = c_n e_n$, the stated inner product gives

$$\langle f, k(\cdot, n)\rangle_{H_k} = f(n).$$

The assertion about $\{\varphi_n\}$ follows immediately. □

For this kernel, the associated kernel expression defines the maximal operator

$$(T_k^w f)(n) = c_n w_n f(n)$$

on

$$\mathrm{dom}(T_k^w) = \left\{ f \in \ell^2(\mathbb{N}, \mu_w) : \sum_{n=1}^{\infty} (c_n w_n)^2 |f(n)|^2 w_n < \infty \right\}.$$

Proposition 3.2. *Let*

$$U_w : \ell^2(\mathbb{N}, \mu_w) \longrightarrow \ell^2(\mathbb{N}), \qquad (U_w f)_n = \sqrt{w_n}\, f(n).$$

Then, with the corresponding maximal domains,

$$U_w T_k^w U_w^{-1} = \mathrm{diag}(c_1 w_1, c_2 w_2, \ldots).$$

Consequently, T_k^w is positive and self-adjoint.

Proof. If $x = U_w f$, then

$$(U_w T_k^w f)_n = \sqrt{w_n}\, c_n w_n f(n) = c_n w_n x_n.$$

The remaining assertions follow from the corresponding properties of the maximal positive diagonal operator on $\ell^2(\mathbb{N})$. □

The preceding representation reduces the required operator properties to the standard theory of diagonal operators.

Theorem 3.3. *Let $1 \leq p < \infty$. Then:*

1. *T_k^w is bounded if and only if $\sup_{n\geq 1} c_n w_n < \infty$, and in this case*
$$\|T_k^w\| = \sup_{n\geq 1} c_n w_n.$$
2. *T_k^w is compact if and only if $c_n w_n \longrightarrow 0$.*
3. *$T_k^w \in S_p \iff \sum_{n=1}^{\infty} (c_n w_n)^p < \infty$. Whenever this holds,*
$$\|T_k^w\|_{S_p}^p = \sum_{n=1}^{\infty} (c_n w_n)^p,$$

and the singular values are the non-increasing rearrangement of $\{c_n w_n\}_{n\geq 1}$.

Proof. By Proposition 3.2, T_k^w is unitarily equivalent to the positive diagonal operator $\mathrm{diag}(c_1 w_1, c_2 w_2, \ldots)$. The boundedness and compactness criteria are therefore immediate. For the Schatten class, the singular values are the non-increasing rearrangement of the diagonal entries, which gives the stated criterion and norm identity. □

For $p = 1$, Theorem 3.3 gives

$$T_k^w \in S_1 \quad \Longleftrightarrow \quad \sum_{n=1}^{\infty} c_n w_n < \infty.$$

Since T_k^w is positive,

$$\operatorname{Tr}(T_k^w) = \sum_{n=1}^{\infty} c_n w_n = \int_{\mathbb{N}} k(n,n)\, d\mu_w(n).$$

Thus, in the present diagonal setting, diagonal integrability is both necessary and sufficient for the induced operator to be trace class; compare [12, Lemma 2.3].

4. The RKHS Embedding Operator

We now consider the natural embedding of the reproducing kernel Hilbert space H_k into the weighted sequence space $\ell^2(\mathbb{N}, \mu_w)$. In the diagonal setting, this operator also admits an explicit representation, which makes its relation to T_k^w transparent.

Proposition 4.1. *The natural identity map on finitely supported sequences extends to a bounded operator $J : H_k \longrightarrow \ell^2(\mathbb{N}, \mu_w)$ if and only if*

$$\sup_{n \geq 1} c_n w_n < \infty. \tag{4.1}$$

In this case,

$$\|J\|^2 = \sup_{n \geq 1} c_n w_n. \tag{4.2}$$

Proof. For $f \in H_k$,

$$\|Jf\|^2_{\ell^2(\mathbb{N},\mu_w)} = \sum_{n=1}^{\infty} |f(n)|^2 w_n = \sum_{n=1}^{\infty} \frac{|f(n)|^2}{c_n}\, c_n w_n \leq \left(\sup_{n \geq 1} c_n w_n \right) \|f\|^2_{H_k}.$$

Hence (4.1) implies that J is bounded.

Conversely, let $\varphi_n = \sqrt{c_n}\, e_n$, which is an orthonormal basis of H_k by Lemma 3.1. Then $\|J\varphi_n\|^2_{\ell^2(\mathbb{N},\mu_w)} = c_n w_n$. Therefore, if J is bounded, $c_n w_n \leq \|J\|^2$ for every n. Taking the supremum gives the converse implication and, together with the preceding estimate, proves (4.2). □

Lemma 4.2. *Assume that $J : H_k \longrightarrow \ell^2(\mathbb{N}, \mu_w)$ is bounded. Then*

$$(J^* g)(n) = c_n w_n g(n), \qquad g \in \ell^2(\mathbb{N}, \mu_w), \qquad n \geq 1.$$

Proof. Let $g \in \ell^2(\mathbb{N}, \mu_w)$ and define

$$h(n) = c_n w_n g(n).$$

Since J is bounded, Proposition 4.1 gives

$$M := \sup_{n \geq 1} c_n w_n < \infty.$$

Therefore

$$\|h\|^2_{H_k} = \sum_{n=1}^{\infty} \frac{|c_n w_n g(n)|^2}{c_n} = \sum_{n=1}^{\infty} c_n w_n^2 |g(n)|^2 \leq M \sum_{n=1}^{\infty} |g(n)|^2 w_n < \infty.$$

Hence $h \in H_k$.

For every $f \in H_k$,

$$\langle Jf, g\rangle_{\ell^2(\mathbb{N},\mu_w)} = \sum_{n=1}^{\infty} f(n)\overline{g(n)}\, w_n = \sum_{n=1}^{\infty} \frac{f(n)\overline{c_n w_n g(n)}}{c_n} = \langle f, h\rangle_{H_k}.$$

Thus $J^*g = h$, and therefore $(J^*g)(n) = c_n w_n g(n)$. □

The embedding J has a diagonal representation analogous to that of the integral operator.

Proposition 4.3. *Assume that $J : H_k \to \ell^2(\mathbb{N}, \mu_w)$ is bounded. Define*

$$V_c : H_k \longrightarrow \ell^2(\mathbb{N}), \qquad (V_c f)_n = \frac{f(n)}{\sqrt{c_n}}.$$

Then V_c is unitary and

$$U_w J V_c^{-1} = D_{\sqrt{\lambda}}, \qquad D_{\sqrt{\lambda}} = \operatorname{diag}\big(\sqrt{c_1 w_1}, \sqrt{c_2 w_2}, \ldots\big). \tag{4.3}$$

Moreover,

$$T_k^w = JJ^*. \tag{4.4}$$

Proof. For $f \in H_k$,

$$\|V_c f\|_{\ell^2}^2 = \sum_{n=1}^{\infty} \frac{|f(n)|^2}{c_n} = \|f\|_{H_k}^2,$$

so V_c is unitary.

If $x \in \ell^2(\mathbb{N})$, then

$$(V_c^{-1}x)(n) = \sqrt{c_n}\, x_n.$$

Consequently,

$$(U_w J V_c^{-1} x)_n = \sqrt{w_n}\sqrt{c_n}\, x_n = \sqrt{c_n w_n}\, x_n,$$

which proves (4.3).

To verify the factorisation, let $g \in \ell^2(\mathbb{N}, \mu_w)$. By Lemma 4.2,

$$(J^*g)(n) = c_n w_n g(n).$$

Hence

$$(JJ^*g)(n) = c_n w_n g(n) = (T_k^w g)(n),$$

and therefore (4.4) holds. □

We can now compare the Schatten properties of the integral operator and of the RKHS embedding.

Theorem 4.4. *Let $1 \le p < \infty$. The following conditions are equivalent:*

1. *$T_k^w \in S_p\big(\ell^2(\mathbb{N}, \mu_w)\big)$;*
2. *the natural identity map extends to an operator $J \in S_{2p}\big(H_k, \ell^2(\mathbb{N}, \mu_w)\big)$;*
3. *$\{c_n w_n\}_{n\ge 1} \in \ell^p$.*

Whenever these conditions hold,

$$\|T_k^w\|_{S_p}^p = \|J\|_{S_{2p}}^{2p} = \sum_{n=1}^{\infty} (c_n w_n)^p.$$

Proof. By Theorem 3.3,

$$T_k^w \in S_p \quad \Longleftrightarrow \quad \{c_n w_n\}_{n\geq 1} \in \ell^p,$$

and, whenever these conditions hold,

$$\|T_k^w\|_{S_p}^p = \sum_{n=1}^{\infty} (c_n w_n)^p.$$

Assume that $\{c_n w_n\}_{n\geq 1} \in \ell^p$. Then $\{c_n w_n\}$ is bounded, so Proposition 4.1 shows that $J : H_k \longrightarrow \ell^2(\mathbb{N}, \mu_w)$ is bounded. By Proposition 4.3, $T_k^w = JJ^*$. Hence Proposition 2.1 gives

$$T_k^w \in S_p \quad \Longleftrightarrow \quad J \in S_{2p},$$

together with

$$\|T_k^w\|_{S_p}^p = \|J\|_{S_{2p}}^{2p}.$$

Conversely, if $J \in S_{2p}$, then J is bounded and $T_k^w = JJ^*$. Proposition 2.1 therefore yields $T_k^w \in S_p$, and Theorem 3.3 implies $\{c_n w_n\}_{n\geq 1} \in \ell^p$.

Combining these relations gives the stated equivalences and the norm identity. □

5. Changes of Measure and Schatten Stability

The preceding sections describe the Schatten properties of a kernel operator for a fixed weighted measure. We now consider how these properties change when the underlying measure is replaced by another one while the kernel remains fixed.

We first record a general measure-change principle for RKHS embeddings and the corresponding kernel operators. Related change-of-measure identities for kernel covariance embeddings appear in [9]. The proposition below records the form needed here for the comparison of RKHS embeddings and Schatten properties.

Proposition 5.1. *Let (X, μ) be a measure space and let*

$$d\nu = h\, d\mu,$$

where $h : X \to (0, \infty)$ is measurable and $h \in L_\infty(\mu)$.

Let k be a measurable positive-definite kernel with associated RKHS H_k, and assume that the natural embedding $J_\mu : H_k \longrightarrow L_2(\mu)$ is bounded.

Then the natural embedding $J_\nu : H_k \longrightarrow L_2(\nu)$ is also bounded. Moreover, the operator

$$U_h : L_2(\nu) \longrightarrow L_2(\mu), \qquad U_h f = \sqrt{h}\, f,$$

is unitary and

$$U_h J_\nu = M_{\sqrt{h}} J_\mu, \tag{5.1}$$

where $M_{\sqrt{h}}$ denotes multiplication by $\sqrt{h}$ on $L_2(\mu)$.

Consequently, for the associated kernel operators

$$T_k^\mu = J_\mu J_\mu^*, \qquad T_k^\nu = J_\nu J_\nu^*,$$

one has

$$U_h T_k^\nu U_h^{-1} = M_{\sqrt{h}}\, T_k^\mu\, M_{\sqrt{h}}. \tag{5.2}$$

In particular, for every $1 \le p < \infty$,

$$T_k^\mu \in S_p \quad \Longrightarrow \quad T_k^\nu \in S_p,$$

and

$$\|T_k^\nu\|_{S_p} \le \|h\|_{L_\infty(\mu)} \|T_k^\mu\|_{S_p}. \tag{5.3}$$

Similarly, for every $1 \le q < \infty$,

$$J_\mu \in S_q \quad \Longrightarrow \quad J_\nu \in S_q,$$

with

$$\|J_\nu\|_{S_q} \le \|h\|_{L_\infty(\mu)}^{1/2} \|J_\mu\|_{S_q}. \tag{5.4}$$

Proof. For $f \in H_k$,

$$\|J_\nu f\|_{L_2(\nu)}^2 = \int_X |f|^2 h \, d\mu \le \|h\|_{L_\infty(\mu)} \|J_\mu f\|_{L_2(\mu)}^2,$$

so J_ν is bounded with

$$\|J_\nu\| \le \|h\|_{L_\infty(\mu)}^{1/2} \|J_\mu\|.$$

For $f \in L_2(\nu)$,

$$\|U_h f\|_{L_2(\mu)}^2 = \int_X h|f|^2 \, d\mu = \|f\|_{L_2(\nu)}^2,$$

while for every $g \in L_2(\mu)$ one has $g/\sqrt{h} \in L_2(\nu)$ and $U_h(g/\sqrt{h}) = g$. Hence U_h is unitary with $U_h^{-1} g = g/\sqrt{h}$. Finally, $U_h J_\nu f = \sqrt{h}\, f = M_{\sqrt{h}} J_\mu f$ for $f \in H_k$, which proves (5.1).

Since $T_k^\mu = J_\mu J_\mu^*$ and $T_k^\nu = J_\nu J_\nu^*$,

$$U_h T_k^\nu U_h^{-1} = (U_h J_\nu)(U_h J_\nu)^* = M_{\sqrt{h}} T_k^\mu M_{\sqrt{h}},$$

which is (5.2). As $\|M_{\sqrt{h}}\| = \|h\|_{L_\infty(\mu)}^{1/2}$, the ideal property of the Schatten classes gives

$$\|T_k^\nu\|_{S_p} = \|M_{\sqrt{h}} T_k^\mu M_{\sqrt{h}}\|_{S_p} \le \|M_{\sqrt{h}}\|^2 \|T_k^\mu\|_{S_p} = \|h\|_{L_\infty(\mu)} \|T_k^\mu\|_{S_p},$$

which is (5.3). Finally, by (5.1) and unitary invariance, $\|J_\nu\|_{S_q} = \|M_{\sqrt{h}} J_\mu\|_{S_q} \le \|h\|_{L_\infty(\mu)}^{1/2} \|J_\mu\|_{S_q}$. □

Theorem 5.2. *Let*

$$w = \{w_n\}_{n \ge 1}, \qquad v = \{v_n\}_{n \ge 1}$$

be two sequences of positive weights, let $1 \le p, q < \infty$, *and put*

$$r_n = \frac{v_n}{w_n}, \qquad n \ge 1.$$

The following conditions are equivalent:

1. For every positive sequence $\{c_n\}_{n \ge 1}$ *and the corresponding diagonal positive-definite kernel* $k(m,n) = c_n \delta_{mn}$, *one has*

$$T_k^w \in S_p \quad \Longrightarrow \quad T_k^v \in S_q.$$

2. For every such diagonal kernel,

$$J_w \in S_{2p} \quad \Longrightarrow \quad J_v \in S_{2q},$$

where

$$J_w : H_k \to \ell^2(\mathbb{N}, \mu_w), \qquad J_v : H_k \to \ell^2(\mathbb{N}, \mu_v)$$

are the corresponding natural embeddings.

3. The ratio sequence $r = \{r_n\}_{n\geq 1}$ *satisfies*

$$r \in \ell^\infty, \qquad \text{if } p \leq q,$$

whereas, if $q < p$,

$$r \in \ell^s, \qquad \frac{1}{s} = \frac{1}{q} - \frac{1}{p},$$

or equivalently,

$$s = \frac{pq}{p-q}.$$

Moreover, whenever these conditions hold,

$$\|T_k^v\|_{S_q} \leq \begin{cases} \|r\|_{\ell^\infty} \|T_k^w\|_{S_p}, & p \leq q, \\ \|r\|_{\ell^s} \|T_k^w\|_{S_p}, & q < p, \end{cases} \tag{5.5}$$

and

$$\|J_v\|_{S_{2q}} \leq \begin{cases} \|r\|_{\ell^\infty}^{1/2} \|J_w\|_{S_{2p}}, & p \leq q, \\ \|r\|_{\ell^s}^{1/2} \|J_w\|_{S_{2p}}, & q < p. \end{cases}$$

In particular, when $p = q$, *preservation of* S_p *for every positive diagonal kernel is equivalent to*

$$\sup_{n\geq 1} \frac{v_n}{w_n} < \infty.$$

Proof. For a fixed positive diagonal kernel, put

$$a_n = c_n w_n, \qquad r_n = \frac{v_n}{w_n}, \qquad n \geq 1.$$

Then

$$c_n v_n = r_n a_n.$$

By Theorem 3.3,

$$T_k^w \in S_p \quad \Longleftrightarrow \quad a \in \ell^p,$$

whereas

$$T_k^v \in S_q \quad \Longleftrightarrow \quad ra \in \ell^q.$$

Since every strictly positive sequence $a \in \ell^p$ can be written as $a_n = c_n w_n$ with $c_n > 0$, condition (1) first implies that $ra \in \ell^q$ for every strictly positive $a \in \ell^p$.

For any $a \in \ell^p$, define

$$b_n = |a_n| + 2^{-n}, \qquad n \geq 1.$$

Then $b = \{b_n\}_{n\geq 1}$ is strictly positive and belongs to ℓ^p, since $\{2^{-n}\}_{n\geq 1} \in \ell^p$. Hence $rb \in \ell^q$, and since

$$|r_n a_n| \leq r_n b_n, \qquad n \geq 1,$$

it follows that $ra \in \ell^q$.

Thus the coordinate multiplier

$$M_r : \ell^p \longrightarrow \ell^q, \qquad M_r(a) = ra,$$

is defined on all of ℓ^p.

Its graph is closed. Indeed, if $a^{(j)} \to a$ in ℓ^p and $M_r a^{(j)} \to b$ in ℓ^q, then coordinatewise convergence gives

$$b_n = r_n a_n, \qquad n \geq 1.$$

Hence, by the closed graph theorem, M_r is bounded. Conversely, boundedness of M_r immediately implies condition (1).

It therefore remains to invoke the classical characterization of coordinate multipliers between sequence spaces, namely

$$M_r \in \mathcal{B}(\ell^p, \ell^q) \quad \Longleftrightarrow \quad \begin{cases} r \in \ell^\infty, & p \leq q, \\ r \in \ell^s, \quad \dfrac{1}{s} = \dfrac{1}{q} - \dfrac{1}{p}, & q < p, \end{cases}$$

together with

$$\|M_r\| = \begin{cases} \|r\|_{\ell^\infty}, & p \leq q, \\ \|r\|_{\ell^s}, & q < p; \end{cases}$$

see [6, p. 60] and [7].

This proves the equivalence between conditions (1) and (3), together with the corresponding estimate for the kernel operators.

Finally, condition (2) is equivalent to condition (1) by Theorem 4.4. Since

$$\|J_w\|^2_{S_{2p}} = \|T^w_k\|_{S_p}, \qquad \|J_v\|^2_{S_{2q}} = \|T^v_k\|_{S_q},$$

the estimates for the embeddings follow as well. □

An immediate consequence of Theorem 5.2 is that the condition for preserving S_p when $p = q$ does not depend on p. Therefore, if one Schatten class S_p is preserved for all positive diagonal kernels, then every Schatten class S_t, $1 \leq t < \infty$, is preserved as well.

The universal quantifier over all positive diagonal kernels is essential in this statement. For a single fixed kernel, Schatten membership may remain unchanged even when the ratio of the two weights is unbounded. For example, if

$$c_n = 2^{-n}, \qquad w_n = 1, \qquad v_n = n,$$

then $v_n/w_n = n$ is unbounded, whereas both $\{c_n w_n\}$ and $\{c_n v_n\}$ belong to every ℓ^p, $1 \leq p < \infty$.

Corollary 5.3. *Let $1 \leq p < \infty$. The following conditions are equivalent:*

1. The weights w and v are equivalent, that is, there exist constants $0 < m \leq M < \infty$ such that

$$m w_n \leq v_n \leq M w_n, \qquad n \geq 1.$$

2. For every positive diagonal kernel,

$$T^w_k \in S_p \quad \Longleftrightarrow \quad T^v_k \in S_p.$$

3. For every positive diagonal kernel,

$$J_w \in S_{2p} \quad \Longleftrightarrow \quad J_v \in S_{2p}.$$

Moreover, whenever these conditions hold,

$$m\|T_k^w\|_{S_p} \le \|T_k^v\|_{S_p} \le M\|T_k^w\|_{S_p}.$$

Proof. Taking $q = p$ in Theorem 5.2, the implication

$$T_k^w \in S_p \quad \Longrightarrow \quad T_k^v \in S_p$$

for every positive diagonal kernel is equivalent to $\sup_{n\ge1} \frac{v_n}{w_n} < \infty$. Applying the same result with w and v interchanged, the reverse implication is equivalent to $\sup_{n\ge1} \frac{w_n}{v_n} < \infty$. Together these conditions are equivalent to the existence of constants $0 < m \le M < \infty$ such that

$$mw_n \le v_n \le Mw_n, \qquad n \ge 1.$$

The norm estimates follow from the corresponding estimates in Theorem 5.2. □

Although T_k^w and T_k^v act on different Hilbert spaces, their unitary realizations on $\ell^2(\mathbb{N})$ can be compared directly.

Theorem 5.4. *Let $1 \le p < \infty$ and let w and v be positive weight sequences. Assume that $T_k^w \in S_p$ and*

$$\sum_{n=1}^{\infty} c_n^p |v_n - w_n|^p < \infty. \tag{5.6}$$

Then $T_k^v \in S_p$. Moreover,

$$\left\| U_v T_k^v U_v^{-1} - U_w T_k^w U_w^{-1} \right\|_{S_p}^p = \sum_{n=1}^{\infty} c_n^p |v_n - w_n|^p. \tag{5.7}$$

Consequently,

$$\|T_k^v\|_{S_p} \le \|T_k^w\|_{S_p} + \left(\sum_{n=1}^{\infty} c_n^p |v_n - w_n|^p \right)^{1/p}.$$

Proof. By Proposition 3.2, $U_w T_k^w U_w^{-1} = \operatorname{diag}(c_n w_n)$ and $U_v T_k^v U_v^{-1} = \operatorname{diag}(c_n v_n)$, so their difference is $\operatorname{diag}\big(c_n(v_n - w_n)\big)$, whose singular values are the non-increasing rearrangement of $\{c_n|v_n - w_n|\}_{n\ge1}$. Hence (5.6) implies that this difference lies in S_p, and (5.7) follows. Since $U_w T_k^w U_w^{-1} \in S_p$, we get $U_v T_k^v U_v^{-1} \in S_p$ and therefore $T_k^v \in S_p$. The final estimate is the triangle inequality for the Schatten norm. □

Corollary 5.5. *Assume that $T_k^w \in S_1$ and*

$$\sum_{n=1}^{\infty} c_n |v_n - w_n| < \infty.$$

Then $T_k^v \in S_1$ and

$$\operatorname{Tr}(T_k^v) - \operatorname{Tr}(T_k^w) = \sum_{n=1}^{\infty} c_n (v_n - w_n).$$

In particular,

$$|\operatorname{Tr}(T_k^v) - \operatorname{Tr}(T_k^w)| \le \sum_{n=1}^{\infty} c_n |v_n - w_n|.$$

Proof. Theorem 5.4 with $p = 1$ gives $T_k^v \in S_1$. Since

$$\mathrm{Tr}(T_k^w) = \sum_{n\geq 1} c_n w_n$$

and

$$\mathrm{Tr}(T_k^v) = \sum_{n\geq 1} c_n v_n,$$

we obtain

$$\mathrm{Tr}(T_k^v) - \mathrm{Tr}(T_k^w) = \sum_{n\geq 1} c_n(v_n - w_n).$$

The series on the right converges absolutely by assumption, which gives the stated estimate. □

6. Schatten Thresholds for Power and Regularly Varying Weights

We conclude by applying the preceding criteria to several model families. These examples illustrate how the decay of the kernel coefficients interacts with the choice of the weighted measure and, in particular, how a change of weight may alter the Schatten class of the induced operator.

We first consider power-type kernel coefficients and weights.

Example 6.1. Let $c_n = n^{-\alpha}$ and $w_n = n^{\beta}$ with $\alpha, \beta \in \mathbb{R}$, and put $d = \alpha - \beta$, so that $c_n w_n = n^{-d}$. By Theorem 3.3, T_k^w is bounded if and only if $d \geq 0$, compact if and only if $d > 0$, and, for $1 \leq p < \infty$,

$$T_k^w \in S_p \quad \Longleftrightarrow \quad \sum_{n=1}^{\infty} n^{-pd} < \infty \quad \Longleftrightarrow \quad d > \frac{1}{p}. \tag{6.1}$$

In particular, $T_k^w \in S_1$ exactly when $d > 1$, and $T_k^w \in S_2$ exactly when $d > \frac{1}{2}$; by Theorem 4.4, $J \in S_{2p}$ under the same condition $pd > 1$. If $d > 0$, the sequence $\{n^{-d}\}$ is decreasing, so $s_n(T_k^w) = n^{-d}$ and $s_n(J) = n^{-d/2}$, whence $\|T_k^w\|_{S_p}^p = \zeta(pd)$ whenever $pd > 1$. At the critical value $pd = 1$ one has $\sum_{n\leq N} s_n(T_k^w)^p = \log N + O(1)$ as $N \to \infty$, so the pure power model fails to belong to S_p, with logarithmic divergence.

The power model can be refined by allowing a slowly varying factor. For background on borderline singular-number estimates and refined operator ideals, see [4]. The following consequence of the standard theory of slowly varying functions gives the corresponding Schatten thresholds in the present diagonal setting (see [3, Chapter 1]).

Proposition 6.2. *Assume that* $c_n w_n = n^{-d} L(n), \qquad n \geq 2,$ *where* $d \in \mathbb{R}$ *and* $L : [2, \infty) \longrightarrow (0, \infty)$ *is a measurable slowly varying function, that is,*

$$\frac{L(tx)}{L(x)} \longrightarrow 1 \qquad (x \to \infty)$$

for every $t > 0$.

Then the following assertions hold.

1. If $d > 0$, then T_k^w is compact.
2. If $d < 0$, then T_k^w is unbounded.
3. If $d = 0$, then T_k^w is bounded if and only if $\sup_{n\geq 2} L(n) < \infty$, and it is compact if and only if $L(n) \longrightarrow 0 \qquad (n \to \infty)$.
4. Let $1 \leq p < \infty$ and $d > 0$. If $pd > 1$, then $T_k^w \in S_p$, whereas if $pd < 1$, then $T_k^w \notin S_p$. At the critical value $pd = 1$,

$$T_k^w \in S_p \quad \Longleftrightarrow \quad \sum_{n=2}^{\infty} \frac{L(n)^p}{n} < \infty.$$

Proof. Since L is positive and slowly varying, for every $\varepsilon > 0$ we have $n^{-\varepsilon}L(n) \to 0$ and $n^{\varepsilon}L(n) \to \infty$ as $n \to \infty$; see, for example, [3].

If $d > 0$, choose $0 < \varepsilon < d$; then $n^{-d}L(n) = n^{-(d-\varepsilon)}\big(n^{-\varepsilon}L(n)\big) \to 0$, so T_k^w is compact by Theorem 3.3. If $d < 0$, write $-d = a > 0$ and choose $0 < \varepsilon < a$; then $n^{-d}L(n) = n^{a-\varepsilon}\big(n^{\varepsilon}L(n)\big) \to \infty$, so T_k^w is unbounded. If $d = 0$, then $c_n w_n = L(n)$ and the assertions follow from the boundedness and compactness criteria in Theorem 3.3.

Finally, let $1 \leq p < \infty$ and $d > 0$. By Theorem 3.3, $T_k^w \in S_p$ if and only if $\sum_{n\geq 2} n^{-pd}L(n)^p < \infty$. Since L^p is slowly varying, Potter's bounds [3, Theorem 1.5.6] give, for every $\varepsilon > 0$, constants $c, C > 0$ and N with $c\,n^{-\varepsilon} \leq L(n)^p \leq C\,n^{\varepsilon}$ for $n \geq N$. If $pd > 1$, choosing ε with $pd - \varepsilon > 1$ yields $n^{-pd}L(n)^p \leq Cn^{-pd+\varepsilon}$ for $n \geq N$, and the comparison series converges. If $pd < 1$, choosing ε with $pd + \varepsilon < 1$ yields $n^{-pd}L(n)^p \geq c\,n^{-pd-\varepsilon}$ for $n \geq N$, and the comparison series diverges. At $pd = 1$ the Schatten condition becomes exactly $\sum_{n\geq 2} L(n)^p/n < \infty$. □

Thus, a slowly varying factor may change the behavior at the critical threshold. The logarithmic correction gives a simple explicit example.

Corollary 6.3. *Let $1 \leq p < \infty$ and assume that*

$$c_n w_n = n^{-d}(\log n)^{\gamma}, \qquad n \geq 2,$$

where $d > 0$ and $\gamma \in \mathbb{R}$. At the critical value $d = \frac{1}{p}$, one has

$$T_k^w \in S_p \quad \Longleftrightarrow \quad \gamma < -\frac{1}{p}.$$

Proof. At $d = 1/p$,

$$\sum_{n=2}^{\infty}(c_n w_n)^p = \sum_{n=2}^{\infty} \frac{(\log n)^{p\gamma}}{n}.$$

The latter series converges if and only if $p\gamma < -1$, which is equivalent to $\gamma < -\frac{1}{p}$. □

We finish with a single example showing how different weighted measures may place the operators generated by the same kernel in different Schatten classes.

Example 6.4. Fix $c_n = 1/n$. For $w_n = 1/n$ one has $c_n w_n = n^{-2}$, so $T_k^w \in S_1$ with $\mathrm{Tr}(T_k^w) = \pi^2/6$ and $J_w \in S_2$. For $w_n = 1$ one has $c_n w_n = 1/n$, hence $T_k^w \in S_p$ exactly when $p > 1$; in particular $T_k^w \in S_2 \setminus S_1$. This kernel with the

counting measure also appears in [12, Example 2.9], where it is used in a different context; here it identifies the precise position of the corresponding operator in the Schatten scale. Finally, for $w_n = n^{1/2}$ one has $c_n w_n = n^{-1/2}$, so T_k^w is compact but $T_k^w \notin S_2$, and $T_k^w \in S_p$ exactly when $p > 2$. Thus a fixed diagonal kernel may generate a trace-class operator, a Hilbert–Schmidt operator which is not trace class, or a compact operator which is not Hilbert–Schmidt, depending only on the choice of the weighted measure, in accordance with the weight-comparison results of Section 5.

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