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CONCAVE-CONVEX PROBLEM FOR THE ADVECTIVE p-LAPLACIAN

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Abstract. In this study, we investigate the existence and multiplicity of nonnegative solutions for a concave-concave type problem involving the p Laplacian and an advection term. By combining minimization on the Nehari manifold with the fibering map method, we prove that, for every sufficiently small parameter $\lambda > 0$, the problem admits at least two distinct nonnegative weak solutions. The solutions are obtained as minimizers of the associated energy functional on the positive and negative components of the Nehari manifold. The analysis also shows that the degenerate component of the manifold is empty for sufficiently small λ.

Keywords. p-Laplacian, advection term, concave-concave problem, nonnegative solutions, Nehari manifold

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1. Introduction and main results

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$, $0 < \alpha < 1 < \gamma < \frac{N+2}{N-2}$, $\lambda > 0$ and $f, g : \Omega \longrightarrow \mathbb{R}$ be smooth functions. The work [6] examined the existence of multiple solutions of the following semilinear elliptic boundary problem

$$\begin{cases} \text{Find } (\lambda, u) \in (0,\infty) \times W_0^{1,2}(\Omega) \text{ s.t.} & \\ -\Delta u(x) = \lambda f(x) u^\alpha + g(x) u^\gamma, & \text{in } \Omega, \\ 0 = u(x), & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

The authors used the Nehari manifold combined with the fibering maps approach to prove the existence of at least two positive solutions for (1.1). Analogous equations have been investigated using a variety of approaches such as Mountain Pass Theorem, the Nehari manifold, sub-supersolution methods, and eigencurve technique (see [9, 12, 19]). The purpose of this paper is to study the multiplicity results of nonnegative solutions for the so-called concave-convex type problem

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involving the p-Laplace operator

$$\begin{cases} \text{Find } (\beta, \lambda, u) \in \mathbb{R}^N \times (0,\infty) \times W_0^{1,p}(\Omega) \text{ s.t.} \\ -\Delta_p u(x) - \beta \cdot |\nabla u|^{p-2}\nabla u = \lambda a(x) u^q + b(x) u^r, & \text{in } \Omega, \\ 0 \leq u, 0 \not\equiv u, & \text{in } \Omega, \\ 0 = u(x), & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where the p-Laplace operator is defined on Ω by $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and $\beta \in \mathbb{R}^N$. The rest of the parameters satisfy the following standing assumptions:

(H) $0 < q < p-1 < r < \frac{N(p-1)+p}{N-p} =: p^* - 1$ and $a, b : \Omega \longrightarrow \mathbb{R}$ are smooth functions which may change sign on Ω such that $a^+ \not\equiv 0, b^+ \not\equiv 0$.

The first fairly complete analysis of (1.2) when $\beta \equiv 0_{\mathbb{R}^N}$ was given in [15] by minimization on the Nehari manifold, an approach that turns out to be an efficient method for critical points of functionals lacking coercivity. Turning back to our concern, the system (1.2) can be regarded as an extension of (1.1) and it is worth mentioning that the presence of $\beta \not\equiv 0_{\mathbb{R}^N}$, the possibly sign-changing and unbounded weights considered in (1.2) prevent the functional energy $J_{\beta,\lambda}$ defined below from being coercive and lead to features different from related topics in the literature. Inspired by previous works, the current one aims to show that problem (1.2) admits at least two nonnegative solutions when the parameter λ is sufficiently small. Our investigation is mainly based on the fibering maps approach combined with the minimization on Nehari manifold. These methods have been successfully used in the literature by several authors to study steady-state solutions of parabolic equations driven by different diffusion operators such as variable $p(\cdot)$-Laplacian operator, p-biharmonic operator, fractal p-Laplacian operator (see for instance [2, 3, 9, 11]). It is well established from [4, 18] that the system (1.2) is equivalent to:

$$\begin{cases} \text{Find } (\lambda, u) \in (0,\infty) \times W_0^{1,p}(\Omega) \text{ s.t.} \\ -\Delta_p^\beta u(x) = \lambda a(x) \mathrm{e}^{\beta\cdot x} u^q + b(x)\mathrm{e}^{\beta\cdot x} u^r, & \text{in } \Omega, \\ 0 = u(x), & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\Delta_p^\beta u(x) := \operatorname{div}(\mathrm{e}^{\beta\cdot x}|\nabla u|^{p-2}\nabla u)$. Hence, the functional energy associated to (1.3) is given for $(\beta, \lambda, u) \in \mathbb{R}^N \times (0,\infty) \times W_0^{1,p}(\Omega)$ by

$$J_{\beta,\lambda}(u) = \frac{1}{p} I_\beta(u) - \frac{\lambda}{q+1} A_\beta(u) - \frac{1}{r+1} B_\beta(u) \tag{1.4}$$

with

$$I_\beta(u) = \int_\Omega \mathrm{e}^{\beta\cdot x}|\nabla u|^p dx;$$

$$A_\beta(u) = \int_\Omega a(x)\mathrm{e}^{\beta\cdot x}|u|^{q+1} dx;$$

and

$$B_\beta(u) = \int_\Omega b(x)\mathrm{e}^{\beta\cdot x}|u|^{r+1} dx.$$

Due to [17], the critical points of $J_{\beta,\lambda}$ solve (1.3) and then give solutions for (1.2). Our main result concerns the existence of at least two nonnegative solutions for (1.2) reads as follows.

Theorem 1.1. *Fix $\beta \in \mathbb{R}^N$ and assume* **(H)**. *There exists $\lambda_* > 0$ such that for $\lambda \in (0, \lambda_*)$, the system (1.2) admits at least two nonnegative solutions that is, there exist $0 \leq u_1, u_2 \in W_0^{1,p}(\Omega)$ such that (β, λ, u_1) and (β, λ, u_2) are solutions of (1.2) for all $\lambda \in (0, \lambda_*)$.*

Remark 1.2. Assume that the assumptions of Theorem (1.1) hold.

(i) When $p = 2$ and $\beta \equiv 0_{\mathbb{R}^N}$, our result in Theorem (1.1) generalizes for the p-Laplacian operator, the result of [6, Corollary 4.3].
(ii) When $\beta \equiv 0_{\mathbb{R}^N}$, our result covers [14, Theorem 1 (a)] when Robin boundary condition is imposed and improves those in [16] and related references therein.

Under an additional assumption, we establish the existence of positive solutions, thereby extending the results of [6] and [13].

Theorem 1.3. *Assume* **(H)** *and suppose, in addition, that $a(x) \geq 0$ in Ω. Then every nontrivial nonnegative weak solution u of (1.2) is strictly positive in Ω.*

The solutions of (1.2) can be viewed as steady-state solutions of the following nonlinear diffusion equation:

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta_p u + f(x,u), & \text{in } \Omega \times (0,\infty), \\ 0 = u(x,\cdot), & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) \geq 0, & \text{in } \Omega, \end{cases} \tag{1.5}$$

where $f : \Omega \times (0,\infty) \to \mathbb{R}$ and $u_0 : \Omega \to \mathbb{R}$ are given functions. In particular, a steady-state solution is a time-independent solution of (1.5) and therefore satisfies the corresponding stationary problem (1.2). Nonlinear reaction-diffusion equations of the form (1.5) have attracted considerable attention due to their relevance in a wide range of applications, including image restoration, the modeling of electrorheological fluids, and more broadly, nonlinear diffusion processes; see for instance, [1, 7, 8] and the references therein. From a mathematical perspective, steady-states play a fundamental role in understanding the long-time behavior of solutions to evolution equations. Indeed, they provide equilibrium configurations around which the dynamics can be analyzed and often serve as key objects in the study of stability, asymptotic behavior, and pattern formation. Consequently, the existence and qualitative properties of steady-state solutions are of particular interest.

Motivated by these considerations, we study the stationary problem (1.2). In recent years, stationary nonlinear diffusion problems have received increasing attention under various types of boundary conditions, including no-flux (Neumann), Robin, and nonlocal boundary conditions such as the Steklov boundary condition. These boundary conditions arise naturally in different physical and mathematical settings and lead to substantially different analytical challenges. In particular, the interplay between the nonlinear diffusion operator, the reaction term, and the

boundary condition can have a decisive influence on the existence and qualitative properties of solutions.

The rest of the paper is organized as follows. Section 2 presents some notations and preliminary results associated with the study of (1.2). Section 3 is devoted to the proof of our main results.

2. Notations and preliminaries

2.1. **Notations and definitions.** Let $\Omega \subset \mathbb{R}^N$ (with $N \geq 1$) be a bounded domain with smooth boundary $\partial\Omega$. For $p > 1$, The p-Laplace operator is a prototype of nonlinear elliptic partial differential operators with non-standard growth. It acts on every function $u : \Omega \longrightarrow \mathbb{R}$ as $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. The Sobolev space $W^{1,p}(\Omega)$ will stand for the space of $u \in L^p(\Omega)$ such that $\nabla u \in (L^p(\Omega))^N$ and be endowed with the norm defined for $u \in W^{1,p}(\Omega)$ by

$$\|u\| := \left(\int_\Omega |\nabla u(x)|^p + |u(x)|^p \, dx\right)^{1/p}.$$

The norm of Lebesgue space $L^s(\Omega)$ with $s \in [1, \infty)$ will be denoted $\|\cdot\|_s$ and defined for $u \in L^s(\Omega)$ by

$$\|u\|_s := \left(\int_\Omega |u(x)|^s \, dx\right)^{1/s}.$$

For the sake of simplicity, we adopt conventions to state two independent assertions using symbols $\mp$ and $\lessgtr$. For example, "$E^{\mp} = \emptyset$ if $x \lessgtr 0$" means "$E^- = \emptyset$ if $x < 0$" and "$E^+ = \emptyset$ if $x > 0$". In addition, the negative and positive parts of a function u are defined, respectively, by $u^{\mp} := \max(\mp u, 0)$. Given $u \in C(\bar{\Omega})$, we use the notations $u_{\min} = \min_{x\in\bar{\Omega}} u(x)$, and $u_{\max} = \max_{x\in\bar{\Omega}} u(x)$. The Sobolev space $W_0^{1,p}(\Omega)$ denotes the closure of $C_c^1(\Omega)$ in $W^{1,p}(\Omega)$, equipped with the $W^{1,p}(\Omega)$ norm, is a separable Banach space and reflexive[5]. We start with the following definition.

Definition 2.1. Let E be a real Banach space and $F : E \longrightarrow \mathbb{R}$ be a C^1 functional. The Nehari manifold of F denoted by $\mathscr{N}_F$ is given by

$$\mathscr{N}_F := \{u \in E \,/\, \langle F'(u), u\rangle = 0\} \tag{2.1}$$

where $\langle\cdot,\cdot\rangle$, denotes the pairing between E^* (the dual space of E) and E.

The technique of our analysis regarding (1.2) will involve the Nehari manifold of the functional in (1.4) and it is known from standard arguments that the solvability of (1.2) requires the critical points of (1.4). Due to the indefinite weights in (1.2), its functional is not bounded from below and to recover this coercivity, we restrict the functional to a suitable chosen submanifold of the Nehari manifold. As we are interested in weak solutions of (1.2), we state the following definition.

Definition 2.2. Let $\beta \in \mathbb{R}^N$, $\lambda \in (0,\infty)$ and suppose **(H)**. We say that $u \in W_0^{1,p}(\Omega)$ is a solution of (1.2) if for all $v \in W_0^{1,p}(\Omega)$, we have

$$\int_\Omega \mathrm{e}^{\beta\cdot x}|\nabla u|^{p-2}\nabla u\nabla v dx = \lambda\int_\Omega a(x)\mathrm{e}^{\beta\cdot x}|u|^{q-1}uvdx + \int_\Omega b(x)\mathrm{e}^{\beta\cdot x}|u|^{r-1}uvdx. \quad (2.2)$$

Under the hypotheses of Definition 2.2, one can give the variational formulation of (1.2) by considering the following C^1 functionals on $W_0^{1,p}(\Omega)$:

$$J_{\beta,\lambda}(u) = \frac{1}{p}I_\beta(u) - \frac{\lambda}{q+1}A_\beta(u) - \frac{1}{r+1}B_\beta(u) \quad (2.3)$$

where

$$I_\beta(u) = \int_\Omega \mathrm{e}^{\beta\cdot x}|\nabla u|^p dx; \quad (2.4)$$

$$A_\beta(u) = \int_\Omega a(x)\mathrm{e}^{\beta\cdot x}|u|^{q+1}dx; \quad (2.5)$$

and

$$B_\beta(u) = \int_\Omega b(x)\mathrm{e}^{\beta\cdot x}|u|^{r+1}dx. \quad (2.6)$$

As a result, relation (2.2) becomes

$$\langle J'_{\beta,\lambda}(u), v\rangle = 0 \quad \forall v \in W_0^{1,p}(\Omega), \quad (2.7)$$

so that the Nehari manifold associated to (1.2) reads:

$$\mathscr{N}_{J_{\beta,\lambda}} = \left\{u \in W_0^{1,p}(\Omega) \,/\, I_\beta(u) = \lambda A_\beta(u) + B_\beta(u)\right\}. \quad (2.8)$$

2.2. **Preliminary results.** We start our variational analysis by considering the energy functional $J_{\beta,\lambda}$ in (1.4) on the Nehari manifold $\mathscr{N}_{J_{\beta,\lambda}}$ defined in (2.8). The next lemma proves that $J_{\beta,\lambda}$ is coercive and bounded below on this manifold, providing a key foundation for the variational method.

Lemma 2.3. *Assume* **(H)**. *Then the energy functional $J_{\beta,\lambda}$ is coercive and bounded below on $\mathscr{N}_{J_{\beta,\lambda}}$.*

Proof. Let $u \in \mathscr{N}_{J_{\beta,\lambda}}$. Since

$$I_\beta(u) = \lambda A_\beta(u) + B_\beta(u),$$

we have

$$J_{\beta,\lambda}(u) = \left(\frac{1}{p} - \frac{1}{r+1}\right) I_\beta(u) - \lambda\left(\frac{1}{q+1} - \frac{1}{r+1}\right) A_\beta(u).$$

Define $\psi(x) = e^{\beta\cdot x}$ for $x \in \overline{\Omega}$. By Poincaré's inequality,

$$I_\beta(u) = \int_\Omega e^{\beta\cdot x}|\nabla u|^p\, dx \geq \frac{\psi_{\min}}{1+C_P^p}\|u\|^p.$$

Moreover, by the Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{q+1}(\Omega)$, there exists $S_{q+1} > 0$ such that

$$A_\beta(u) \leq \psi_{\max}\|a\|_\infty\|u\|_{q+1}^{q+1} \leq \psi_{\max}\|a\|_\infty S_{q+1}^{q+1}\|u\|^{q+1}.$$

Consequently,

$$J_{\beta,\lambda}(u) \geq C_1\|u\|^p - C_2\|u\|^{q+1},$$

where

$$C_1 = \frac{\psi_{\min}}{1 + C_P^p}\left(\frac{1}{p} - \frac{1}{r+1}\right) > 0,$$

and

$$C_2 = \lambda\left(\frac{1}{q+1} - \frac{1}{r+1}\right)\psi_{\max}\|a\|_\infty S_{q+1}^{q+1} > 0.$$

Since $p > q+1$, it follows that

$$\lim_{\substack{\|u\|\to\infty \\ u\in\mathcal{N}_{J_{\beta,\lambda}}}} J_{\beta,\lambda}(u) = \infty.$$

Thus, $J_{\beta,\lambda}$ is coercive on $\mathcal{N}_{J_{\beta,\lambda}}$. Furthermore, the function $t \mapsto C_1 t^p - C_2 t^{q+1}$ is bounded below on $[0,\infty)$; hence $J_{\beta,\lambda}$ is bounded below on $\mathcal{N}_{J_{\beta,\lambda}}$. □

The geometry of the Nehari manifold can be understood through the behavior of the one-dimensional maps

$$\Phi_u : (0,\infty) \to \mathbb{R}, \qquad \Phi_u(t) = J_{\beta,\lambda}(tu),$$

for $u \in W_0^{1,p}(\Omega)$. These maps, commonly called fibering maps, were introduced by [10]. For $u \in W_0^{1,p}(\Omega)$, the fibering map is given by

$$\Phi_u(t) = \frac{t^p}{p} I_\beta(u) - \frac{\lambda t^{q+1}}{q+1} A_\beta(u) - \frac{t^{r+1}}{r+1} B_\beta(u). \tag{2.9}$$

Its first and second derivatives are

$$\Phi_u^{'}(t) = t^{p-1} I_\beta(u) - \lambda t^q A_\beta(u) - t^r B_\beta(u), \tag{2.10}$$

and

$$\Phi_u^{''}(t) = (p-1)t^{p-2} I_\beta(u) - q\lambda t^{q-1} A_\beta(u) - r t^{r-1} B_\beta(u). \tag{2.11}$$

A function u belongs to $\mathcal{N}_{J_{\beta,\lambda}}$ if and only if $\Phi_u'(1) = 0$. More generally, for every $t > 0$,

$$\Phi_u'(t) = 0 \quad \Longleftrightarrow \quad tu \in \mathcal{N}_{J_{\beta,\lambda}}.$$

Hence, points on the Nehari manifold correspond precisely to the critical points of the associated fibering maps. This observation motivates the following decomposition of $\mathcal{N}_{J_{\beta,\lambda}}$, according to the nature of the critical point $t = 1$:

$$\mathcal{N}_{J_{\beta,\lambda}}^{\mp} = \{u \in \mathcal{N}_{J_{\beta,\lambda}} : \Phi_u^{''}(1) \lessgtr 0\},$$

and

$$\mathcal{N}_{J_{\beta,\lambda}}^{0} = \{u \in \mathcal{N}_{J_{\beta,\lambda}} : \Phi_u^{''}(1) = 0\}.$$

Thus, $\mathcal{N}_{J_{\beta,\lambda}}^{-}$, $\mathcal{N}_{J_{\beta,\lambda}}^{+}$ and $\mathcal{N}_{J_{\beta,\lambda}}^{0}$ correspond, respectively, to local minima, local maxima, and degenerate critical points of the fibering maps. Moreover, if $u \in \mathcal{N}_{J_{\beta,\lambda}}$, then $\Phi_u'(1) = 0$, and consequently

$$\Phi_u''(1) = (p-q-1)I_\beta(u) + (q-r)B_\beta(u).$$

Equivalently,

$$\Phi_u''(1) = (p-r-1)I_\beta(u) - \lambda(q-r)A_\beta(u).$$

The next result, which is well known in the Nehari manifold framework, ensures that nondegenerate local extrema of $J_{\beta,\lambda}$ on the Nehari manifold are critical points of $J_{\beta,\lambda}$ in the ambient space. Precisely, the following result holds.

Proposition 2.4. *Let u_0 be a local minimizer or a local maximizer of $J_{\beta,\lambda}$ on $\mathcal{N}_{J_{\beta,\lambda}}$. If $u_0 \notin \mathcal{N}^0_{J_{\beta,\lambda}}$, then u_0 is a critical point of $J_{\beta,\lambda}$.*

Proof. Define $G(u) := \langle J'_{\beta,\lambda}(u), u\rangle$, with $u \in W^{1,p}_0(\Omega)$. Then

$$\mathcal{N}_{J_{\beta,\lambda}} = \left\{u \in W^{1,p}_0(\Omega) \setminus \{0\} : G(u) = 0\right\}.$$

Since u_0 is a local extremum of $J_{\beta,\lambda}$ subject to the constraint $G(u) = 0$, the Lagrange multiplier rule yields the existence of $\mu \in \mathbb{R}$ such that $J'_{\beta,\lambda}(u_0) = \mu G'(u_0)$. Taking the duality pairing with u_0, we obtain

$$\left\langle J'_{\beta,\lambda}(u_0), u_0\right\rangle = \mu \left\langle G'(u_0), u_0\right\rangle.$$

Because $u_0 \in \mathcal{N}_{J_{\beta,\lambda}}$, the left-hand side vanishes. Thus, $0 = \mu\langle G'(u_0), u_0\rangle$. Let $\Phi_{u_0}(t) = J_{\beta,\lambda}(tu_0)$ for $t > 0$. Since $G(u_0) = \Phi'_{u_0}(1) = 0$, we have $\langle G'(u_0), u_0\rangle = \Phi''_{u_0}(1)$. Moreover, the assumption $u_0 \notin \mathcal{N}^0_{J_{\beta,\lambda}}$ implies that $\Phi''_{u_0}(1) \neq 0$. Hence, $\langle G'(u_0), u_0\rangle \neq 0$, and therefore $\mu = 0$. It follows that $J'_{\beta,\lambda}(u_0) = 0$. Thus, u_0 is a critical point of $J_{\beta,\lambda}$. □

We now analyze the fibering maps associated with the problem. Their behavior is determined by the signs of $A_\beta(u)$ and $B_\beta(u)$. For $u \in W^{1,p}_0(\Omega) \setminus \{0\}$, define

$$n_u(t) = t^{p-q-1} I_\beta(u) - t^{r-q} B_\beta(u), \qquad t > 0.$$

Then

$$tu \in \mathcal{N}_{J_{\beta,\lambda}} \iff n_u(t) = \lambda A_\beta(u). \tag{2.12}$$

Moreover,

$$n'_u(t) = (p-q-1)\, t^{p-q-2} I_\beta(u) - (r-q) t^{r-q-1} B_\beta(u). \tag{2.13}$$

Hence, $n'_u(t) \lesseqgtr 0$ imply that $tu \in \mathcal{N}^{\mp}_{J_{\beta,\lambda}}$. It follows directly from (2.13) that n_u is strictly increasing on $(0,\infty)$ whenever $B_\beta(u) \leq 0$, as illustrated in Figure 1a. If instead $B_\beta(u) > 0$, then n_u increases near $t = 0$, attains a unique maximum, and decreases thereafter, as shown in Figure 1b. We distinguish the following four cases.

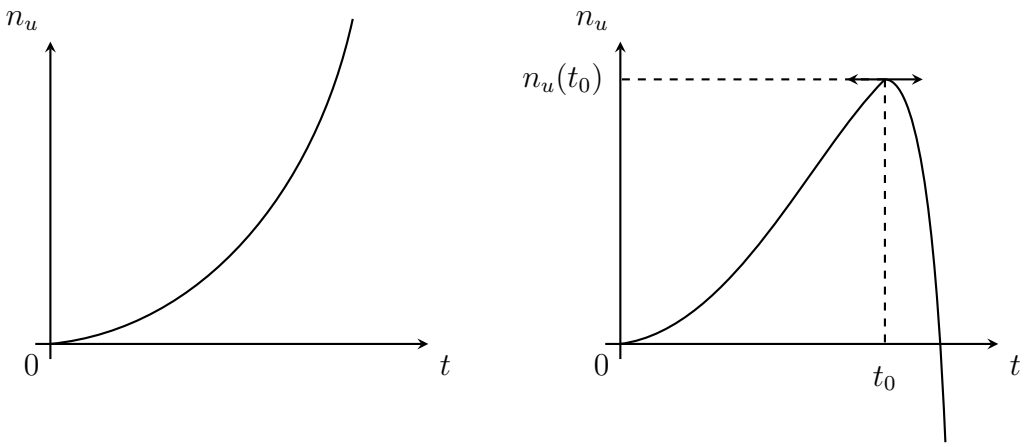
Case 1. First, suppose that $A_\beta(u) \leq 0$ and $B_\beta(u) \leq 0$. Then, Φ_u is strictly increasing on $(0,\infty)$ and has graph as shown in Figure 2a. Therefore, no positive multiple of u belongs to $\mathcal{N}_{J_{\beta,\lambda}}$.

Case 2. Second, suppose that $A_\beta(u) > 0$ and $B_\beta(u) \leq 0$. In this case, n_u is strictly increasing, and there exists a unique solution $t(u) > 0$ such that $n_u(t(u)) = \lambda A_\beta(u)$. Hence, $t(u)u \in \mathcal{N}_{J_{\beta,\lambda}}$. Since $n'_u(t(u)) > 0$, we have $t(u)u \in \mathcal{N}^+_{J_{\beta,\lambda}}$. Thus, Φ_u has a unique critical point, which is a local minimum. Moreover, $\lim_{t\to\infty} \Phi_u(t) = \infty$ and consequently, the fibering map Φ_u follows the profile depicted in Figure 2b.

Case 3. Next, suppose that $A_\beta(u) \leq 0$ and $B_\beta(u) > 0$. Then n_u first increases, reaches a unique maximum, and subsequently decreases to $-\infty$. And again,

there is a unique solution $t(u) > 0$ satisfying $n_u(t(u)) = \lambda A_\beta(u)$. Furthermore, $n_u'(t(u)) < 0$, and hence $t(u)u \in \mathcal{N}^-_{J_{\beta,\lambda}}$. Accordingly, Φ_u has a unique critical point, which is a local maximum, and $\lim_{t\to\infty} \Phi_u(t) = -\infty$. This establishes that the graph of Φ_u takes the form shown in Figure 2c.

Case 4. Finally, suppose that $A_\beta(u) > 0$ and $B_\beta(u) > 0$. Here, n_u again has a unique maximum. If $\lambda > 0$ is sufficiently large, the equation $n_u(t) = \lambda A_\beta(u)$ has no solution; equivalently, Φ_u has no critical point and is strictly decreasing on $(0, \infty)$. Therefore, no positive multiple of u belongs to $\mathcal{N}_{J_{\beta,\lambda}}$. On the other hand, if $\lambda > 0$ is sufficiently small, the equation $n_u(t) = \lambda A_\beta(u)$ admits exactly two solutions $0 < t_1(u) < t_2(u)$, satisfying $n_u'(t_1(u)) > 0$ and $n_u'(t_2(u)) < 0$. Consequently, $t_1(u)u \in \mathcal{N}^+_{J_{\beta,\lambda}}$, $t_2(u)u \in \mathcal{N}^-_{J_{\beta,\lambda}}$. Thus, Φ_u has exactly two critical points (see Figure 2d below): a local minimum at $t = t_1(u)$ and a local maximum at $t = t_2(u)$. In addition, Φ_u is decreasing on $(0, t_1(u))$, increasing on $(t_1(u), t_2(u))$, and decreasing on $((t_2(u), \infty)$.



(A) The case where $B_\beta(u) \le 0$. (B) The case where $B_\beta(u) > 0$.

FIGURE 1. Illustrative forms of the function n_u depending on the sign of $B_\beta(u)$.

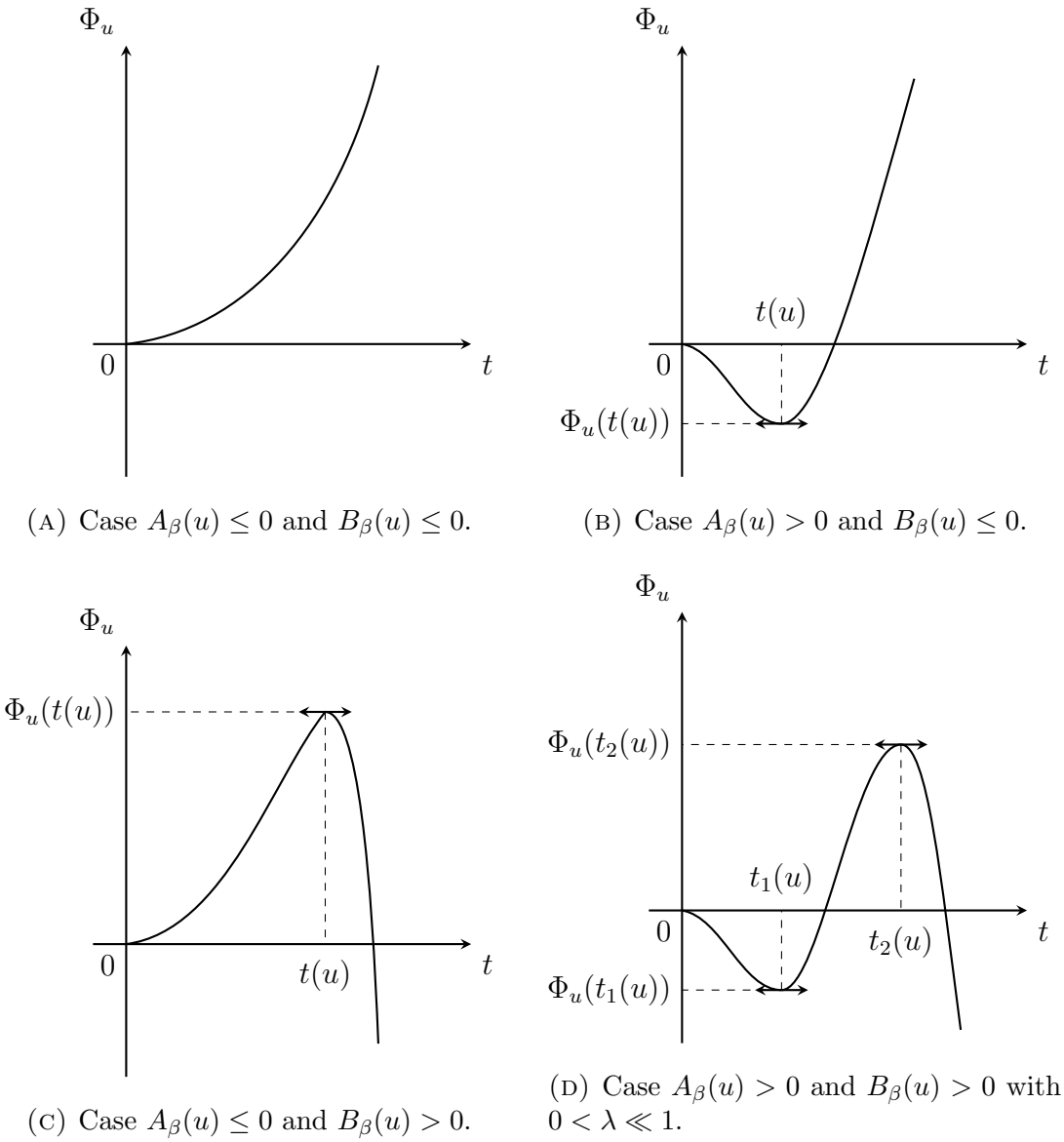
3. Proof of Theorem 1.1 and Theorem 1.3

We need a few lemmas.

Lemma 3.1. *Assume* **(H)**. *There exists $\lambda_* > 0$ such that, for every $0 < \lambda < \lambda_*$ and every $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, the fibering map Φ_u attains a positive value on $(0, \infty)$.*

Proof. Let $u \in W_0^{1,p}(\Omega) \setminus \{0\}$. For $t > 0$, the fibering map is given by

$$\Phi_u(t) = \frac{t^p}{p} I_\beta(u) - \frac{\lambda t^{q+1}}{q+1} A_\beta(u) - \frac{t^{r+1}}{r+1} B_\beta(u).$$



(A) Case $A_\beta(u) \le 0$ and $B_\beta(u) \le 0$.

(B) Case $A_\beta(u) > 0$ and $B_\beta(u) \le 0$.

(C) Case $A_\beta(u) \le 0$ and $B_\beta(u) > 0$.

(D) Case $A_\beta(u) > 0$ and $B_\beta(u) > 0$ with $0 < \lambda \ll 1$.

FIGURE 2. Comprehensive breakdown of the four qualitative forms of $\Phi_u(t)$.

By Poincaré's inequality and the Sobolev embeddings, there exist positive constants C_β, K_a, and K_b such that

$$I_\beta(u) \ge C_\beta \|u\|^p, \qquad A_\beta(u) \le K_a \|u\|^{q+1}, \qquad B_\beta(u) \le K_b \|u\|^{r+1},$$

where, for instance,

$$C_\beta = \frac{\psi_{\min}}{1 + C_P^p}, \qquad K_a = \psi_{\max} \|a\|_\infty S_{q+1}^{q+1}, \qquad K_b = \psi_{\max} \|b\|_\infty S_{r+1}^{r+1}.$$

with ψ as in the proof of Lemma 2.3. We proceed in two cases.

Case 1: $B_\beta(u) \leq 0$. In this case,

$$\Phi_u(t) \geq \frac{t^p}{p} I_\beta(u) - \frac{\lambda t^{q+1}}{q+1} A_\beta(u).$$

Since $p > q+1$ and $I_\beta(u) > 0$, it follows that $\Phi_u(t) \longrightarrow \infty$ as $t \longrightarrow \infty$. Hence, Φ_u takes positive values on $(0, \infty)$.

Case 2: $B_\beta(u) > 0$. Define

$$h_u(t) := \frac{t^p}{p} I_\beta(u) - \frac{t^{r+1}}{r+1} B_\beta(u).$$

Then

$$h_u'(t) = t^{p-1} \left(I_\beta(u) - t^{r-p+1} B_\beta(u) \right).$$

Thus, h_u attains its unique maximum at

$$t_{\max} = \left(\frac{I_\beta(u)}{B_\beta(u)} \right)^{\frac{1}{r-p+1}}.$$

Moreover,

$$h_u(t_{\max}) = \frac{r-p+1}{p(r+1)} \left(\frac{[I_\beta(u)]^{r+1}}{[B_\beta(u)]^p} \right)^{\frac{1}{r-p+1}}.$$

Using the estimates above, we obtain

$$[B_\beta(u)]^p \leq K_b^p \|u\|^{p(r+1)} \leq \frac{K_b^p}{C_\beta^{r+1}} [I_\beta(u)]^{r+1}.$$

Consequently, $h_u(t_{\max}) \geq \delta$, where

$$\delta := \frac{r-p+1}{p(r+1)} \left(\frac{C_\beta^{r+1}}{K_b^p} \right)^{\frac{1}{r-p+1}} > 0.$$

Importantly, δ does not depend on u. On the other hand,

$$A_\beta(u) \leq K_a \|u\|^{q+1} \leq K_a c_\beta^{-\frac{q+1}{p}} [C_\beta(u)]^{\frac{q+1}{p}}.$$

Therefore,

$$\frac{t_{\max}^{q+1}}{q+1} A_\beta(u) \leq \theta \left[h_u(t_{\max}) \right]^{\frac{q+1}{p}},$$

where

$$\theta := \frac{K_a C_\beta^{-\frac{q+1}{p}}}{q+1} \left(\frac{p(r+1)}{r-p+1} \right)^{\frac{q+1}{p}} > 0.$$

Hence,

$$\begin{aligned} \Phi_u(t_{\max}) &= h_u(t_{\max}) - \frac{\lambda t_{\max}^{q+1}}{q+1} A_\beta(u) \\ &\geq [h_u(t_{\max})]^{\frac{q+1}{p}} \left([h_u(t_{\max})]^{\frac{p-q-1}{p}} - \lambda\theta \right) \\ &\geq \delta^{\frac{q+1}{p}} \left(\delta^{\frac{p-q-1}{p}} - \lambda\theta \right). \end{aligned} \tag{3.1}$$

Define

$$\lambda_* := \frac{\delta^{\frac{p-q-1}{p}}}{\theta} > 0. \tag{3.2}$$

Then, for every $0 < \lambda < \lambda_*$, $\Phi_u(t_{\max}) > 0$. Combining the two cases, we conclude that, for every $0 < \lambda < \lambda_*$ and every $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, $\sup_{t>0} \Phi_u(t) > 0$. □

Corollary 3.2. *Assume* **(H)**. *Let* $0 < \lambda < \lambda_*$. *Then* $\mathcal{N}^0_{J_{\beta,\lambda}} \setminus \{0\} = \emptyset$. *Moreover, there exists a constant* $\eta > 0$ *such that* $J_{\beta,\lambda}(u) \geq \eta$ *for every* $u \in \mathcal{N}^-_{J_{\beta,\lambda}}$.

Proof. We first proceed by contradiction to establish that $\mathcal{N}^0_{J_{\beta,\lambda}} \setminus \{0\} = \emptyset$. To this end, suppose that there exists $u \in \mathcal{N}^0_{J_{\beta,\lambda}} \setminus \{0\}$. Since $u \in \mathcal{N}_{J_{\beta,\lambda}}$ and $\Phi''_u(1) = 0$, we have

$$I_\beta(u) = \lambda A_\beta(u) + B_\beta(u),$$

and

$$(p - q - 1)\lambda A_\beta(u) = (r - p + 1)B_\beta(u).$$

As $I_\beta(u) > 0$ for every $u \neq 0$, these identities imply $A_\beta(u) > 0$ and $B_\beta(u) > 0$. For $0 < \lambda < \lambda_*$, the equation $n_u(t) = \lambda A_\beta(u)$ has exactly two positive solutions $t_1(u) < t_2(u)$, satisfying

$$t_1(u)u \in \mathcal{N}^+_{J_{\beta,\lambda}} \qquad \text{and} \qquad t_2(u)u \in \mathcal{N}^-_{J_{\beta,\lambda}}.$$

However, since $u \in \mathcal{N}^0_{J_{\beta,\lambda}}$, the value $t = 1$ is also a solution of this equation. Hence, either $1 = t_1(u)$ or $1 = t_2(u)$, which is impossible. Thus, $\mathcal{N}^0_{J_{\beta,\lambda}} \setminus \{0\} = \emptyset$.

Now let $u \in \mathcal{N}^-_{J_{\beta,\lambda}}$. Then $B_\beta(u) > 0$, and $t = 1$ is the global maximum point of Φ_u. Hence,

$$J_{\beta,\lambda}(u) = \Phi_u(1) \geq \Phi_u(t_{\max})$$

and the result follows from (3.1). □

Next, we establish the existence of minimizers on the two nondegenerate components of the Nehari manifold. The fact that $\mathcal{N}^0_{J_{\beta,\lambda}} = \emptyset$ for $0 < \lambda < \lambda_*$ allows us to treat $\mathcal{N}^-_{J_{\beta,\lambda}}$ and $\mathcal{N}^+_{J_{\beta,\lambda}}$ separately. These minimizers will subsequently yield distinct critical points of $J_{\beta,\lambda}$.

Lemma 3.3. *Assume* **(H)** *and let* $0 < \lambda < \lambda_*$. *Then the functional* $J_{\beta,\lambda}$ *attains its infimum on both components of the Nehari manifold. More precisely, there exist* $u^\mp_\lambda \in \mathcal{N}^\mp_{J_{\beta,\lambda}}$ *such that*

$$J_{\beta,\lambda}(u^\mp_\lambda) = \inf_{u \in \mathcal{N}^\mp_{J_{\beta,\lambda}}} J_{\beta,\lambda}(u).$$

Proof. Set

$$c^-_{\beta,\lambda} := \inf_{u \in \mathcal{N}^-_{J_{\beta,\lambda}}} J_{\beta,\lambda}(u),$$

and

$$c^+_{\beta,\lambda} := \inf_{u \in \mathcal{N}^+_{J_{\beta,\lambda}}} J_{\beta,\lambda}(u).$$

From Lemma 2.3, the numbers $c^{\mp}_{\beta,\lambda}$ are finite. We first claim that $c^{+}_{\beta,\lambda} < 0$. Indeed, choose $w \in W^{1,p}_0(\Omega) \setminus \{0\}$ such that $A_\beta(w) > 0$. Then

$$\frac{\Phi_w(t)}{t^{q+1}} = \frac{t^{p-q-1}}{p} I_\beta(w) - \frac{\lambda}{q+1} A_\beta(w) - \frac{t^{r-q}}{r+1} B_\beta(w).$$

Since $q+1 < p < r+1$, we have

$$\lim_{t\to 0^+} \frac{\Phi_w(t)}{t^{q+1}} = -\frac{\lambda}{q+1} A_\beta(w) < 0.$$

Thus, $\Phi_w(t) < 0$ for $t > 0$ sufficiently small. By the fibering map analysis, there exists $t^+(w) > 0$ such that

$$t^+(w)w \in \mathcal{N}^+_{J_{\beta,\lambda}},$$

and

$$\Phi_w(t^+(w)) < 0.$$

Consequently, $c^{+}_{\beta,\lambda} < 0$. Moreover, by Corollary 3.2, there exists $\eta > 0$ such that

$$J_{\beta,\lambda}(u) \geq \eta \qquad \text{for every } u \in \mathcal{N}^-_{J_{\beta,\lambda}}.$$

Hence, $c^{-}_{\beta,\lambda} \geq \eta > 0$. Let $\{u^+_n\} \subset \mathcal{N}^+_{J_{\beta,\lambda}}$ and $\{u^-_n\} \subset \mathcal{N}^-_{J_{\beta,\lambda}}$ be minimizing sequences; namely,

$$J_{\beta,\lambda}(u^+_n) \longrightarrow c^+_{\beta,\lambda},$$

and

$$J_{\beta,\lambda}(u^-_n) \longrightarrow c^-_{\beta,\lambda}.$$

The coercivity of $J_{\beta,\lambda}$ on $\mathcal{N}_{J_{\beta,\lambda}}$ implies that both sequences are bounded in $W^{1,p}_0(\Omega)$. By Ekeland's variational principle, we may suppose that

$$\left(J_{\beta,\lambda}|_{\mathcal{N}^{\mp}_{\beta,\lambda}}\right)'(u^{\mp}_n) \longrightarrow 0.$$

Recall

$$G(u) = \langle J'_{\beta,\lambda}(u), u\rangle.$$

Since

$$\mathcal{N}_{J_{\beta,\lambda}} = \left\{u \in W^{1,p}_0(\Omega) \setminus \{0\} : G(u) = 0\right\},$$

the Lagrange multiplier rule yields numbers $\mu^{\mp}_n \in \mathbb{R}$ such that

$$J'_{\beta,\lambda}(u^{\mp}_n) = \mu^{\mp}_n G'(u^{\mp}_n) + o(1).$$

Taking the duality pairing with $u^{\mp}_n$, we obtain

$$0 = \mu^{\mp}_n \langle G'(u^{\pm}_n), u^{\mp}_n\rangle + o(1).$$

Since $\langle G'(u^{\mp}_n), u^{\mp}_n\rangle = \Phi''_{u^{\mp}_n}(1)$ and $\mathcal{N}^0_{J_{\beta,\lambda}} = \emptyset$, the standard nondegeneracy estimate on $\mathcal{N}^-_{J_{\beta,\lambda}}$ and $\mathcal{N}^+_{J_{\beta,\lambda}}$ implies that $\mu^{\mp}_n \to 0$. Therefore,

$$J'_{\beta,\lambda}(u^{\mp}_n) \longrightarrow 0 \qquad \text{in } \left(W^{1,p}_0(\Omega)\right)^*.$$

Up to a subsequence, there exist $u^-_\lambda, u^+_\lambda \in W^{1,p}_0(\Omega)$ such that

$$u^{\mp}_n \rightharpoonup u^{\mp}_\lambda \qquad \text{weakly in } W^{1,p}_0(\Omega).$$

Since the embeddings

$$W_0^{1,p}(\Omega) \hookrightarrow L^{q+1}(\Omega) \qquad \text{and} \qquad W_0^{1,p}(\Omega) \hookrightarrow L^{r+1}(\Omega)$$

are compact, we have

$$u_n^{\mp} \longrightarrow u_\lambda^{\mp} \quad \text{in } L^{q+1}(\Omega) \text{ and } L^{r+1}(\Omega).$$

Using $J'_{\beta,\lambda}(u_n^{\mp}) \longrightarrow 0$ and testing against $u_n^{\mp} - u_\lambda^{\mp}$, we obtain

$$\int_\Omega e^{\beta\cdot x}|\nabla u_n^{\mp}|^{p-2}\nabla u_n^{\mp} \cdot \nabla(u_n^{\mp} - u_\lambda^{\mp})\, dx \longrightarrow 0.$$

Since $x \mapsto e^{\beta\cdot x}$ is bounded below by a positive constant, the (S_+) property of the p-Laplacian implies that

$$u_n^{\mp} \longrightarrow u_\lambda^{\mp} \qquad \text{strongly in } W_0^{1,p}(\Omega).$$

We claim that the limits are nontrivial. Indeed, if $u_\lambda^+ = 0$, then $J_{\beta,\lambda}(u_n^+) \longrightarrow J_{\beta,\lambda}(0) = 0$, contradicting $c_{\beta,\lambda}^+ < 0$. Similarly, if $u_\lambda^- = 0$, then $J_{\beta,\lambda}(u_n^-) \longrightarrow J_{\beta,\lambda}(0) = 0$, contradicting $c_{\beta,\lambda}^- > 0$. Next, by strong convergence, $u_\lambda^{\mp} \in \mathcal{N}_{J_{\beta,\lambda}}$. Since $\mathcal{N}_{J_{\beta,\lambda}}^-$, $\mathcal{N}_{J_{\beta,\lambda}}^0$ and $\mathcal{N}_{J_{\beta,\lambda}}^+$ are disjoint, we obtain $u_\lambda^{\mp} \in \mathcal{N}_{J_{\beta,\lambda}}^{\mp}$. Finally, the continuity of $J_{\beta,\lambda}$ gives

$$J_{\beta,\lambda}(u_\lambda^+) = \lim_{n\to\infty} J_{\beta,\lambda}(u_n^+) = c_{\beta,\lambda}^+,$$

and

$$J_{\beta,\lambda}(u_\lambda^-) = \lim_{n\to\infty} J_{\beta,\lambda}(u_n^-) = c_{\beta,\lambda}^-.$$

This completes the proof. □

Combining the previous results, we now prove Theorem 1.1.

Proof of Theorem 1.1. Take λ_* as in (3.2) and fix $\lambda \in (0, \lambda_*)$. By Lemma 3.3, there exist $u_\lambda^{\mp} \in \mathcal{N}_{J_{\beta,\lambda}}^{\mp}$ such that $J_{\beta,\lambda}(u_\lambda^{\mp}) = \inf_{u\in\mathcal{N}_{J_{\beta,\lambda}}^{\mp}} J_{\beta,\lambda}(u)$. Since $\mathcal{N}_{J_{\beta,\lambda}}^0$ is empty, the minimizers $u_\lambda^{\mp}$ are nondegenerate. Hence, by Proposition 2.4, $J'_{\beta,\lambda}(u_\lambda^{\mp}) = 0$. Therefore, both u_λ^- and u_λ^+ are weak solutions of (1.2). Now observe that for every $u \in W_0^{1,p}(\Omega)$, $J_{\beta,\lambda}(|u|) = J_{\beta,\lambda}(u)$ and $|u| \in \mathcal{N}_{J_{\beta,\lambda}}^{\mp}$ is equivalent to $u \in \mathcal{N}_{J_{\beta,\lambda}}^{\mp}$. Thus, replacing u_λ^- and u_λ^+ by their absolute values if necessary, we may assume that $u_\lambda^{\mp} \geq 0$ in Ω. The resulting functions remain minimizers on their respective components of the Nehari manifold. Once again, Proposition 2.4 implies that they are weak solutions of (1.2). Finally, the two nonnegative solutions are distinct as $\mathcal{N}_{J_{\beta,\lambda}}^- \cap \mathcal{N}_{J_{\beta,\lambda}}^+ = \emptyset$. □

We complete this section with a proof of Theorem 1.3.

Proof of Theorem 1.3. Let $u \geq 0$ be a nontrivial weak solution of (1.2). By the regularity theory for the p-Laplacian and the subcritical assumptions in **(H)**, we have $u \in L^\infty(\Omega)$. Since $a \geq 0$ and $u \geq 0$, we have $\lambda a(x)u^q \geq 0$. Moreover, since $r > p-1$,

$$b(x)u^r \geq -\|b^-\|_\infty u^r \geq -\|b^-\|_\infty \|u\|_\infty^{r-p+1} u^{p-1}.$$

and it follows that

$$\lambda a(x)u^q + b(x)u^r \geq -\kappa u^{p-1},$$

where $\kappa := \|b^-\|_\infty \|u\|_\infty^{r-p+1} \geq 0$. Therefore, u satisfies

$$-\operatorname{div}\left(e^{\beta\cdot x}|\nabla u|^{p-2}\nabla u\right) + \kappa\, e^{\beta\cdot x}u^{p-1} \geq 0 \qquad \text{in } \Omega.$$

Since $e^{\beta\cdot x} > 0$ in Ω, then we can invoke the strong maximum principle for the weighted p-Laplacian to conclude the proof, as $u \geq 0$ and $u \not\equiv 0$. □

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References

1. G. Akagi, G., & Matsuura, K. (2013). Nonlinear diffusion equations driven by the $p(\cdot)$-Laplacian. Nonlinear Differ. Equ. Appl., 20, 37-64. https://doi.org/10.1007/s00030-012-0153-6
2. Asso, O., Cuesta, M., Doumatè, J. T., & Leadi, L. (2023). Principal eigenvalues for the fractal p-Laplacian with unbounded sign-changing weights. Electronic Journal of Differential Equations, 2023(38), 1–29. https://doi.org/10.58997/ejde.2023.38
3. Benedikt, J. (2002). Uniqueness theorem for p-biharmonic equations. Electron. J. Differential Equations, 53, 1-17. https://ejde.math.txstate.edu/Volumes/2002/53/abstr.html
4. Ben Haddouch, K., Tsouli, N., & El Allali, Z. (2014). The third order spectrum of p-biharmonic operator with weight. Applicationes Mathematicae, 41(2/3), 247-255. https://eudml.org/doc/280051
5. Brezis, H. (2011). Functional analysis, Sobolev spaces and partial differential equations. Springer Science & Business Media, New York, NY. https://link.springer.com/book/10.1007/978-0-387-70914-7
6. Brown, K. J., & Wu, T. -F. (2007). A fibering map approach to a semilinear elliptic boundary value problem. Electronic Journal of Differential Equations, 69, 1-9. https://ejde.math.txstate.edu/Volumes/2007/69/abstr.html
7. Chen, Y., Levine, S., & Rao, M. (2006). Variable exponent, linear growth functionals in image restoration. SIAM J. Appl. Math., 66, 1383–1406. https://doi.org/10.1137/05062452
8. Diening, L., Ettwein, F., & Růžička, M. (2007). $C^{1,\alpha}$-regularity for electrorheological fluids in two dimensions. Nonlinear Differ. Equ. Appl., 14, 207–217. https://doi.org/10.1007/s00030-007-5026-z
9. Doumate, J. T., Toyou, L. R., & Leadi, L. A. (2022). On eigensurface of p-biharmonic operator and associated concave-convex type equation. Gulf Journal of Mathematics, 13(1), 54-87. https://doi.org/10.56947/gjom.v13i1.927

10. Drabek, P., & Pohozaev, S. I. (1997). Positive solutions for the p-Laplacian: application of the fibering method. Proc. Royal Soc. Edinburgh Sect A, 127, 703-726. https://doi.org/10.1017/S0308210500023787
11. El Khalil, A., Kellati, S., & Touzani, A. (2002). On the spectrum of the p-biharmonic operator. In: 2002-Fez Conference on Partial Differential Equations, Electron. J. Differential Equations, Conference 09, 161-170. https://ejde.math.txstate.edu/conf-proc/09/k1/abstr.html
12. Il'yasov, Y. (2005). On non-local existence results for elliptic operators with convex-concave nonlinearities. Nonlinear Analysis, 61, 211-236. https://doi.org/10.1016/j.na.2004.10.022
13. Li, H. -Y. (2019). The Nehari manifold for a p-Laplacian equation with concave-convex nonlinearities and sign-changing potential. Differential Equations and Applications, 11(3), 349–366. https://dx.doi.org/10.7153/dea-2019-11-16
14. Papageorgiou, N. S., & Scapellato, A. (2020). Conca-convex problems for the Robin p-Laplacian plus an indefinite potential. Mathematics, 8(3), 421. https://doi.org/10.3390/math8030421
15. Quoirin, H. R. (2010). Lack of coercivity in a concave-convex type equation. Calc. Var., 37, 523–546. https://doi.org/10.1007/s00526-009-0275-9
16. Quoirin, H. R., & Ubilla, P. (2013). On some indefinite and non-powerlike elliptic equations. Nonlinear Analysis, 79, 190–203. https://doi.org/10.1016/j.na.2012.10.022
17. Rabinowitz, P. H. (1986). Minimax methods in critical point theory with applications to differential equations. American Mathematical Society. https://doi.org/10.1090/cbms/065
18. Toyou, L. R., Doumate, T. J., & Leadi, L. A. (2025). Existence of principal eigensurfaces for cooperative $(p_1, \ldots, p_n)$-biharmonic systems. Malaya J. Mat., 13(01), 26-36. http://doi.org/10.26637/mjm1301/004
19. Wu, T. -F. (2009). Multiplicity Results for a Semi-Linear Elliptic Equation Involving Sign-Changing Weight Function. Rocky Mountain Journal of Mathematics, 39, 1-14. https://doi.org/10.1216/RMJ-2009-39-3-995

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