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VANDERMONDE DETERMINANTS FOR GENERALIZED BOUNDED-TURNING FUNCTIONS

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Abstract. In this paper, we investigate the Vandermonde determinants for a class of generalized bounded-turning functions associated with a q-Limaçon domain. Using the q-derivative, we establish coefficient bounds for these determinants. Several special cases are also discussed.

Keywords. Analytic functions; q-derivative; q-starlike functions; q-Limaçon domain; Vandermonde determinant.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad z \in \mathbb{E}. \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$ and satisfy the normalization conditions $f(0) = 0, f'(0) = 1$. Furthermore, let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbb{E}$. The two well-known subclasses of $\mathcal{S}$, namely the classes of starlike and convex functions will be denoted by $\mathcal{S}^*$ and $\mathcal{C}$ respectively. Refer to [17] for formal definitions of various subclasses of $\mathcal{S}$.

A fundamental tool in geometric function theory is the concept of *subordination* [3]. Given two analytic functions f and g in $\mathbb{E}$, we say that f is *subordinate* to g, written $f(z) \prec g(z)$, if there exists a Schwartz function $w : \mathbb{E} \to \mathbb{E}$ satisfying $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{E}$, such that

$$f(z) = g(w(z)), \qquad z \in \mathbb{E}.$$

Further, g is univalent in $\mathbb{E}$, the subordination $f(z) \prec g(z)$ simplifies to the conditions $f(0) = g(0)$ and $f(\mathbb{E}) \subseteq g(\mathbb{E})$.

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The q-calculus, introduced by Jackson [6], generalizes classical differential calculus by replacing the ordinary derivative with a difference quotient. For a function f analytic in $\mathbb{E}$, the *q-derivative* is defined as

$$\partial_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}, \qquad 0 < q < 1, \quad z \neq 0.$$

As $q \to 1^-$, the q-derivative reduces to the classical derivative

$$\lim_{q \to 1^-} \partial_q f(z) = f'(z). \tag{1.2}$$

This shows that q-calculus is a natural extension of classical calculus. For a function of the form (1.1), the q-derivative takes the form

$$\partial_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q \, a_n z^{n-1},$$

where $[n]_q$ is the *q-number* defined by

$$[n]_q = \frac{1-q^n}{1-q}.$$

The q-numbers and q-derivative provide a flexible and powerful framework for defining new subclasses of analytic functions and studying their geometric properties [2, 1, 4, 5, 7, 9, 16].

A curve of particular interest in this study is the *Limaçon of Pascal* [5], which in polar coordinates is given by

$$r = a + b\cos\theta, \qquad a, b \in \mathbb{R}, \quad 0 \leq \theta < 2\pi.$$

The shape of this curve depends on the ratio a/b. Specifically, when $a \geq 2b$, the Limaçon is convex, and when $a = b$, it reduces to a *cardioid* (refer Fig. 1). The interior of this curve defines the *Limaçon domain*, which serves as the image domain for the function classes studied in this paper (refer Fig. 2). The classical Limaçon domain is defined analytically by

$$L_\zeta(z) = (1+\zeta z)^2, \qquad -1 \leq \zeta \leq 1, \ \zeta \neq 0,$$

which maps the open unit disk $\mathbb{E}$ onto a region bounded by a Limaçon curve [19].

Using the framework of q-calculus, the q-analog of this domain, called the *q-Limaçon domain*, is defined by

$$L_{q,\zeta}(z) = \left(\frac{2(1+\zeta z)}{2 + (1-q)\zeta z}\right)^2, \qquad -1 \leq \zeta \leq 1, \ \zeta \neq 0, \ 0 < q < 1, \tag{1.3}$$

which reduces to the classical Limaçon domain when $q \to 1$ (see [5, 13]). Fig. 3 illustrates the image of $\mathbb{E}$ under the transformation $L_{q,\xi}(z)$.

A particularly interesting class of analytic functions associated with such domains is the class of *generalized bounded-turning functions* [8]. A function $f \in \mathcal{A}$ belongs to this class if the real part of a suitable linear combination of $f(z)$ and its derivatives remains positive in $\mathbb{E}$. This condition controls the angular rotation of $f(z)$ along the boundary of the unit disk and ensures that the image of $\mathbb{E}$ under f exhibits controlled geometric behavior.

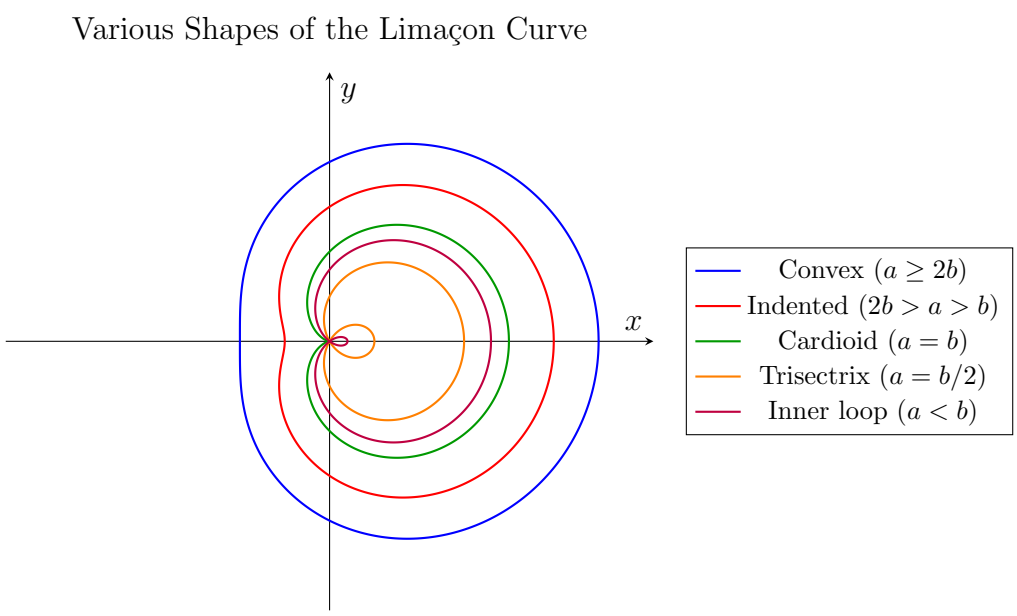


Fig. 1. Different forms of the Limaçon curve corresponding to various ratios of a and b.

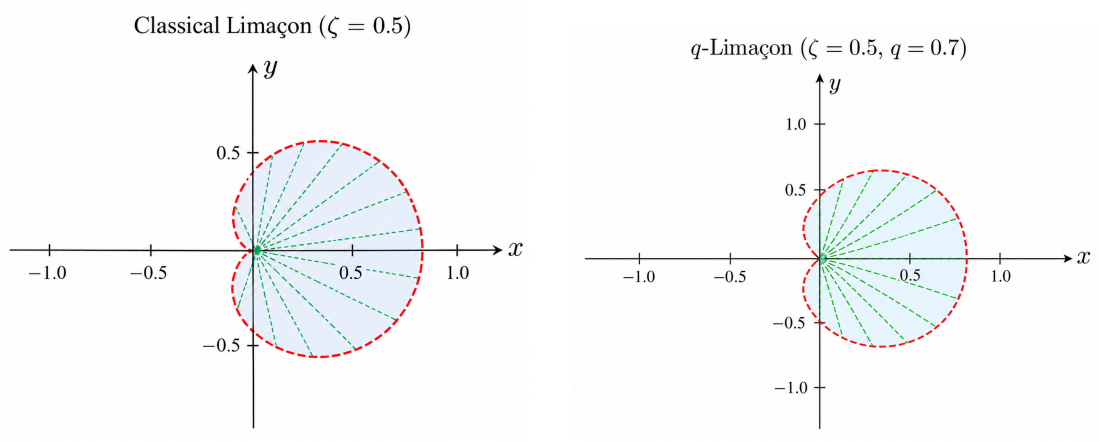


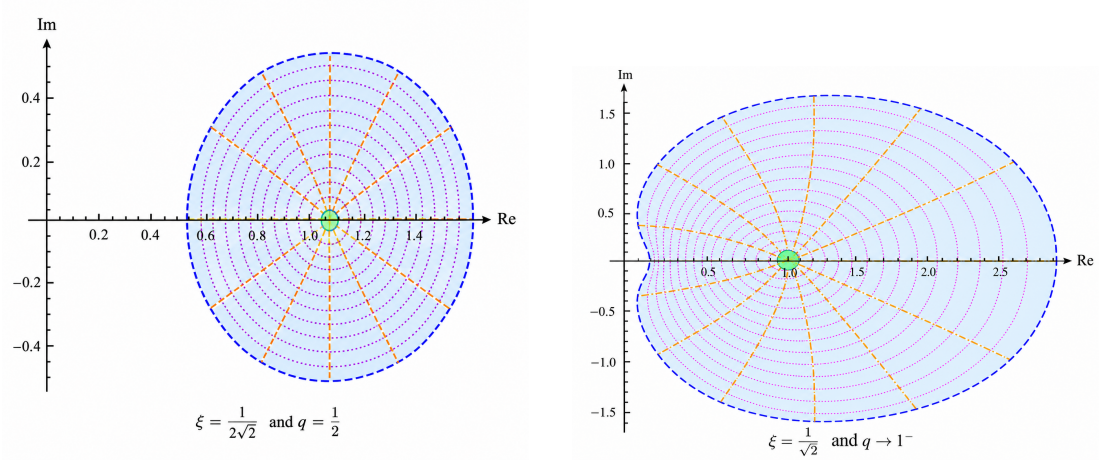
Fig. 2. The image of $\mathbb{E}$ under the transformation $L_{q,\xi}(z)$.

Definition 1.1. An analytic function f is said to belong to the class of *starlike q-generalized bounded turning functions* $ST_q^{\delta,\alpha}(L_{q,\zeta}(z))$ if

$$\frac{e^{i\alpha}\partial_q f(z) - i\sin\alpha - \delta}{\tau_{\alpha\delta}} \prec L_{q,\zeta}(z), \qquad z \in \mathbb{E}, \tag{1.4}$$

where the *q-Limaçon domain* $L_{q,\zeta}(z)$ is defined in equation (1.3)

$$L_{q,\zeta}(z) = \left(\frac{2(1+\zeta z)}{2+(1-q)\zeta z}\right)^2, \qquad -1 \leq \zeta \leq 1,\ \zeta \neq 0,\ 0 < q < 1.$$



(a) $\xi = \frac{1}{2\sqrt{2}}$ and $q = \frac{1}{2}$ (b) $\xi = \frac{1}{\sqrt{2}}$ and $q \to 1^-$

Fig. 3. The image of $\mathbb{E}$ under the transformation $L_{q,\xi}(z)$.

Remark 1.2. We observe that for particular values of the parameters the class $ST_q^{\delta,\alpha}(L_{q,\zeta}(z))$ reduces to the following existing classes

(i) If $\delta = 0$ then

$$ST_q^{0,\alpha}(L_{q,\zeta}(z)) = \left\{ f \in S : \frac{e^{i\alpha}\partial_q f(z) - i\sin\alpha}{\cos\alpha} \prec L_{q,\zeta}(z) \right\}.$$

(ii) If $\alpha = 0$ then

$$ST_q^{\delta,0}(L_{q,\zeta}(z)) = \left\{ f \in S : \frac{\partial_q f(z) - \delta}{1-\delta} \prec L_{q,\zeta}(z) \right\}.$$

(iii) If $\alpha = \delta = 0$ then

$$ST_q^{0,0}(L_{q,\zeta}(z)) = \{ f \in S : \partial_q f(z) \prec L_{q,\zeta}(z) \}.$$

Definition 1.3. A function $f \in \mathcal{A}$ defined (1.1) is said to belong to the class $\mathcal{CV}_q^{\alpha,\delta}(L_{q,\zeta}(z))$ if

$$\frac{e^{i\alpha}\left(1 + \frac{z\,\partial_q^2 f(z)}{\partial_q f(z)}\right) - i\sin\alpha - \delta}{\cos\alpha - \delta} \prec L_{q,\zeta}(z), \qquad z \in \mathbb{E}, \tag{1.5}$$

where $|\alpha| < \pi$, $0 \le \delta < 1$, $0 < q < 1$, and $0 < \zeta \le 1/\sqrt{2}$.

Remark 1.4. We observe that for particular values of the parameters the class $\mathcal{CV}_q^{\alpha,\delta}(L_{q,\zeta}(z))$ reduces to the following existing classes

(i) If $q \to 1^-$ then

$$\mathcal{CV}^{\alpha,\delta}(L_\zeta(z)) = \left\{ f \in S : \frac{e^{i\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right) - i\sin\alpha - \delta}{\cos\alpha - \delta} \prec L_\zeta(z) \right\}.$$

(ii) If $\alpha = 0$ then

$$\mathcal{CV}_q^{\delta}(L_{q,\zeta}(z)) = \left\{ f \in S : \frac{1 + \dfrac{z\,\partial_q^2 f(z)}{\partial_q f(z)} - \delta}{1-\delta} \prec L_{q,\zeta}(z) \right\}.$$

(iii) If $\alpha \neq 0,\ \delta = 0$ then

$$\mathcal{CV}_q^{\alpha}(L_{q,\zeta}(z)) = \left\{ f \in S : \frac{e^{i\alpha}\left(1 + \dfrac{z\,\partial_q^2 f(z)}{\partial_q f(z)}\right) - i\sin\alpha}{\cos\alpha} \prec L_{q,\zeta}(z) \right\}.$$

(iv) $\alpha = 0,\ \delta = 0,\ 0 < q < 1$:

$$\mathcal{CV}_q(L_{q,\zeta}(z)) = \left\{ f \in S : 1 + \frac{z\,\partial_q^2 f(z)}{\partial_q f(z)} \prec L_{q,\zeta}(z) \right\}.$$

Vandermonde matrices appear naturally in many areas of mathematics and engineering. One important application is in digital signal processing, where they are used to compute the Discrete Fourier Transform (DFT) and its inverse (IDFT). They also play a useful role in polynomial interpolation and other approximation problems, largely because of their well-known determinant properties. Generalized bounded turning functions extend the classical idea of bounded turning functions. These functions have been widely studied because of their close relationships with geometric function theory, q-calculus, and coefficient inequalities. In recent years, considerable attention has been devoted to the study of the Hankel determinant [11, 14, 15] for various classes of analytic functions.

A *Vandermonde determinant* [10, 12, 20] is a special type of determinant in which each row forms a geometric progression. For a $p \times p$ determinant, the pth Vandermonde determinant, denoted by $V_p(n)$ with $p \geq 1$ and $n \geq 1$, is defined as

$$V_p(n) = \begin{vmatrix} 1 & a_n & a_n^2 & \cdots & a_n^{p-1} \\ 1 & a_{n+1} & a_{n+1}^2 & \cdots & a_{n+1}^{p-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_{n+p-1} & a_{n+p-1}^2 & \cdots & a_{n+p-1}^{p-1} \end{vmatrix} = \prod_{n \leq i < j \leq n+p-1} (a_j - a_i),$$

where a_i are the coefficients from the Taylor series of a function $f \in \mathcal{A}$ as in (1.1), and in particular $a_1 = 1$. By construction, each row (and column) of a Vandermonde determinant forms a geometric sequence with the first element equal to one. Some specific examples illustrate the structure of Vandermonde determinants:

$$V_2(1) = \begin{vmatrix} 1 & a_1 \\ 1 & a_2 \end{vmatrix} = a_2 - a_1, \quad V_2(2) = \begin{vmatrix} 1 & a_2 \\ 1 & a_3 \end{vmatrix} = a_3 - a_2,$$

$$V_3(1) = \begin{vmatrix} 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \\ 1 & a_3 & a_3^2 \end{vmatrix} = (a_3 - a_2)(a_3 - a_1)(a_2 - a_1).$$

we present some lemmas that will be used in deriving the main results. We first recall the concept of a Schwartz function. A function u is called a *Schwartz function* if it has the form

$$u(z) = \sum_{n=1}^{\infty} u_n z^n, \quad z \in \mathbb{E}, \tag{1.6}$$

and satisfies the conditions

$$u(0) = 0, \quad |u(z)| \leq 1 \text{ for all } z \in \mathbb{E}.$$

Lemma 1.5. [3] *If $u \in \mathbb{E}$ is a Schwartz function as above, then*

$$|u_n| \leq 1 \quad \textit{for all } n = 1, 2, \cdots .$$

Lemma 1.6. [18] *If $u \in \mathbb{E}$ is a Schwartz function, then*

$$|u_2| \leq 1 - |u_1|^2.$$

Inspired by the aforementioned works, we investigate coefficient bounds and Vandermonde determinants for functions belonging to the class $\mathcal{ST}_q^{\alpha,\delta}(L_{q,\zeta}(z))$.

2. Main Results

Theorem 2.1. *Let $f \in \mathcal{ST}_q^{\alpha,\delta}(L_{q,\zeta}(z))$. Then for $|\alpha| < \pi$, $0 \leq \delta < 1$, $0 < q < 1$, $0 < \zeta \leq 1/\sqrt{2}$*

(i) If $\sigma_q^{\alpha,\delta} \neq 0$, $\tau_q^{\alpha,\delta} \neq 0$, $q \neq \frac{1}{3}$, and

$$\zeta \geq \frac{4\,\sigma_q^{\alpha,\delta}}{|3q-1|\,\sigma_q^{\alpha,\delta} + 4(1+q)|\cos\alpha - \delta|},$$

then

$$|V_2(2)| = |a_3 - a_2| \leq \frac{1+q}{[3]_q}\,\tau_q^{\alpha,\delta}\left(\frac{|3q-1|}{4} + \frac{(1+q)|\cos\alpha - \delta|}{\sigma_q^{\alpha,\delta}}\right)\zeta^2 + \frac{1+q}{\sigma_q^{\alpha,\delta}}\,\zeta. \tag{2.1}$$

(ii) If $\sigma_q^{\alpha,\delta} \neq 0$, $\tau_q^{\alpha,\delta} \neq 0$, $q \neq \frac{1}{3}$, and

$$\zeta < \frac{4\,\sigma_q^{\alpha,\delta}}{|3q-1|\,\sigma_q^{\alpha,\delta} + 4(1+q)|\cos\alpha - \delta|},$$

then

$$|V_2(2)| \leq \begin{cases} \dfrac{\tau_q^{\alpha,\delta}(1+q)\,\zeta}{4\,(\sigma_q^{\alpha,\delta})^2}\left(1 - \left(\dfrac{|3q-1|}{4} + \dfrac{(1+q)|\cos\alpha - \delta|}{\sigma_q^{\alpha,\delta}}\right)\zeta\right) + \dfrac{(1+q)\,\tau_q^{\alpha,\delta}}{[3]_q}\,\zeta, & \zeta < \Xi_0, \\ \dfrac{1+q}{[3]_q}\,\tau_q^{\alpha,\delta}\left(\frac{|3q-1|}{4} + \frac{(1+q)|\cos\alpha-\delta|}{\sigma_q^{\alpha,\delta}}\right)\zeta^2 + \dfrac{1+q}{\sigma_q^{\alpha,\delta}}\,\zeta, & \zeta \geq \Xi_0, \end{cases} \tag{2.2}$$

where

$$\Xi_0 = \frac{2}{2\sigma_q^{\alpha,\delta} - \tau_q^{\alpha,\delta}} \cdot \frac{1}{|3q-1|\sigma_q^{\alpha,\delta} + 4(1+q)|\cos\alpha - \delta|}.$$

(iii) If $\sigma_q^{\alpha,\delta} \neq 0$, $\tau_q^{\alpha,\delta} \neq 0$, $q = \frac{1}{3}$, *and* $\cos\alpha \neq \delta$, *and if*

$$\zeta \;\geq\; \frac{3\,\sigma_{1/3}^{\alpha,\delta}}{4\,|\cos\alpha - \delta|},$$

then

$$|V_2(2)| \;\leq\; \frac{16|\cos\alpha - \delta|}{9\,\sigma_{1/3}^{\alpha,\delta}}\,\tau_{1/3}^{\alpha,\delta}\,\zeta^2 + \frac{4}{3\,\sigma_{1/3}^{\alpha,\delta}}\,\zeta. \tag{2.3}$$

(iv) If $\sigma_q^{\alpha,\delta} \neq 0$, $\tau_q^{\alpha,\delta} \neq 0$, $q = \frac{1}{3}$, *and* $\cos\alpha \neq \delta$, *and*

$$\zeta < \frac{3\,\sigma_{1/3}^{\alpha,\delta}}{4\,|\cos\alpha - \delta|},$$

then

$$|V_2(2)| \leq \begin{cases} \dfrac{\tau_{1/3}^{\alpha,\delta}\,\zeta}{\sigma_{1/3}^{\alpha,\delta}}\left(3\sigma_{1/3}^{\alpha,\delta} - |\cos\alpha - \delta|\,\zeta\right) + \frac{4}{3}\,\tau_{1/3}^{\alpha,\delta}\,\zeta, & \zeta < \Xi_1, \\ \dfrac{16|\cos\alpha - \delta|}{9\,\sigma_{1/3}^{\alpha,\delta}}\,\tau_{1/3}^{\alpha,\delta}\,\zeta^2 + \frac{4}{3}\,\frac{\zeta}{\sigma_{1/3}^{\alpha,\delta}}, & \zeta \geq \Xi_1, \end{cases} \tag{2.4}$$

where

$$\Xi_1 = \frac{6\sigma_{1/3}^{\alpha,\delta} - 3\tau_{1/3}^{\alpha,\delta}}{8|\cos\alpha - \delta|}.$$

(v) If $q = \frac{1}{3}$ *and* $\cos\alpha = \delta$, *then*

$$|V_2(2)| \;\leq\; \left(\frac{13 - 9\cos^2\alpha}{48} + \frac{12}{|13 - 9\cos^2\alpha|}\right)\zeta, \tag{2.5}$$

where $B = \frac{e^{i\alpha}}{\cos\alpha - \delta}$, $\sigma = B\,[2]_q$, $\tau = B\,[3]_q$, $|\sigma| = \frac{[2]_q}{|S|}$, $|\tau| = \frac{[3]_q}{|S|}$.

Proof. Let $f \in \mathcal{ST}_q^{\delta,\alpha}(L_{q,\zeta}(z))$. Then there exists a Schwartz function u with $u(0) = 0$ and $|u(z)| < 1$ in $\mathbb{E}$ such that

$$\frac{e^{i\alpha}\,\partial_q f(z) - i\sin\alpha - \delta}{\cos\alpha - \delta} = L_{q,\zeta}(u(z)). \tag{2.6}$$

By elementary calculations, we get

$$\frac{e^{i\alpha}\,\partial_q f(z) - i\sin\alpha - \delta}{\cos\alpha - \delta} = 1 + \sigma a_2 z + \left(\tau a_3 - S\,\sigma\tfrac{a_2^2}{2}\right)z^2 + \cdots,$$

where

$$\sigma = \sigma_q^{\alpha,\delta} = [2]_q - (\cos\alpha - \delta), \qquad \tau = \tau_q^{\alpha,\delta} = [3]_q - (\cos^2\alpha - \delta).$$

Also, we have

$$L_{q,\zeta}(u(z)) = 1 + h_1(q)\zeta u_1 z + \left(h_1(q)\zeta u_2 + h_2(q)\zeta^2 u_1^2\right)z^2 + \cdots.$$

By equating the coefficients in (2.6), we get

$$a_2 = \frac{h_1(q)\zeta}{\sigma}\,u_1, \tag{2.7}$$

$$a_3 = \frac{h_1(q)}{\tau}\,\zeta\,u_2 + \frac{h_2(q)}{\tau}\,\zeta^2 u_1^2 + \frac{S(h_1(q))^2}{2\tau\sigma}\,\zeta^2 u_1^2. \tag{2.8}$$

Using (2.7) and the expression for (2.8), we obtain

$$V_2(2) = \frac{h_1(q)}{\tau}\,\zeta\,u_2 + \left(\frac{h_2(q)}{\tau} + \frac{S(h_1(q))^2}{2\tau\sigma}\right)\zeta^2 u_1^2 - \frac{h_1(q)}{\sigma}\,\zeta\,u_1.$$

Using Lemma 1.6 and $|u_1| = c$, we get

$$|V_2(2)| \le \left(\left|\frac{h_2(q)}{\tau} + \frac{S(h_1(q))^2}{2\tau\sigma}\right|\zeta^2 - \frac{h_1(q)}{|\tau|}\zeta\right)c^2 + \frac{h_1(q)}{|\sigma|}\,\zeta\,c + \frac{h_1(q)}{|\tau|}\zeta$$
$$=: \Phi(c). \tag{2.9}$$

As $|u_1| < 1$ by lemma 1.5, we get

Case I. For $\zeta \ge \dfrac{2h_1(q)|\sigma|}{2h_2(q)|\sigma| + |S|(h_1)^2}$, $\Phi(c)$ is increasing function on $[0,1]$, since $\Phi'(c) \ge 0,$ for $0 \le c \le 1$. Hence $\Phi(c)$ attains its maximum value at $c = 1$.
Thus

$$|V_2(2)| \le \Phi(1).$$

Case II. For $\zeta < \dfrac{2h_1(q)|\sigma|}{2h_2(q)|\sigma| + |S|(h_1)^2}$, $\Phi(c)$ attains its maximum at

$$c_0 = \frac{\frac{h_1(q)}{|\sigma|}}{2\left[\frac{h_1(q)}{|\tau|} - \left(\frac{h_2(q)}{|\tau|} + \frac{|S(h_1(q))^2|}{2|\tau||\sigma|}\right)\zeta\right]}.$$

Thus $\Phi'(c_0) = 0$ and $\Phi''(c_0) < 0$. After Simple computations, it easy to see that $\Phi(c)$ attains its maximum value at $c = 1$.
Thus

$$|V_2(2)| \le \Phi(1).$$

Case III. : $S \neq 0$, large ζ. Require

$$\left|\frac{h_2(q)}{\tau} + \frac{S(h_1(q))^2}{2\tau\sigma}\right| - \frac{h_1(q)}{|\tau|}\zeta \ge 0. \tag{2.10}$$

This yields the bound

$$|V_2(2)| \le \frac{16|S|}{9\sigma}\,\tau\,\zeta^2 + \frac{4}{3\sigma}\,\zeta.$$

Case IV. : $S \neq 0$, small ζ. If $\zeta < 3\sigma/(4|S|)$ then $\Phi_1(c)$ attains its maximum value either $c_0 \in (0,1)$ or $c_0 = 1$. Hence $\zeta < \frac{6|\sigma|-3|\tau|}{8|S|}$. Thus

$$|V_2(2)| \le \Phi_1(c_1),$$

where $c_1 = \frac{3|\tau|}{6\sigma - 8|S|\zeta}$. For $\zeta \ge \frac{6|\sigma|-3|\tau|}{8|S|}$, $\Phi_1(c)$ attains its maximum value at $c = 1$.
Thus

$$|V_2(2)| \le \Phi_1(1).$$

Case V. Special symmetric case: $S = 0$ (i.e. $\cos\alpha = \delta$). Using Lemma 1.6 and taking $|u_1| = c$, we get

$$|V_2(2)| \leq \frac{12}{13}\zeta(1-c^2) + \zeta c.$$

Define $\Psi(c) = \frac{12}{13}\zeta(1-c^2) + \zeta c$. Differentiate

$$\Psi'(c) = -\frac{24}{13}\zeta c + \zeta = \zeta\Big(1 - \frac{24}{13}c\Big).$$

Solve $\Psi'(c) = 0$ to find $c_* = 13/24 \in (0,1)$. Then

$$\Psi(c_*) = \frac{12}{13}\zeta\Big(1 - \frac{169}{576}\Big) + \frac{13}{24}\zeta.$$

Thus

$$|V_2(2)| \leq \Big(\frac{13 - 9\cos^2\alpha}{48} + \frac{12}{|13 - 9\cos^2\alpha|}\Big)\zeta.$$

□

Theorem 2.2. *Let $f \in \mathcal{ST}_q^{\alpha,\delta}(L_{q,\zeta}(z))$ under the same conditions as in Theorem 2.1. Then*

$$|V_2(1)| = |a_2 - a_1| \leq \frac{1+q}{|\sigma_q^{\alpha,\delta}|}\,\zeta + 1. \tag{2.11}$$

Proof. We know $|V_2(1)| = |a_2 - a_1|$. Using equation (2.7), we obtain

$$\begin{aligned} |a_2 - a_1| &= \Big|\frac{h_1(q)\,\zeta}{\sigma_q^{\alpha,\delta}}\,u_1 - 1\Big| \\ &\leq \frac{h_1(q)\,\zeta}{|\sigma_q^{\alpha,\delta}|}\,|u_1| + 1 \\ &\leq \frac{1+q}{|\sigma_q^{\alpha,\delta}|}\,\zeta + 1, \end{aligned} \tag{2.12}$$

which establishes (2.11). □

Theorem 2.3. *Let f defined in (1.1) belong to the class $\mathcal{ST}_q^{\alpha,\delta}(L_{q,\zeta}(z))$ defined in Theorem 2.1. Then*

(i) If $\sigma_q^{\alpha,\delta} \neq 0$ and $\tau_q^{\alpha,\delta} \neq 0$, then

$$|a_3 - a_1| \;\leq\; \begin{cases} \dfrac{1+q}{[3]_q}\,\tau_q^{\alpha,\delta}\left(\dfrac{|3q-1|}{4} + \dfrac{(1+q)|\cos\alpha - \delta|}{\sigma_q^{\alpha,\delta}}\right)\zeta^2 + 1, & \zeta \geq \Xi_2, \\ \dfrac{1+q}{[3]_q}\,\tau_q^{\alpha,\delta}\,\zeta + 1, & \zeta < \Xi_2, \end{cases} \tag{2.13}$$

where

$$\Xi_2 = \frac{4\,\sigma_q^{\alpha,\delta}}{|3q-1|\,\sigma_q^{\alpha,\delta} + 4(1+q)\,|\cos\alpha - \delta|}.$$

(ii) If $q = \frac{1}{3}$ *and* $\cos\alpha \neq \delta$, *then*

$$|a_3 - a_1| \leq \begin{cases} \dfrac{16\,|\cos\alpha - \delta|}{9\,\sigma_{1/3}^{\alpha,\delta}}\,\tau_{1/3}^{\alpha,\delta}\,\zeta^2 + 1, & \zeta \geq \dfrac{3\,\sigma_{1/3}^{\alpha,\delta}}{4|\cos\alpha - \delta|}, \\ \dfrac{4}{3}\,\tau_{1/3}^{\alpha,\delta}\,\zeta + 1, & \zeta < \dfrac{3\,\sigma_{1/3}^{\alpha,\delta}}{4|\cos\alpha - \delta|}. \end{cases} \tag{2.14}$$

(iii) If $q = \frac{1}{3}$ *and* $\cos\alpha = \delta$, *then*

$$|a_3 - a_1| \leq 12\left|13 - 9\delta^2\right|\zeta + 1. \tag{2.15}$$

Proof. From the Theorem 2.1, we get

$$a_2 = \frac{(1+q)\zeta}{\sigma_q^{\alpha,\delta}}\,u_1,$$

$$a_3 = \frac{h_1(q)}{\tau}\zeta u_2 + \frac{h_2(q)}{\tau}\zeta^2 u_1^2 + \frac{s(h_1(q))^2}{2\tau\sigma}\zeta^2 u_1^2. \tag{2.16}$$

By Theorem 2.1, for $q \neq \frac{1}{3}$, and using Lemma (1.6), and taking $|u_1| = c$, we obtain

$$|a_3 - a_1| \leq \frac{|h_1(q)|}{|\tau|}\zeta|u_2| + \frac{|h_2(q)|}{|\tau|}\zeta^2|u_1|^2 + \frac{|s|\,|h_1(q)|^2}{2|\tau|\,|\sigma|}\zeta^2|u_1|^2 + 1.$$

But

$$|u_2| \leq 1 - |u_1|^2.$$

Therefore

$$|a_3 - a_1| \leq \frac{|h_1(q)|}{|\tau|}\zeta(1 - |u_1|^2) + \frac{|h_2(q)|}{|\tau|}\zeta^2|u_1|^2 + \frac{|s|\,|h_1(q)|^2}{2|\tau|\,|\sigma|}\zeta^2|u_1|^2 + 1.$$

Putting $|u_1| = c$, we get

$$\begin{aligned} |a_3 - a_1| &\leq \frac{|h_1(q)|}{|\tau|}\zeta(1 - c^2) + \frac{|h_2(q)|}{|\tau|}\zeta^2 c^2 + \frac{|s|\,|h_1(q)|^2}{2|\tau|\,|\sigma|}\zeta^2 c^2 + 1 \\ &= \left(\frac{|h_2(q)|}{|\tau|}\zeta^2 + \frac{|s|\,|h_1(q)|^2}{2|\tau|\,|\sigma|}\zeta^2 - \frac{|h_1(q)|}{|\tau|}\zeta\right)c^2 + \frac{|h_1(q)|}{|\tau|}\zeta + 1 \\ &= \phi_q(c). \end{aligned}$$

As $|u_1| \leq 1$, by Lemma (1.5), we observe that $c \in [0, 1]$. Hence, we get the following cases.

Case I. When

$$\zeta \geq \frac{2|h_1(q)|\,|\sigma|}{2|h_2(q)|\,|\sigma| + |s|\,|h_1(q)|^2},$$

$\phi_q(c)$ is an increasing function of c in $[0, 1]$, since

$$\phi_q'(c) = 2c\left(\frac{|h_2(q)|}{|\tau|}\zeta^2 + \frac{|s|\,|h_1(q)|^2}{2|\tau|\,|\sigma|}\zeta^2 - \frac{|h_1(q)|}{|\tau|}\zeta\right) \geq 0,$$

for $0 \leq c \leq 1$. Hence, $\phi_q(c)$ attains its maximum value at $c = 1$. Thus

$$|a_3 - a_1| \leq \phi_q(1).$$

Case II. When

$$\zeta < \frac{2|h_1(q)|\,|\sigma|}{2|h_2(q)|\,|\sigma| + |s|\,|h_1(q)|^2},$$

$\phi_q(c)$ is a decreasing function of c in $[0,1]$, because

$$\phi_q'(c) \leq 0, \qquad 0 \leq c \leq 1.$$

Hence, $\phi_q(c)$ attains its maximum value at $c = 0$. Thus

$$|a_3 - a_1| \leq \phi_q(0).$$

Now, for $q = \frac{1}{3}$ and $s \neq 0$, we obtain

$$|a_3 - a_1| \leq \frac{16|s|}{9|\sigma|\,|\tau|}\zeta^2 + 1 = \phi_s(c).$$

Case III. When $\zeta \geq \frac{3|\sigma|}{4|s|}$, $\phi_s(c)$ is an increasing function of c in $[0,1]$. Hence

$$|a_3 - a_1| \leq \phi_s(1).$$

Otherwise, when $\zeta < \frac{3|\sigma|}{4|s|}$, $\phi_s(c)$ is a decreasing function of c in $[0,1]$. Hence

$$|a_3 - a_1| \leq \phi_s(0).$$

Now, for $q = \frac{1}{3}$ and when $S = 0$, we obtain

$$|a_3 - a_1| \leq \frac{4}{3|\tau|}\zeta + 1 = \phi_0(c).$$

Since $\phi_0(c)$ is a decreasing function of c in $[0,1]$, we have

$$|a_3 - a_1| \leq \phi_0(0).$$

This completes the proof.

□

Theorem 2.4. *Let f defined in (1.1) belong to the class $\mathcal{ST}_q^{\alpha,\delta}(L_{q,\zeta}(z))$ defined in Theorem 2.1. Then*

(i) If $q \neq \frac{1}{3}$ and

$$\zeta \geq \frac{4\,\sigma_q^{\alpha,\delta}}{|3q-1|\,\sigma_q^{\alpha,\delta} + 4(1+q)\,\tau_{\alpha,\delta}},$$

then

$$|V_3(1)| \leq (1+q)\,\tau_q^{\alpha,\delta}\left(\frac{|3q-1|}{4} + \frac{(1+q)\,\tau_{\alpha,\delta}}{\sigma_q^{\alpha,\delta}}\right)\zeta^2 + \frac{(1+q)\,\zeta}{\sigma_q^{\alpha,\delta}} \cdot (1+q)\,\zeta. \quad (2.17)$$

(ii) If $q \neq \frac{1}{3}$ and

$$\zeta < \frac{4\,\sigma_q^{\alpha,\delta}}{|3q-1|\,\sigma_q^{\alpha,\delta} + 4(1+q)\,\tau_{\alpha,\delta}},$$

then

$$|V_3(1)| \leq \left(\tau_q^{\alpha,\delta}\frac{(1+q)\,\zeta}{4\,(\sigma_q^{\alpha,\delta})^2}\left[1-\left(\frac{|3q-1|}{4}+\frac{(1+q)\,\tau_{\alpha,\delta}}{\sigma_q^{\alpha,\delta}}\right)\zeta\right]+\frac{(1+q)\,\zeta}{\sigma_q^{\alpha,\delta}}\right) \times\left(\frac{(1+q)\,\zeta}{\sigma_q^{\alpha,\delta}}+1\right)\left(\tau_q^{\alpha,\delta}\zeta+1\right). \tag{2.18}$$

(iii) If $q=\frac{1}{3}$, $\cos\alpha\neq\delta$ *and*

$$\zeta\geq\frac{3\,\sigma_{1/3}^{\alpha,\delta}}{4\,\tau_{\alpha,\delta}},$$

then

$$|V_3(1)| \leq \frac{16\,\tau_{\alpha,\delta}}{9\,\sigma_{1/3}^{\alpha,\delta}}\,\tau_{1/3}^{\alpha,\delta}\,\zeta^2\left(\frac{16\,\tau_{\alpha,\delta}}{9\,\sigma_{1/3}^{\alpha,\delta}}\,\zeta^2+1\right)\left(\tfrac{4}{3}\,\sigma_{1/3}^{\alpha,\delta}\,\zeta+1\right). \tag{2.19}$$

(iv) If $q=\frac{1}{3}$, $\cos\alpha\neq\delta$ *and*

$$\zeta<\frac{3\,\sigma_{1/3}^{\alpha,\delta}}{4\,\tau_{\alpha,\delta}},$$

then

$$|V_3(1)| \leq \left(\tau_{1/3}^{\alpha,\delta}\zeta-\frac{|\cos\alpha-\delta|}{\sigma_{1/3}^{\alpha,\delta}}\zeta+\tfrac{4}{3}\,\tau_{1/3}^{\alpha,\delta}\zeta\right)\left(\tfrac{4}{3}\,\sigma_{1/3}^{\alpha,\delta}\,\zeta+1\right)\left(\tfrac{4}{3}\,\tau_{1/3}^{\alpha,\delta}\zeta+1\right). \tag{2.20}$$

(v) If $q=\frac{1}{3}$ *and* $\cos\alpha=\delta$, *then*

$$|V_3(1)| \leq \left(\frac{13-9\delta^2}{48}+\frac{12}{|13-9\delta^2|}\right)\zeta(\zeta+1)\left(\frac{12}{|13-9\delta^2|}\,(\zeta+1)\right).$$

Proof. Utilizing the results from Theorem 2.1 to Theorem 2.3, we obtain the desired result. □

Corollary 2.5. *Let* $f\in\mathcal{ST}_q^{\alpha,\delta}(L_{q,\zeta}(z))$ *as in Theorem 2.1. If* $\sigma_q^{\alpha,\delta}\neq 0$, *then*

$$|V_2(1)|=|a_2-a_1|\leq\frac{1+q}{\sigma_q^{\alpha,\delta}}\zeta+1.$$

Corollary 2.6. *Under the conditions of Theorem 2.3, for* $\sigma_q^{\alpha,\delta}\neq 0$ *and* $\tau_q^{\alpha,\delta}\neq 0$, *then*

$$|a_3-a_1|\leq\begin{cases}\dfrac{1+q}{[3]_q}\tau_q^{\alpha,\delta}\left(\dfrac{|3q-1|}{4}+\dfrac{(1+q)\tau_{\alpha,\delta}}{\sigma_q^{\alpha,\delta}}\right)\zeta^2+1, & \zeta\geq\Xi_2,\\ \dfrac{1+q}{[3]_q}\tau_q^{\alpha,\delta}\zeta+1, & \zeta<\Xi_2,\end{cases}$$

where Ξ_2 *is as defined in Theorem 2.3.*

Corollary 2.7. *For* $f\in\mathcal{ST}_q^{\alpha,\delta}(L_{q,\zeta}(z))$ *with* $q=\frac{1}{3}$ *and* $\cos\alpha=\delta$, *then*

$$|V_3(1)| \leq \left(\frac{13-9\delta^2}{48}+\frac{12}{|13-9\delta^2|}\right)\zeta(\zeta+1)\left(\frac{12}{|13-9\delta^2|}(\zeta+1)\right).$$

Theorem 2.8. *Let $f \in \mathcal{CV}_q^{\alpha,\delta}(L_{q,\zeta}(z))$ and $S = \cos\alpha - \delta \neq 0$, $|\alpha| < \pi$, $0 \leq \delta < 1$, $0 < q < 1$, $0 < \zeta \leq 1/\sqrt{2}$. Then*

(i) for $q \neq \frac{1}{3}$ and $\zeta \geq \Xi_2 := \dfrac{h_1(q)}{|\chi|\, P}$, we have

$$|V_2(2)| = |a_3 - a_2| \leq \left(\frac{|h_2(q)|}{|\chi|} + \frac{[2]_q}{[3]_q}\frac{(h_1(q))^2}{|\varsigma|^2}\right)\zeta^2 + \frac{h_1(q)}{|\varsigma|}\zeta. \tag{2.21}$$

(ii) for $q \neq \frac{1}{3}$ and $\zeta < \Xi_2$, we have

$$|V_2(2)| \leq \begin{cases} \dfrac{h_1(q)}{|\chi|}\zeta + \dfrac{(h_1(q))^2\,\zeta}{4|\varsigma|^2\left(\dfrac{h_1(q)}{|\chi|} - P\zeta\right)}, & \zeta < \Xi_3, \\ P\,\zeta^2 + \dfrac{h_1(q)}{|\varsigma|}\zeta, & \Xi_3 \leq \zeta < \Xi_2, \end{cases} \tag{2.22}$$

where

$$\Xi_3 = \frac{1}{2P}\left(\frac{2h_1(q)}{|\chi|} - \frac{h_1(q)}{|\varsigma|}\right);$$

(iii) for $q = \frac{1}{3}$ and $\zeta \geq \dfrac{13|\varsigma_{1/3}|^2}{16|\chi_{1/3}|}$, we have

$$|V_2(2)| \leq \frac{64}{39|\varsigma_{1/3}|^2}\zeta^2 + \frac{4}{3|\varsigma_{1/3}|}\zeta. \tag{2.23}$$

(iv) for $q = \frac{1}{3}$ and $\zeta < \dfrac{13|\varsigma_{1/3}|^2}{16|\chi_{1/3}|}$, we have

$$|V_2(2)| \leq \begin{cases} \dfrac{13|\chi_{1/3}|\,\zeta}{39|\varsigma_{1/3}|^2 - 48|\chi_{1/3}|\zeta} + \dfrac{4}{3|\chi_{1/3}|}\zeta, & \zeta < \Xi_4, \\ \dfrac{64}{39|\varsigma_{1/3}|^2}\zeta^2 + \dfrac{4}{3|\varsigma_{1/3}|}\zeta, & \Xi_4 \leq \zeta < \dfrac{13|\varsigma_{1/3}|^2}{16|\chi_{1/3}|}, \end{cases} \tag{2.24}$$

where $\Xi_4 = \dfrac{13|\varsigma_{1/3}|\left(2|\varsigma_{1/3}| - |\chi_{1/3}|\right)}{32|\chi_{1/3}|}$. *Also* $J = \dfrac{e^{i\alpha}}{S}$,

$$\varsigma = J[2]_q, \qquad \chi = J[3]_q[2]_q, \qquad \Gamma_q = J[2]_q^2,$$

$$h_1(q) = 1 + q,\ h_2(q) = \frac{(3q-1)(1+q)}{4},$$

$$P = \frac{|h_2(q)|}{|\chi|} + \frac{|\Gamma_q|}{|\chi|}\frac{(h_1(q))^2}{|\varsigma|^2} = \frac{|h_2(q)|}{|\chi|} + \frac{[2]_q}{[3]_q}\frac{(h_1(q))^2}{|\varsigma|^2}.$$

Proof. Let $f \in \mathcal{CV}_q^{\alpha,\delta}(L_{q,\zeta}(z))$ be given by (1.1). Then there exists an Schwartz function $v(z)$ such that such that

$$\frac{e^{i\alpha}\left(1 + \dfrac{z\partial_q^2 f(z)}{\partial_q f(z)}\right) - i\sin\alpha - \delta}{\cos\alpha - \delta} = L_{q,\zeta}(v(z)) = \left(\frac{2(1+\zeta v(z))}{2 + (1-q)\zeta v(z)}\right)^2. \tag{2.25}$$

By elementry calculations from equation (2.25), we get

$$1+\varsigma a_2 z+\left(\chi a_3-\Gamma_q a_2^2\right)z^2+\cdots \\ =1+h_1(q)\zeta v_1 z+\left(h_1(q)\zeta v_2+h_2(q)\zeta^2 v_1^2\right)z^2+\cdots. \quad (2.26)$$

Equating the coefficients of (2.26) on both sides, we get

$$a_2=\frac{h_1(q)}{\varsigma}\zeta v_1,$$
$$a_3=\frac{h_1(q)}{\chi}\zeta v_2+\frac{h_2(q)}{\chi}\zeta^2 v_1^2+\frac{\Gamma_q}{\chi}\frac{(h_1(q))^2}{\varsigma^2}\zeta^2 v_1^2, \quad (2.27)$$

where $\Gamma_q/\chi=[2]_q/[3]_q$. Using Lemma 1.6 and taking $|v_1|=d$ we obtain

$$|V_2(2)|\leq\left(P\zeta^2-\frac{h_1(q)}{|\chi|}\zeta\right)d^2+\frac{h_1(q)}{|\varsigma|}\zeta d+\frac{h_1(q)}{|\chi|}\zeta=:\beta_0(d).$$

As $|v_1|\leq 1$ by Lemma 1.5, we observe that $d\in[0,1]$. Hence, we get

Case 1. When $\zeta\geq\Xi_2=\dfrac{h_1(q)}{|\chi|P}$, $\beta_0(d)$ is an increasing function of d in $[0,1]$, since

$$\beta_0'(d)=2\left(P\zeta^2-\frac{h_1(q)}{|\chi|}\zeta\right)d+\frac{h_1(q)}{|\varsigma|}\zeta\geq 0,$$

for $0\leq d\leq 1$. Hence $\beta_0(d)$ attains its maximum value at $d=1$. Thus $|V_2(2)|\leq\beta_0(1)$, which is (2.21).

Case 2. When $\zeta<\Xi_2$, $\beta_0(d)$ attains its maximum value either at some $d_0\in(0,1)$ or at $d=1$. Indeed, when

$$\zeta<\frac{1}{2P}\left(\frac{2h_1(q)}{|\chi|}-\frac{h_1(q)}{|\varsigma|}\right)=\Xi_3,$$

$\beta_0(d)$ attains its maximum value at

$$d_0=\frac{\dfrac{h_1(q)}{|\varsigma|}}{2\left(\dfrac{h_1(q)}{|\chi|}-P\zeta\right)},\qquad(0<d_0<1),$$

since $\beta_0'(d_0)=0$ and $\beta_0''(d_0)<0$. Thus

$$|V_2(2)|\leq\beta_0(d_0)=\frac{h_1(q)}{|\chi|}\zeta+\frac{(h_1(q))^2\zeta}{4|\varsigma|^2\left(\dfrac{h_1(q)}{|\chi|}-P\zeta\right)},$$

which follows (2.22). On the other hand, when $\Xi_3\leq\zeta<\Xi_2$, $\beta_0'(d)\geq 0$ on $[0,1]$, so $\beta_0(d)$ attains its maximum value at $d=1$. Thus $|V_2(2)|\leq\beta_0(1)$, which follows (2.22). Also

$$1+\varsigma_{\frac{1}{3}}a_2 z+\left(\chi_{\frac{1}{3}}a_3-\Gamma_{\frac{1}{3}}a_2^2\right)z^2+\cdots=1+\tfrac{4}{3}\zeta v_1 z+\tfrac{4}{3}\zeta v_2 z^2+\cdots. \quad (2.28)$$

Equating the coefficients of z^n on both sides, we get

$$a_2 = \frac{4}{3\varsigma_{\frac{1}{3}}}\zeta v_1, \qquad a_3 = \frac{4}{3\chi_{\frac{1}{3}}}\zeta v_2 + \frac{64}{39\,\varsigma_{\frac{1}{3}}^2}\zeta^2 v_1^2. \tag{2.29}$$

Using Lemma 1.6 and taking $|v_1| = d$ we obtain

$$|V_2(2)| \le \left(\frac{64}{39|\varsigma_{\frac{1}{3}}|^2}\zeta^2 - \frac{4}{3|\chi_{\frac{1}{3}}|}\zeta\right) d^2 + \frac{4}{3|\varsigma_{\frac{1}{3}}|}\zeta d + \frac{4}{3|\chi_{\frac{1}{3}}|}\zeta =: \beta_1(d).$$

Case 3. When $\zeta \ge \dfrac{13|\varsigma_{1/3}|^2}{16|\chi_{1/3}|}$, the coefficient of d^2 is non-negative, so $\beta_1(d)$ is an increasing function of d in $[0,1]$. Hence $|V_2(2)| \le \beta_1(1)$, which is (2.23).

Case 4. When $\zeta < \dfrac{13|\varsigma_{1/3}|^2}{16|\chi_{1/3}|}$, $\beta_1(d)$ attains its maximum value either at some $d_1 \in (0,1)$ or at $d = 1$. Hence, when $\zeta < \Xi_4 = \dfrac{13|\varsigma_{1/3}|\left(2|\varsigma_{1/3}| - |\chi_{1/3}|\right)}{32|\chi_{1/3}|}$, $|V_2(2)| \le \beta_1(d_1)$ where

$$d_1 = \frac{13|\chi_{\frac{1}{3}}|}{39|\varsigma_{\frac{1}{3}}|^2 - 48|\chi_{\frac{1}{3}}|\zeta} \cdot |\varsigma_{\frac{1}{3}}|, \qquad (0 < d_1 < 1)$$

which gives the first line of (2.24). On the other hand, when $\Xi_4 \le \zeta < \frac{13|\varsigma_{1/3}|^2}{16|\chi_{1/3}|}$, $\beta_1(d)$ attains its maximum value at $d = 1$. Thus $|V_2(2)| \le \beta_1(1)$, which is the second line of (2.24). This completes the proof. □

Theorem 2.9. *Given the same conditions as in Theorem 2.8,then*

$$|V_2(1)| = |a_2 - a_1| \le \frac{h_1(q)}{|\varsigma|}\,\zeta + 1. \tag{2.30}$$

Proof. Using Theorem 2.8, we have

$$a_2 = \frac{h_1(q)}{\varsigma}\,\zeta v_1 = S\,e^{-i\alpha}\,\zeta v_1,$$

where $h_1(q) = 1 + q = [2]_q$. Since $a_1 = 1$, it follows that

$$|V_2(1)| = |a_2 - a_1| = \left|\frac{h_1(q)}{\varsigma}\,\zeta v_1 - 1\right|.$$

Using the triangle inequality, Lemma 1.5 and taking $|v_1| = d$, we obtain

$$|V_2(1)| \le \frac{h_1(q)}{|\varsigma|}\,\zeta d + 1 =: \beta_3(d), \qquad d \in [0,1].$$

Since $\beta_3'(d) = \dfrac{h_1(q)}{|\varsigma|}\zeta > 0$, $\beta_3(d)$ attains its maximum value at $d = 1$ in $[0,1]$. Hence

$$|V_2(1)| \le \beta_3(1) = \frac{h_1(q)}{|\varsigma|}\,\zeta + 1.$$

□

Theorem 2.10. *Given the same conditions as in Theorem 2.8, then*

(i) if $q \neq \frac{1}{3}$, *then*

$$|a_3 - a_1| \leq \begin{cases} \left(\dfrac{|h_2(q)|}{|\chi|} + \dfrac{[2]_q}{[3]_q} \dfrac{(h_1(q))^2}{|\varsigma|^2} \right) \zeta^2 + 1, & \zeta \geq \Xi_2, \\ \dfrac{h_1(q)}{|\chi|} \zeta + 1, & \zeta < \Xi_2, \end{cases} \tag{2.31}$$

where $\Xi_2 = \dfrac{h_1(q)}{|\chi| P}$.

(ii) if $q = \frac{1}{3}$, *then*

$$|a_3 - a_1| \leq \begin{cases} \dfrac{64}{39|\varsigma_{1/3}|^2} \zeta^2 + 1, & \zeta \geq \dfrac{13|\varsigma_{1/3}|^2}{16|\chi_{1/3}|}, \\ \dfrac{4}{3|\chi_{1/3}|} \zeta + 1, & \zeta < \dfrac{13|\varsigma_{1/3}|^2}{16|\chi_{1/3}|}. \end{cases} \tag{2.32}$$

Proof. Using Theorem 2.8 and Lemma 1.6, we obtain

$$\begin{aligned} |a_3 - a_1| &\leq \frac{h_1(q)}{|\chi|} \zeta (1 - d^2) + \left(\frac{|h_2(q)|}{|\chi|} + \frac{[2]_q}{[3]_q} \frac{(h_1(q))^2}{|\varsigma|^2} \right) \zeta^2 d^2 + 1 \\ &= \left(P\zeta^2 - \frac{h_1(q)}{|\chi|} \zeta \right) d^2 + \frac{h_1(q)}{|\chi|} \zeta + 1 =: \beta_4(d). \end{aligned}$$

As $|v_1| \leq 1$ by Lemma 1.5, we observe that $d \in [0, 1]$. Hence, we get

Case 1. When $\zeta \geq \Xi_2 = \dfrac{h_1(q)}{|\chi| P}$, $\beta_4(d)$ is an increasing function of d in $[0, 1]$, since

$$\beta_4'(d) = 2 \left(P\zeta^2 - \frac{h_1(q)}{|\chi|} \zeta \right) d \geq 0,$$

for $0 \leq d \leq 1$. Hence $\beta_4(d)$ attains its maximum value at $d = 1$. Thus

$$|a_3 - a_1| \leq \beta_4(1) = P\zeta^2 + 1.$$

Case 2. When $\zeta < \Xi_2$, $\beta_4(d)$ is a decreasing function of d in $[0, 1]$, since

$$\beta_4'(d) = 2 \left(P\zeta^2 - \frac{h_1(q)}{|\chi|} \zeta \right) d \leq 0,$$

for $0 \leq d \leq 1$. Hence $\beta_4(d)$ attains its maximum value at $d = 0$. Thus

$$|a_3 - a_1| \leq \beta_4(0) = \frac{h_1(q)}{|\chi|} \zeta + 1.$$

From equation (2.29), we obtain

$$|a_3 - a_1| \leq \left(\frac{64}{39|\varsigma_{\frac{1}{3}}|^2} \zeta^2 - \frac{4}{3|\chi_{\frac{1}{3}}|} \zeta \right) d^2 + \frac{4}{3|\chi_{\frac{1}{3}}|} \zeta + 1 =: \beta_5(d).$$

Case 3. When $\zeta \geq \dfrac{13|\varsigma_{1/3}|^2}{16|\chi_{1/3}|}$, the coefficient of d^2 is non-negative, so $\beta_5(d)$ is an

increasing function of d in $[0,1]$. Hence $|a_3-a_1| \le \beta_5(1) = \dfrac{64}{39|\varsigma_{1/3}|^2}\zeta^2+1$. On the other hand, when $\zeta < \dfrac{13|\varsigma_{1/3}|^2}{16|\chi_{1/3}|}$, $\beta_5(d)$ is a decreasing function of d in $[0,1]$. Hence $|a_3-a_1| \le \beta_5(0) = \dfrac{4}{3|\chi_{1/3}|}\zeta+1$. This proves (2.32). □

Theorem 2.11. *Given the same conditions as in Theorem 2.8, the determinant $V_3(1)$ satisfies the following bounds*

(i) If $q \neq \frac{1}{3}$ and $\zeta \ge \Xi_2$, then

$$|V_3(1)| \le \left(P\zeta^2 + \frac{h_1(q)}{|\varsigma|}\zeta\right)\left(P\zeta^2+1\right)\left(\frac{h_1(q)}{|\varsigma|}\zeta+1\right). \tag{2.33}$$

(ii) If $q \neq \frac{1}{3}$ and $\zeta < \Xi_2$, then

$$|V_3(1)| \le \begin{cases} \left(\dfrac{h_1(q)}{|\chi|}\zeta + \dfrac{(h_1(q))^2\zeta}{4|\varsigma|^2\left(\dfrac{h_1(q)}{|\chi|}-P\zeta\right)}\right) \\ \quad \times \left(\dfrac{h_1(q)}{|\varsigma|}\zeta+1\right)\left(\dfrac{h_1(q)}{|\chi|}\zeta+1\right), \quad \zeta < \Xi_3, \\ \left(P\zeta^2+\dfrac{h_1(q)}{|\varsigma|}\zeta\right) \\ \quad \times \left(\dfrac{h_1(q)}{|\varsigma|}\zeta+1\right)\left(\dfrac{h_1(q)}{|\chi|}\zeta+1\right), \quad \Xi_3 \le \zeta < \Xi_2. \end{cases} \tag{2.34}$$

(iii) If $q = \frac{1}{3}$ and $\zeta \ge \Xi_5 = \dfrac{13|\varsigma_{1/3}|^2}{16|\chi_{1/3}|}$, then

$$|V_3(1)| \le \left(\frac{64}{39|\varsigma_{1/3}|^2}\zeta^2+\frac{4}{3|\varsigma_{1/3}|}\zeta\right)\left(\frac{64}{39|\varsigma_{1/3}|^2}\zeta^2+1\right)\left(\frac{4}{3|\varsigma_{1/3}|}\zeta+1\right). \tag{2.35}$$

(iv) If $q = \frac{1}{3}$ and $\zeta < \Xi_5$, then

$$|V_3(1)| \le \begin{cases} \left(\dfrac{13|\chi_{1/3}|\,\zeta}{39|\varsigma_{1/3}|^2-48|\chi_{1/3}|\zeta}+\dfrac{4}{3|\chi_{1/3}|}\zeta\right) \\ \quad \times \left(\dfrac{4}{3|\varsigma_{1/3}|}\zeta+1\right)\left(\dfrac{4}{3|\chi_{1/3}|}\zeta+1\right), \quad \zeta < \Xi_4, \\ \left(\dfrac{64}{39|\varsigma_{1/3}|^2}\zeta^2+\dfrac{4}{3|\varsigma_{1/3}|}\zeta\right) \\ \quad \times \left(\dfrac{4}{3|\varsigma_{1/3}|}\zeta+1\right)\left(\dfrac{4}{3|\chi_{1/3}|}\zeta+1\right), \quad \Xi_4 \le \zeta < \Xi_5. \end{cases} \tag{2.36}$$

Proof. Since $V_3(1) = (a_3 - a_2)(a_3 - a_1)(a_2 - a_1)$, we have

$$|V_3(1)| = |a_3 - a_2|\,|a_3 - a_1|\,|a_2 - a_1|.$$

Utilizing the results from Theorem 2.8, Theorem 2.9 and Theorem 2.10, we obtain a desired result.This completes the proof. □

3. Conclusion

Recent developments in the literature have generated considerable interest in the study of Vandermonde determinants. Motivated by these advances, we investigate the Vandermonde determinants associated with q-starlike generalized bounded turning functions, defined over a q-Limaçon domain. Using tools from q-calculus and subordination, we found bounds for the second- and third-order Vandermonde determinants. Our results build on and extend earlier work on Sakaguchi-type functions, and offer new coefficient estimates for functions in the generalized bounded turning class.

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