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ASYMPTOTIC NORMALITY OF THE NONPARAMETRIC S-GINI ESTIMATOR

ESOSSOLAM KPATCHA[1*], ESSOMANDA TCHANDAO MANGAMANA[1] and TCHILABALO ABOZOU KPANZOU[1]

Abstract. Statistical inference is fundamentally based on parameter estimation. Once an estimator is proposed, studying its asymptotic behavior becomes crucial, since knowledge of its limiting distribution makes it possible to construct confidence intervals and perform hypothesis testing. In this work, we establish a General Asymptotic Representation (GAR) for s-Gini inequality indices by using the functional empirical process (fep). This representation then allows us to characterize the asymptotic distribution of their nonparametric estimators. A simulation study confirms the theoretical results related to asymptotic normality by assessing estimator performance through bias, mean squared error, skewness, and kurtosis.

Keywords. nonparametric estimators, functional empirical process, general asymptotic representation, simulation.
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1. Introduction

Parametric estimation and the analysis of the asymptotic behavior of estimators are central concerns in statistics. From this perspective, methodologies based on the functional empirical process (fep) provide a unifying framework applicable to a wide range of statistics used both in mathematical statistics and in various applied disciplines (see [2], [6], [8], [19] and [23], the first of which is a recent application to copula goodness-of-fit testing).
The idea of developing a common framework for the asymptotic theory of these statistics was developed by [11], where real-valued empirical processes were used. Subsequently, the combination of the fep with the residual empirical process, which was then transformed into a functional form called the residual functional empirical process, gave rise to a new global approach providing the GAR as functions of the fep for a broad class of indices appearing in many fields (see, for example, [17] for the Gini inequality index). Further developments and more details can also be found in [13].

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* Corresponding author
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In this paper, we provide the GAR for the s-Gini index based on the fep and the residual functional empirical process (rfep), introduced by Lo and Sall (see [14] and [15]). Then, using this representation, we study the asymptotic normality of this index.

To situate our contribution, it should be mentioned that the asymptotic theory of J(ν) has been the subject of recent investigations, notably in [21] for the calculation of the asymptotic variance, and in [24] for a general central limit theorem based on the expression:

$$J(\nu) = 1 - \frac{\nu}{\mu} \int_0^1 F^{-1}(u)(1-u)^{\nu-1} du \tag{1.1}$$

(see also [22]). Unlike the approach in [24], which established a general central limit theorem directly, our approach derives asymptotic normality as a consequence of the GAR framework, which additionally yields the explicit form of the asymptotic variance and provides a unified treatment applicable to other indices beyond J(ν).

Consider an i.i.d. sample $X_1, X_2, \ldots, X_n$ with the same distribution as X.

Thus, the discretization of the s-Gini index gives us:

$$J_n(\nu) = \frac{1}{n\mu_n} \sum_{j=1}^{n} \left(1 + n\left\{\left(1 - \frac{j}{n}\right)^{\nu} - \left(1 - \frac{j-1}{n}\right)^{\nu}\right\}\right) X_{j,n}. \tag{1.2}$$

(see [10])

The main contribution of this work lies in the study of the asymptotic normality of this index and in the validation of the theoretical results through numerical simulations.

The structure of the remainder of the paper is organized as follows: Section 2 recalls the GAR of the s-Gini index as well as the hypotheses and regularity conditions that enabled its derivation. Section 3 establishes the asymptotic normality of the considered index, and Section 4 details the simulation studies and interpretation of results, ending with a conclusion.

2. General asymptotic representation of the s-Gini index

Based on the notations above and other previous work (see [10]), we obtain the statistic $J_n(\nu)$ in the form:

$$J_n(\nu) = \frac{1}{\mu_n}\left\{\frac{1}{n}\sum_{1\le j\le n} h(X_j) + \frac{1}{n}\sum_{1\le j\le n}\Big(\mathbb{F}_n(X_j) - F(X_j)\Big)\, q(X_j)\right\} + o_{\mathbb{P}}(n^{-1/2}), \tag{2.1}$$

From there, as in [13], we need the following conditions:

(CRe1) $\mathbb{E}q(X) < +\infty$;

(CRe2) When $n \to +\infty$, we have:

$$\textit{(H2G)} \quad \int_0^1 \sqrt{n}\,(s - \mathbb{V}_n(s))\,\Delta_n(q,s)\,ds \to 0,$$

with

$$\Delta_n(q,s) = \left\{ q\left(F^{-1}\left(\mathbb{V}_n(s)\right)\right) - q\left(F^{-1}(s)\right)\right\}, \quad s \in (0,1).$$

Therefore, under conditions (CRe1) and (CRe2), the resulting GAR is:

$$J_n(\nu) = \frac{\mathbb{E}H(X)}{\mu} + n^{-1/2}\mathbb{G}_n(h) + n^{-1/2}\int_0^1 \mathbb{G}_n(\tilde{f}_s)\ \ell(s)\ ds + o_{\mathbb{P}}(n^{-1/2}), \quad (2.2)$$

with

$$\begin{aligned} H(x) &= \left\{1-\nu\left(1-F(x)\right)^{\nu-1}\right\}x \quad (2.3)\\ h(x) &= \frac{H}{\mu} - \frac{\mathbb{E}H(X)}{\mu^2}h_1(x), \quad h_1(x) = x \\ h_p(x) &= x^p, \quad p>0 \\ q(x) &= \frac{\nu(\nu-1)}{\mu}(1-F(x))^{\nu-2}x \\ \ell(s) &= -q(F^{-1}(s)) = \frac{-\nu(\nu-1)}{\mu}(1-s)^{\nu-2}F^{-1}(s). \end{aligned}$$

and

$$\tilde{f}_s = 1_{(-\infty,F^{-1}(s)]},\ s \in (0,1). \quad (2.4)$$

3. **Asymptotic normality of the s-Gini index** $J(\nu)$

Based on the asymptotic representation of this index given in the previous section, we now deduce its asymptotic normality.
Consider the following assumptions:

(**Ha**): $\mathbb{E}q(X) < +\infty$

(**Hb**): $\int_0^1 \sqrt{n}(s-\mathbb{V}_n(s))\Delta_n(s)ds \to 0, n \to +\infty$ then

(**Hc**): $\frac{1}{\sqrt{n}}\left(\frac{1}{n}\sum_{j=1}^{n} q(X_j)\right) = \circ_{\mathbb{P}}(1)$, when $n \to +\infty$

(**Hd**): $\int_0^1 [s(1-s)]^{1/2}\Delta_n(s) = \circ_{\mathbb{P}}(1), n \to +\infty.$

(**He**): $\int_0^1 \Gamma_{(1)}(h,\tilde{f}_s)\ell(s)ds = 0.$

The main theoretical result obtained can be stated as follows:

Theorem 3.1. *If $h \in L^2(\mathbb{P}_X)$ and if (**Ha**) or (**Hb**) and (**Hc**) or (**Hd**) are verified, the sequence of indices J_n satisfies:*

$$\sqrt{n}\left(J_n(\nu) - J(\nu)\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0,\Gamma^{(J)}), \quad \textit{when} \quad n \to +\infty,$$

where: $J(\nu) = \mathbb{E}h(X)$ and

$$\Gamma^{(J)}(h_1,h_2)=\int(h_1-\mathbb{E}h_1(X))(h_2-\mathbb{E}h_2(X))d\mathbb{P}_X \tag{3.1}$$

and

$$\begin{aligned}\Gamma^{(J)}&=\gamma_1+\gamma_2+2\gamma_3,\\ \gamma_1&=\Gamma(h,h);\\ \gamma_2&=\int_0^1\int_0^1\Gamma_{(1)}(\tilde{f}_s,\tilde{f}_t)\ell(s)\ell(t)dsdt;\\ \gamma_3&=\int_0^1\Gamma_{(1)}(h,\tilde{f}_s)\ell(s)ds.\end{aligned}$$

*Finally, if (**He**) is verified then:*

$$\Gamma^{(J)}=\gamma_1+\gamma_2$$

Proof. The objective is to show that the considered index is asymptotically normal (see [14] and [15]); it is a matter of proving that:

$$\sqrt{n}(J_n-\mathbb{E}h(X))\rightsquigarrow\mathcal{N}(0,\Gamma^{(J)}),\quad\text{when}\quad n\to+\infty.$$

If we consider an estimator T_n such that $T_n=T+n^{-1/2}\mathbb{G}_n(h)+o_{\mathbb{P}}(n^{-1/2})$, then we have:

$$\begin{aligned}T_n-T=n^{-1/2}\mathbb{G}_n(h)+o_{\mathbb{P}}(n^{-1/2}) &\Longrightarrow\ T_n-T=n^{-1/2}(\mathbb{G}_n(h)+o_{\mathbb{P}}(1))\\ &\Longrightarrow\ \sqrt{n}(T_n-T)=\mathbb{G}_n(h)+o_{\mathbb{P}}(1),\\ &\Longrightarrow\ \sqrt{n}(T_n-T)\longrightarrow\mathcal{N}(0,\Gamma(h,h))\end{aligned}$$

with

$$\Gamma(h,h)=\int\left(h-\mathbb{P}(h)\right)^2d\mathbb{P}_X(x). \tag{3.2}$$

In our case, we have the following GAR for the s-Gini index:

$$J_n(\nu)=\frac{\mathbb{E}(H(X))}{\mu}+n^{-1/2}\left(\mathbb{G}_n(h)+\int_0^1\mathbb{G}_n(\tilde{f}_s)\ell(s)ds\right)+o_{\mathbb{P}}(n^{-1/2}). \tag{3.3}$$

with

$$\begin{aligned}H(x)&=\left\{1-\nu\left(1-F(x)\right)^{\nu-1}\right\}x \qquad (3.4)\\ h(x)&=\frac{H}{\mu}-\frac{\mathbb{E}H(X)}{\mu^2}h_1(x),\quad h_1(x)=x\\ h_p(x)&=x^p,\quad p>0\\ q(x)&=\frac{\nu(\nu-1)}{\mu}(1-F(x))^{\nu-2}x\\ \ell(s)&=-q(F^{-1}(s))=\frac{-\nu(\nu-1)}{\mu}(1-s)^{\nu-2}F^{-1}(s).\end{aligned}$$

Thus, for the s-Gini index, under the verified hypotheses (**Ha**) or (**Hb**) and (**Hc**) or (**Hd**), the J_n index satisfies:

$$J_n(\nu) = \mathbb{E}(H(X)) + n^{-1/2}\left(\mathbb{G}_n(h) + \int_0^1 \mathbb{G}_n(\tilde{f}_s)\ell(s)ds\right) + o_{\mathbb{P}}(n^{-1/2}). \tag{3.5}$$

with $f_s = 1_{[0,s]}$ and $\tilde{f}_s = 1_{(-\infty,F^{-1}(s)]}$, $s \in (0;1)$.

From relation (3.5), we have:

$$J_n(\nu) - \mathbb{E}(H(X)) = n^{-1/2}\left(\mathbb{G}_n(h) + \int_0^1 \mathbb{G}_n(\tilde{f}_s)\ell(s)ds\right) + o_{\mathbb{P}}(n^{-1/2}).$$

Then,

$$\sqrt{n}\left(J_n(\nu) - \mathbb{E}(H(X))\right) = \mathbb{G}_n(h) + \int_0^1 \mathbb{G}_n(\tilde{f}_s)\ell(s)ds + o_{\mathbb{P}}(1) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \Gamma^{(J)}),$$

where

$$\Gamma^{(J)} = Var\left(\mathbb{G}_n(h) + \int_0^1 \mathbb{G}_n(\tilde{f}_s)\ell(s)ds\right). \tag{3.6}$$

Using the formula for the variance of a sum of two Gaussian random variables, $Var(A+B) = Var(A) + Var(B) + 2Cov(A,B)$,

we have:

$$\begin{aligned} Var\left(\mathbb{G}_n(h) + \int_0^1 \mathbb{G}_n(\tilde{f}_s)\ell(s)ds\right) &= Var(\mathbb{G}_n(h)) + Var\left(\int_0^1 \mathbb{G}_n(\tilde{f}_s)\ell(s)ds\right) \\ &+ 2Cov\left(\mathbb{G}_n(h), \int_0^1 \mathbb{G}_n(\tilde{f}_s)\ell(s)ds\right). \end{aligned}$$

Which implies that:

$$\Gamma^{(J)} = \sigma^2_{J(\nu)} = \gamma_1 + \gamma_2 + 2\gamma_3,$$

with:

$$\begin{aligned} \gamma_1 &= Var(\mathbb{G}_n(h)) \\ \gamma_2 &= Var\left(\int_0^1 \mathbb{G}_n(\tilde{f}_s)\ell(s)ds\right) \\ \gamma_3 &= Cov\left(\mathbb{G}_n(h), \int_0^1 \mathbb{G}_n(\tilde{f}_s)\ell(s)ds\right). \end{aligned}$$

We now determine γ_1, γ_2 and γ_3.

$$\begin{aligned} \gamma_1 = Var(\mathbb{G}_n(h)) &= \Gamma(h,h) \\ &= \mathbb{E}[h(X)^2] - [\mathbb{E}h(X)]^2 \\ \gamma_1 &= \mathbb{E}[h(X)^2]. \end{aligned}$$

Using $h(x) = \frac{H}{\mu} - \frac{\mathbb{E}H(X)}{\mu^2}h_1(x)$ where $h_1(x) = x$, we get:

$$\mathbb{E}h(X) = \frac{\mathbb{E}H(X)}{\mu} - \frac{\mathbb{E}H(X)}{\mu^2} \times \mathbb{E}(X).$$

Let $\mathbb{E}(X) = \mu$ and $\mathbb{E}H(X) = \theta$, we have $\mathbb{E}h(X) = \frac{\theta}{\mu} - \frac{\theta}{\mu} = 0$.
Let us find $h(x)^2$, using $h(x) = \frac{H}{\mu} - \frac{\mathbb{E}H(X)}{\mu^2}h_1(x)$.

$$h(x)^2 = \frac{H^2(x)}{\mu^2} - 2\frac{H(x)\mathbb{E}H(X)h_1(x)}{\mu^3} + \frac{[\mathbb{E}H(X)]^2h_1^2(x)}{\mu^4}.$$

Hence

$$\begin{aligned}\mathbb{E}[h(X)^2] &= \mathbb{E}\left(\frac{H^2(x)}{\mu^2} - 2\frac{H(x)\mathbb{E}H(X)h_1(x)}{\mu^3} + \frac{[\mathbb{E}H(X)]^2h_1^2(x)}{\mu^4}\right) \\ &= \frac{1}{\mu^2}\mathbb{E}(H^2(X)) - 2\frac{\mathbb{E}H(X)}{\mu^3}\mathbb{E}\left(H(X)h_1(X)\right) + \frac{[\mathbb{E}H(X)]^2}{\mu^4}\mathbb{E}h_1^2(X).\end{aligned} \tag{3.7}$$

Setting:

$$\mathbb{E}H(X) = \theta,\ \mathbb{E}H^2(X) = m_H^{(2)},\ \mathbb{E}h_1^2(X) = m_2 \text{ and } \mathbb{E}\left(H(X)h_1(X)\right) = m_{Hh_1}.$$

We have:

$$\gamma_1 = \Gamma(h,h) = \frac{m_H^{(2)}}{\mu^2} - 2\frac{\theta m_{Hh_1}}{\mu^3} + \frac{\theta^2 m_2}{\mu^4}. \tag{3.8}$$

Let us determine γ_2

$$\begin{aligned}\gamma_2 &= Var\left(\int_0^1 \mathbb{G}_n(\tilde{f}_s)\ell(s)ds\right),\ s \in (0,1) \\ &= \int_0^1\int_0^1 Cov\left(\mathbb{G}_n(\tilde{f}_s), \mathbb{G}_n(\tilde{f}_t)\right)\ell(s)\ell(t)dsdt \\ &= \int_0^1\int_0^1 Cov\left(\tilde{f}_s), (\tilde{f}_t)\right)\ell(s)\ell(t)dsdt \\ \gamma_2 &= \int_0^1\int_0^1 \Gamma\left(\tilde{f}_s), (\tilde{f}_t)\right)\ell(s)\ell(t)dsdt.\end{aligned} \tag{3.9}$$

Let us calculate $\Gamma\left(\tilde{f}_s, \tilde{f}_t\right)$

$$\begin{aligned}\Gamma\left(\tilde{f}_s, \tilde{f}_t\right) &= \mathbb{E}[\tilde{f}_s(X), \tilde{f}_t(X)] - \mathbb{E}[\tilde{f}_s(X)]\mathbb{E}[\tilde{f}_t(X)],\ \text{with } \tilde{f}_s = 1_{(-\infty, F^{-1}(s)]} \text{ and } \tilde{f}_t = 1_{(-\infty, F^{-1}(t)]} \\ \Gamma\left(\tilde{f}_s, \tilde{f}_t\right) &= min(s,t) - st.\end{aligned}$$

Returning to γ_2 we find:

$$\gamma_2 = \int_0^1\int_0^1 [min(s,t) - st]\ell(s)\ell(t)dsdt,\ \text{with } \ell(s) = -q(F^{-1}(s)) = \frac{-\nu(\nu-1)}{\mu}(1-s)^{\nu-2}F^{-1}(s).$$

So:

$$\gamma_2 = \frac{\nu^2(\nu-1)^2}{\mu^2}\int_0^1\int_0^1 [min(s,t)-st](1-s)^{\nu-2}(1-t)^{\nu-2}F^{-1}(s)F^{-1}(t)dsdt$$

but

$$min(s,t) = \begin{cases} s(1-t) & \text{if } s \leq t \\ t(1-s) & \text{if } s > t, \ s,t \in (0;1) \end{cases}$$

Ultimately we have:

$$\begin{aligned}\gamma_2 = \frac{\nu^2(\nu-1)^2}{\mu^2}\int_0^1 \Bigg[\int_0^s t(1-s)(1-s)^{\nu-2}(1-t)^{\nu-2}F^{-1}(s)F^{-1}(t)dsdt \\ + \int_s^1 s(1-t)(1-s)^{\nu-2}(1-t)^{\nu-2}F^{-1}(s)F^{-1}(t)dsdt\Bigg] \\ \gamma_2 = \frac{2\nu^2(\nu-1)^2}{\mu^2}\int_0^1\int_0^1 s(1-s)^{\nu-2}(1-t)^{\nu-1}F^{-1}(s)F^{-1}(t)dsdt.\end{aligned}$$

Let's now find γ_3 :

$$\begin{aligned}\gamma_3 &= Cov\left(\mathbb{G}_n(h), \int_0^1 \mathbb{G}_n(\tilde{f}_s)\ell(s)ds\right) \\ \gamma_3 &= \int_0^1 \Gamma(h,\tilde{f}_s)\ell(s)ds. \qquad (3.10)\end{aligned}$$

Since $\mathbb{E}[h(X)] = 0$,

$$\begin{aligned}\Gamma(h,\tilde{f}_s) &= \mathbb{E}[h(X),\tilde{f}_s(X)], \\ &= \int_{\mathbb{R}} h(x)1_{(-\infty,F^{-1}(s)]}(x)dF(x) \\ \Gamma(h,\tilde{f}_s) &= \int_{-\infty}^{F^{-1}(s)} h(x)dF(x). \qquad (3.11)\end{aligned}$$

Thus returning to equation (3.10), we have:

$$\gamma_3 = \int_0^1 \left(\int_{-\infty}^{F^{-1}(s)} h(x)dF(x)\right)\ell(s)ds,$$

with $h(x) = \frac{H}{\mu} - \frac{\mathbb{E}H(X)}{\mu^2}x$ and $\ell(s) = \frac{-\nu(\nu-1)}{\mu}(1-s)^{\nu-2}F^{-1}(s)$.

This implies that:

$$\gamma_3 = \frac{-\nu(\nu-1)}{\mu}\int_0^1 \left(\int_{-\infty}^{F^{-1}(s)} \left[\frac{H(x)}{\mu} - \frac{\mathbb{E}H(X)}{\mu^2}x\right]dF(x)\right)(1-s)^{\nu-2}F^{-1}(s)ds.$$

In conclusion:

$$\sqrt{n}\left(J_n(\nu) - J(\nu)\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0,\Gamma^{(J)}),$$

with

$$\Gamma^{(J)} = \sigma^2_{J(\nu)} = \gamma_1 + \gamma_2 + 2\gamma_3,$$

where:

$$\begin{cases} \gamma_1 = \Gamma(h,h) = \frac{m_H^{(2)}}{\mu^2} - 2\frac{\theta m_{Hh_1}}{\mu^3} + \frac{\theta^2 m_2}{\mu^4} \\ \gamma_2 = \frac{2\nu^2(\nu-1)^2}{\mu^2} \int_0^1 \int_0^1 s(1-s)^{\nu-2}(1-t)^{\nu-1} F^{-1}(s) F^{-1}(t) ds dt. \\ \gamma_3 = \frac{-\nu(\nu-1)}{\mu} \int_0^1 \left(\int_{-\infty}^{F^{-1}(s)} \left[\frac{H(x)}{\mu} - \frac{\mathbb{E}H(X)}{\mu^2} x \right] dF(x) \right) (1-s)^{\nu-2} F^{-1}(s) ds. \end{cases} \tag{3.12}$$

Considering the hypothesis (**He**), we have

$$\Gamma^{(J)} = \sigma^2_{J(\nu)} = \gamma_1 + \gamma_2,$$

where:

$$\begin{cases} \gamma_1 = \Gamma(h,h) = \frac{m_H^{(2)}}{\mu^2} - 2\frac{\theta m_{Hh_1}}{\mu^3} + \frac{\theta^2 m_2}{\mu^4} \\ \gamma_2 = \frac{2\nu^2(\nu-1)^2}{\mu^2} \int_0^1 \int_0^1 s(1-s)^{\nu-2}(1-t)^{\nu-1} F^{-1}(s) F^{-1}(t) ds dt. \end{cases} \tag{3.13}$$

□

4. Simulation Studies

The asymptotic normality of the s-Gini index $J(\nu)$ with parameter $\nu \geq 2$ is illustrated using simulated data. To ensure consistency with the theoretical requirements of finite mean and variance, we simulate data from a Pareto distribution with a shape parameter of 2.5 and a scale parameter of 0.1. This choice is consistent with [18], who emphasizes that shape parameters around 2.5 and scale values as low as 0.1 are particularly suitable for modeling income, wealth, or other skewed variables in social science research. Note that a random variable X follows a Pareto distribution with shape parameter β and scale parameter x_0 (denoted Pareto(β, x_0)) if its distribution function is given by:

$$F(x;\beta,x_0) = 1 - \left(\frac{x_0}{x}\right)^{\beta}, \quad \text{for } \beta > 0,\ x \geq x_0 > 0.$$

Regarding the choice of parameters, we use $\nu = 2$ and $\nu = 3.5$ for the s-Gini index, as values between 2 and 5 place greater emphasis on low-income observations (see, for example, [3, 7]). We conduct simulations with sample sizes $n = 100,\ 200,\ 500,\ 1000$ over $N = 1000$ replications. To facilitate reproducibility, the random number generator seed is set to 10.

The simulation results are presented in figures and tables. Figure 1 (with Table 1) illustrates the convergence of $J(2)$ and Figure 2 (with Table 2) that of $J(3.5)$. In the graphs, we plot the densities of $\frac{\sqrt{n}(J_n(\nu)-J(\nu))}{\sigma_J}$, along with the standard normal distribution, to visually assess convergence. The results clearly show an increasing alignment with the standard normal distribution as the sample size increases. This is further confirmed by the numerical summaries in the tables, which report bias, mean squared error (MSE), skewness, and kurtosis. For reference, the standard normal distribution has a skewness of 0 and a kurtosis of 3. Overall,

the simulations provide compelling empirical support for the theoretical results, reinforcing their practical applicability.

In each case considered, for sample sizes $n = 100$, slight deviations can be observed between the simulated density (blue curve) and the standard normal density $\mathcal{N}(0, 1)$ (red curve), particularly in the tails of the distribution, especially when the data are generated from asymmetric or heavy-tailed distributions. However, these discrepancies quickly diminish as the sample size n increases, and from $n = 500$ onward, the simulated densities are practically superimposed on the normal distribution, confirming the theoretical result of asymptotic normality of the s-Gini index estimator (see Figures 1, 2).

The analysis of numerical indicators confirms these graphical observations. The empirical biases are small for all indices and decrease with sample size, indicating that the estimators are asymptotically unbiased. Similarly, the mean squared errors (MSE) systematically decrease as the sample size n increases, confirming the consistency of the studied estimators.

The skewness and kurtosis coefficients provide an assessment of the quality of the normal approximation. The skewness tends toward zero as the sample size increases, reflecting good progressive symmetry of the empirical distributions. The kurtosis values remain close to the theoretical value of 3 associated with the normal distribution, with slight fluctuations due to the inherent variability of the simulations.

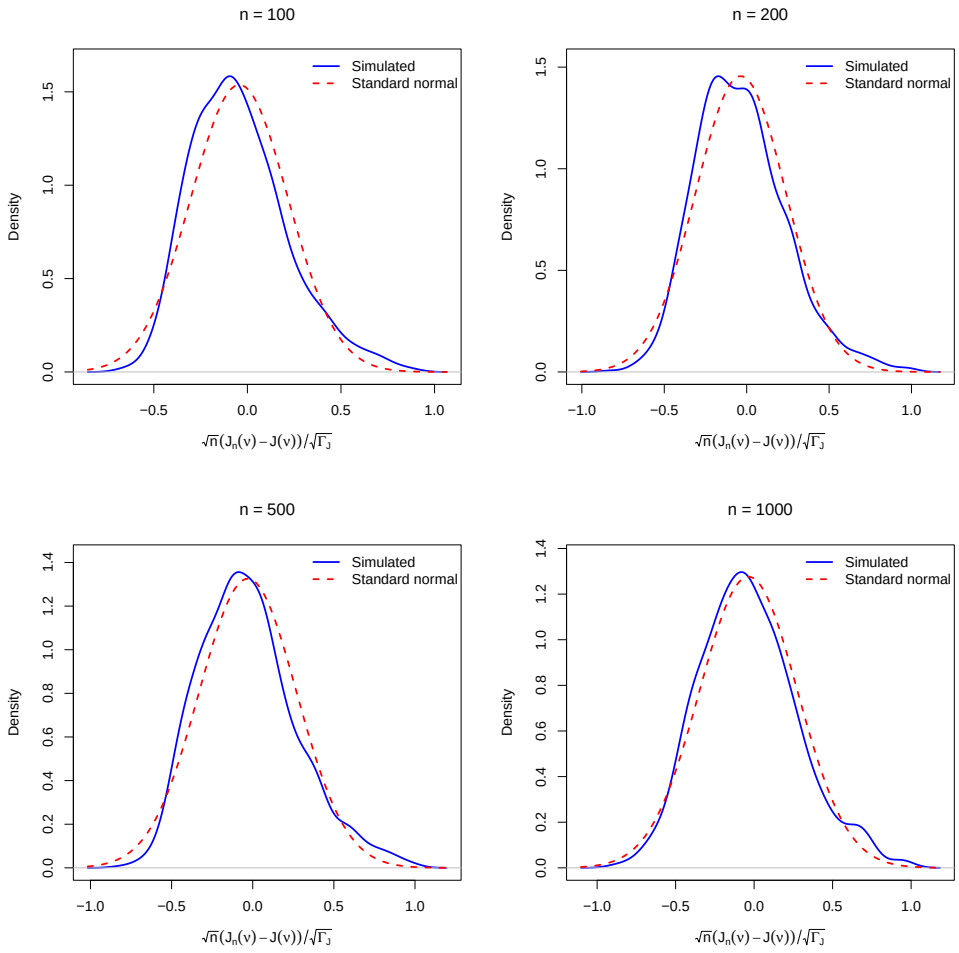


FIGURE 1. Densities illustrating the asymptotic normality of the s-Gini index with $\nu = 2$ (Gini), and samples generated from Pareto ($\beta = 2.5, x_0 = 0.1$).

TABLE 1. Quality measures for the asymptotic normality of the s-Gini index with $\nu = 2$ (Gini) under the Pareto distribution ($\beta = 2.5, x_0 = 0.1$).

J(2)	n=100	n=200	n=500	n=1000
Bias	-0.004958	-0.002726	-0.001220	-0.001041
MSE	0.002103	0.001140	0.000556	0.000314
Skewness	0.657033	0.568056	0.564650	0.396423
Kurtosis	3.404205	3.522445	3.260967	3.145408

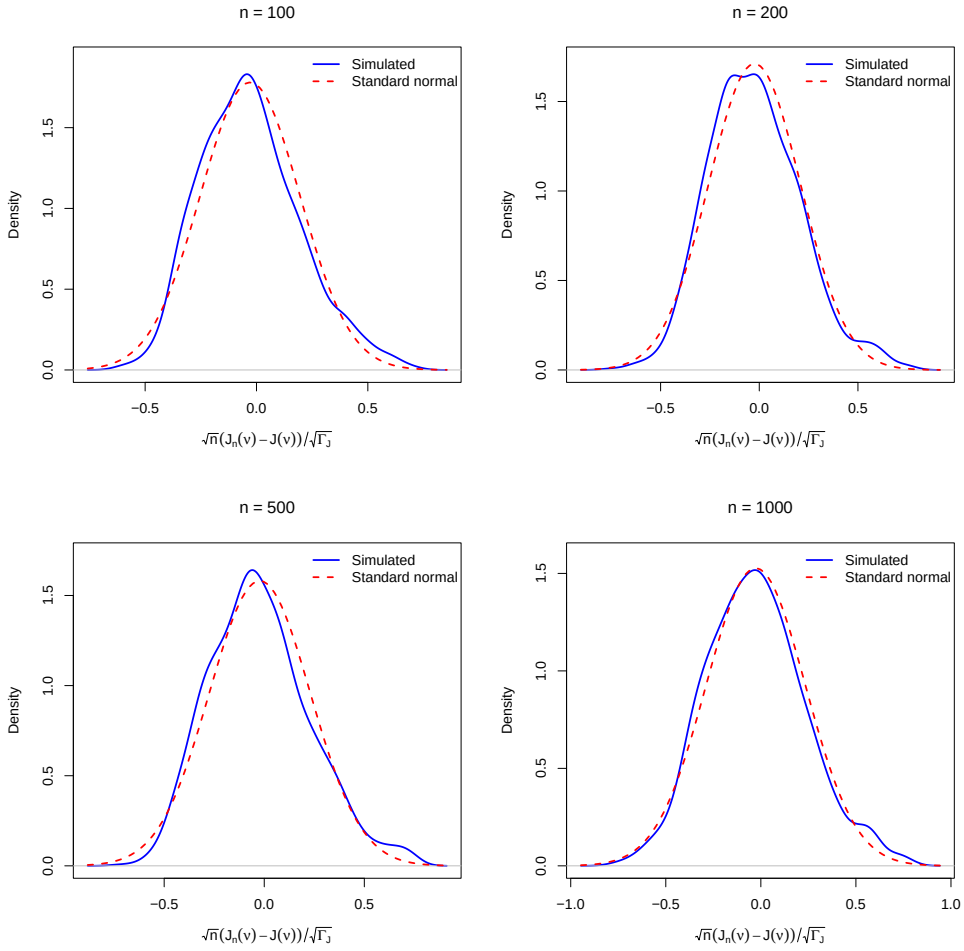


FIGURE 2. Densities illustrating the asymptotic normality of the s-Gini index with $\nu = 3.5$, and samples generated from Pareto $(\beta = 2.5, x_0 = 0.1)$.

TABLE 2. Quality measures for the asymptotic normality of the s-Gini index with $\nu = 3.5$, under the Pareto distribution $(\beta = 2.5, x_0 = 0.1)$.

J(3.5)	n=100	n=200	n=500	n=1000
Bias	-0.003774	-0.002010	-0.001047	-0.000917
MSE	0.002203	0.001167	0.000551	0.000308
Skewness	0.443544	0.378796	0.429891	0.283228
Kurtosis	3.051073	3.24222	3.051276	3.050845

5. Conclusion

In this work, we have established the GAR for the s-Gini inequality index by using the theoretical framework of the functional empirical process and the residual functional empirical process. This approach, which takes advantage of the linearity property of the fep, allowed us to overcome the limitations of traditional methods based on real-valued processes.

The obtained GAR constitutes an important tool for studying the asymptotic behavior of the nonparametric estimator of the s-Gini index. Indeed, under appropriate regularity conditions, we demonstrated that this estimator converges to a normal distribution in large samples. This asymptotic normality result is fundamental for statistical practice, as it theoretically justifies the construction of confidence intervals and the implementation of hypothesis tests related to the s-Gini index.

The simulation studies conducted in Section 4 confirmed the theoretical predictions. The calculation of bias, mean squared error, skewness, and kurtosis showed satisfactory behavior of the estimators, with convergence to normality as the sample size increases. These empirical results reinforce the validity of the proposed approach and demonstrate its usefulness for practical applications.

Future work could investigate bias-reduction techniques for the nonparametric estimator of $J(\nu)$, in the spirit of jackknife-based approaches recently proposed for other statistical indices (see [16]).

References

[1] Babou, E. H., Niang, A. B., & Sangaré, H. (2023). The GAR of the unified welfare Foster-Thorbecke-Greer-Sen-Kakwani index. *Afrika Statistika*, *18*(3), 3623–3650. https://doi.org/10.16929/as/2023.3623.321.sFEP

[2] Bian, N. H., Kanga, S.-H. A., Okou, C., & Hili, O. (2025). Robust goodness-of-fit test for copulas by the maximum mean discrepancy estimator. *Gulf Journal of Mathematics*, *20*(1), 1–21. https://doi.org/10.56947/gjom.v20i.2857

[3] Biewen, M., & Jenkins, S. P. (2006). Variance estimation for generalized entropy and Atkinson inequality indices: The complex survey data case. *Oxford Bulletin of Economics and Statistics*, *68*(3), 371–383. https://doi.org/10.1111/j.1468-0084.2006.00166.x

[4] Camara, A., Fall, A. M., & Diallo, M. (2023). Asymptotic statistical theory for the samples problems using the functional empirical process. *Afrika Statistika*, *18*(3), 3567–3602. https://doi.org/10.16929/as/2023.3567.319.sFEP

[5] Da, G. B. O., Gning, G., & Ngom, M. (2023). Explicit fourth order generalized Jarque-Bera tests and applications to recent laws. *Afrika Statistika*, *18*(3), 3551–3566. https://doi.org/10.16929/as/2023.3551.318.sFEP

[6] Foster, J., Greer, J., & Shorrocks, A. (1984). A class of decomposable poverty measures. *Econometrica*, *52*, 761–766. https://doi.org/10.2307/1913475

[7] Glassman, B. (2019). *Multidimensional inequality: Measurement and analysis using the American Community Survey* (SEHSD Working Paper No. 2019-17). U.S. Census Bureau. https://www.census.gov/content/dam/Census/library/working-papers/2019/demo/sehsd-wp2019-17

[8] Kakwani, N. (1980). On a class of poverty measures. *Econometrica*, *48*, 437–446.

[9] Kpanzou, T. A., Ba, D., Da, G. B. O., & Lo, G. S. (2025). A multinomial asymptotic representation of Zenga's discrete index, its influence function and data-driven applications. *REVSTAT-Statistical Journal*, *23*(2), 189–205. https://doi.org/10.57805/revstat.v23i2.523

[10] Kpatcha, E., Kpanzou, T. A., & Ba, D. (2023). Asymptotic representation of s-Gini index in the fep-lrfep and application. *Afrika Statistika*, *18*(3), 3603–3621. https://doi.org/10.16929/as/2023.3603.320.sFEP

[11] Lo, G. S. (2013). On the general poverty index. *Far East Journal of Theoretical Statistics*, *42*(1), 1–22.

[12] Lo, G. S., Ngom, M., & Kpanzou, T. A. (2016). *Weak convergence (IA). Sequences of random vectors*. SPAS Books Series. https://doi.org/10.16929/sbs/2016.0001,arXiv:1610.05415

[13] Lo, G. S., Kpanzou, T. A., & Da, B. O. G. (2023). General asymptotic representations of indexes based on the functional empirical process and the residual functional empirical process and applications. *Afrika Statistika*, *18*(3), 3521–3548. https://doi.org/10.16929/as/3521.2023.317.ssFEPLREP

[14] Lo, G. S., & Sall, S. T. (2010a). Asymptotic representation theorems for poverty indices. *Afrika Statistika*, Special Vol. 5, 238–244.

[15] Lo, G. S., & Sall, S. T. (2010b). Uniform weak convergence of the time-dependent poverty measures for continuous longitudinal data. *Brazilian Journal of Probability and Statistics*, *24*(3), 457–467. https://doi.org/10.1214/08-BJPS101

[16] Mahamat, A., Diop, A., Dabye, A., & Djabbi, M. M. (2024). Méthodologie du jackknife pour la réduction du biais dans l'estimation de l'indice de queue sous troncature aléatoire. *Gulf Journal of Mathematics*, *18*(1), 38–53. https://doi.org/10.56947/gjom.v18i1.2386

[17] Mergane, P. D., Lo, G. S., & Kpanzou, T. A. (2018). On the joint distribution of variations of the Gini index and welfare indices. In *A collection of papers in mathematics and related sciences: A festschrift in honour of the late Galaye Dia* (pp. 163–194). Spas Editions. https://doi.org/10.16929/sbs/2018.100-02-07

[18] Moral de la Rubia, J. (2025). Pareto distribution: A probability model in social research. *Open Journal of Social Sciences*, *13*(1), 86–121. https://doi.org/10.4236/jss.2025.131007

[19] Sen, A. K. (1976). Poverty: An ordinal approach to measurement. *Econometrica*, *44*, 219–231.

[20] Shorack, G. R., & Wellner, J. A. (1986). *Empirical processes with applications to statistics*. Wiley-Interscience. https://doi.org/10.1137/1.9780898719017

[21] Xu, K. (2000). Inference for generalized Gini indices using the iterated-bootstrap method. *Journal of Business and Economic Statistics*, *18*, 223–227. https://doi.org/10.1080/07350015.2000.10524864

[22] Yitzhaki, S. (1983). On an extension of the Gini inequality index. *International Economic Review*, *24*, 617–628.

[23] Zheng, B. (1997). Aggregate poverty measures. *Journal of Economic Surveys*, *11*, 123–162. https://doi.org/10.1111/1467-6419.00028

[24] Zitikis, R., & Gastwirth, J. L. (2002). The asymptotic distribution of the s-Gini index. *Australian and New Zealand Journal of Statistics*, *44*(4), 439–446. https://doi.org/10.1111/1467-842X.00245

[1] Department of Mathematics, University of Kara, Kara, TOGO.
Email address: kpatcharodrigue2@gmail.com, tchandao.mangamana@univkara.tg, kpanzout@univkara.tg