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EXISTENCE AND UNIQUENESS RESULTS FOR HADAMARD FRACTIONAL BOUNDARY VALUE PROBLEMS

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Abstract. This paper investigates a class of nonlinear Hadamard-type fractional differential equations involving the p-Laplacian operator and multi-point fractional integral boundary conditions. The nonlinear source term depends on both the unknown function and its lower-order Hadamard fractional derivative. By means of the properties of Hadamard fractional calculus, the boundary value problem is converted into an equivalent integral equation whose kernel is given explicitly. For $p > 2$, sufficient conditions ensuring the existence and uniqueness of solutions are established via the Banach contraction principle. An illustrative example demonstrates the applicability of the theoretical result.

Keywords. Hadamard fractional derivative, p-Laplacian operator, multi-point boundary value problem, Banach contraction principle, existence and uniqueness
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1. Introduction

Fractional calculus has become an important branch of mathematical analysis due to its ability to describe memory effects, hereditary properties, and nonlocal interactions that cannot be adequately represented by classical integer-order models. Fractional differential equations have consequently found applications in various areas of science and engineering, including viscoelasticity, anomalous diffusion, control theory, signal processing, and mathematical biology. The fundamental theory of fractional integrals and derivatives, together with a systematic treatment of fractional differential equations, can be found in the monographs [7, 8, 10, 11].

In recent years, considerable attention has been devoted to the qualitative analysis of fractional differential and boundary value problems, particularly to the establishment of the existence, uniqueness, multiplicity, and stability of solutions [6, 13]. Among the various fractional operators introduced in the literature, the Hadamard fractional derivative occupies a distinctive position because its definition involves a logarithmic kernel. In contrast to the power-law kernels

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appearing in the Riemann–Liouville and Caputo operators, the Hadamard operator is naturally expressed in terms of the logarithmic ratio of its arguments. This feature makes it particularly suitable for problems governed by multiplicative or scale-dependent structures. As a result, Hadamard fractional differential equations and their associated boundary value problems have received increasing attention. Several analytical techniques, including fixed point theory, monotone iterative methods, upper and lower solution techniques, and variational methods, have been employed to investigate such problems. A number of existence results have been established for Hadamard fractional boundary value problems subject to integral and multi-point conditions. Positive solutions on unbounded domains have received particular attention: Yang and Qin [19] and Thiramanus et al. [15] established existence results via integral boundary conditions, while Zhang and Liu [21, 22] extended these techniques to higher-order equations and multiplicity results using successive approximation [9]. These contributions, however, are confined to domains without upper bound and do not address the coupling between the equation and a nonlinear operator acting on the highest-order derivative. Moreover, Wang et al. [17] examined a nonlocal Hadamard fractional boundary value problem involving both Hadamard integral and discrete boundary conditions on a half-line. Further existence, uniqueness, and multiplicity results for positive solutions on infinite intervals were presented by Zhang and Liu [22].

Boundary value problems involving the p-Laplacian operator constitute another important class of nonlinear problems. The nonlinear mapping $\phi_p(s) = |s|^{p-2}s$ $(p > 1)$ arises naturally in nonlinear diffusion, non-Newtonian fluid models, elasticity theory, and various other nonlinear phenomena. The presence of this operator generally makes the analysis more complicated because the resulting differential equation is nonlinear even before the nonlinear source term is taken into account. In the Hadamard fractional setting, Wang and Wang [16] studied a nonlinear differential equation involving a p-Laplacian operator and a strip boundary condition. Beyond the Hadamard setting, p-Laplacian fractional boundary value problems have been extensively studied for Riemann–Liouville and Caputo-type derivatives, including anti-periodic problems [5] and related boundary conditions, establishing a range of techniques—fixed point index theory, upper–lower solutions, and topological degree methods—for handling the nonlinearity of ϕ_p. Further developments include coupled systems with fully Hadamard-type integral conditions [1], integro-differential boundary problems [2], integral conditions on unbounded domains for systems [14], and Hadamard fractional systems with integral conditions [20]. These studies demonstrate that Hadamard fractional integral conditions provide a flexible framework for describing boundary interactions determined by information distributed over the underlying interval.

More closely related to the present work, Şenlik Çerdik and Yörük Deren [12] investigated a higher-order Hadamard-type fractional boundary value problem on the half-line in which the nonlinear term depends on both the unknown function and its lower-order Hadamard fractional derivative. They considered a problem

of the form

$$\begin{cases} {}^{H}D_{1^+}^{\vartheta}u(t) + p(t)f\left(t, u(t), {}^{H}D_{1^+}^{\vartheta-1}u(t)\right) = 0, & t \in (1,\infty), \\ u^{(k)}(1) = 0, & 0 \le k \le n-2, \\ {}^{H}D_{1^+}^{\vartheta-1}u(\infty) = \displaystyle\int_1^{\infty} g(t)u(t)\frac{dt}{t} + \sum_{i=1}^{m} \lambda_i {}^{H}I_{1^+}^{\beta_i}u(\varsigma). \end{cases}$$

Thus, their boundary condition combines an integral term with multi-point Hadamard fractional integral terms. In another related contribution, Xu, Cui, and O'Regan [18] studied the Hadamard-type fractional-order three-point boundary value problem

$$\begin{cases} D_{1^+}^{\sigma}z(t) = -\theta(t)f(t, z(t)), \quad t \in (1,\infty), \\ z(1) = \delta z(1) = 0, \\ D_{1^+}^{\sigma-1}z(\infty) = bz(\xi), \end{cases}$$

where $\sigma \in (2,3)$ and $\delta = t\frac{d}{dt}$. By employing a fixed point method and conditions involving the spectral radius of an associated linear operator, they established the existence and multiplicity of positive solutions. These contributions provide important results for higher-order Hadamard fractional problems on unbounded domains and demonstrate the significance of nonlinearities depending on lower-order fractional derivatives and of nonlocal boundary conditions.

Despite these developments, the problem considered in the present paper differs essentially from the aforementioned problems. In particular, our differential equation contains the composite nonlinear operator

$${}^{H}D_{1^+}^{\gamma}\left[\phi_p\left({}^{H}D_{1^+}^{\alpha}u(t)\right)\right],$$

whereas the problems studied in [12, 18] do not involve this composed Hadamard fractional p-Laplacian structure. This composite structure is not merely a technical generalization: since ϕ_p is nonlinear for $p \neq 2$, the reduction of boundary conditions on u to conditions on the auxiliary variable $\phi_p\left({}^{H}D_{1^+}^{\alpha}u\right)$ requires careful justification, unlike the linear case $p = 2$ addressed implicitly in prior work. Moreover, allowing the nonlinearity to depend on the lower-order derivative ${}^{H}D_{1^+}^{\beta}u$ reflects situations in which the rate of local change of the state—rather than the state alone—governs the underlying process, as in models with hereditary damping or rate-dependent diffusion. Furthermore, the nonlinear term in the present problem depends on the lower-order Hadamard fractional derivative ${}^{H}D_{1^+}^{\beta}u$, and the problem is posed on the bounded interval $(1,T)$ subject to a multi-point Hadamard fractional integral boundary condition. The simultaneous presence of the composed Hadamard operators, the nonlinear mapping ϕ_p, the derivative-dependent nonlinearity, and the multi-point fractional integral boundary condition leads to a different integral representation and requires careful estimates for the associated solution operator.

Building on these considerations, in this paper we investigate the following nonlinear Hadamard-type fractional boundary value problem involving the p-Laplacian operator:

$$ {}^{H}D_{1^+}^{\gamma}\left[\phi_p\left({}^{H}D_{1^+}^{\alpha}u(t)\right)\right] + a(t)f\left(t, u(t), {}^{H}D_{1^+}^{\beta}u(t)\right) = 0, \qquad t \in (1,T), \tag{1.1} $$

$$ u(1) = \delta u(1) = \cdots = \delta^{n-2}u(1) = 0, \tag{1.2} $$

$$ {}^{H}D_{1^+}^{\alpha-1}u(T) = \sum_{i=1}^{l-2} \lambda_i {}^{H}I_{1^+}^{\eta}u(\xi_i), \tag{1.3} $$

$$ \begin{aligned} \phi_p\left({}^{H}D_{1^+}^{\alpha}u(t)\right)\Big|_{t=1} &= \delta\left[\phi_p\left({}^{H}D_{1^+}^{\alpha}u(t)\right)\right]\Big|_{t=1} \\ &= \cdots = \delta^{n-2}\left[\phi_p\left({}^{H}D_{1^+}^{\alpha}u(t)\right)\right]\Big|_{t=1} = 0. \end{aligned} \tag{1.4} $$

Here, ${}^{H}D_{1^+}^{\gamma}$, ${}^{H}D_{1^+}^{\alpha}$, and ${}^{H}D_{1^+}^{\beta}$ denote the Hadamard fractional derivatives, ${}^{H}I_{1^+}^{\eta}$ denotes the Hadamard fractional integral, and $\delta = t\frac{d}{dt}$ is the standard delta derivative operator. The orders of the operators satisfy

$$ n-2 < \gamma \le n-1, \qquad n-1 < \alpha \le n, \qquad n-3 < \beta \le n-2, $$

where $n \ge 3$ and $l \ge 3$ are integers. Moreover, $\eta > 0$, $\xi_i \in (1,T)$, and $\lambda_i \in \mathbb{R}^+$ for $i = 1, \ldots, l-2$. The functions $a\colon [1,T] \to \mathbb{R}$ and $f\colon [1,T] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous. The nonlinear mapping ϕ_p is defined by $\phi_p(s) = |s|^{p-2}s$ for $p > 1$. It is invertible, and its inverse is given by $\phi_p^{-1} = \phi_q$, where $\frac{1}{p} + \frac{1}{q} = 1$. Throughout this paper, we assume that $p > 2$ (and hence $1 < q < 2$).

The remainder of the paper is organized as follows. Section 2 briefly recalls the necessary background on Hadamard fractional calculus and the p-Laplacian operator. In Section 3, an equivalent integral representation of the boundary value problem is derived and the main existence and uniqueness result is established. Section 4 presents an illustrative example confirming the applicability of the theoretical results. Finally, Section 5 concludes the paper with some remarks and potential future directions.

2. Preliminaries

In this section, we introduce the basic definitions, notations, and auxiliary results that will be used throughout the paper.

For the fundamental theory and standard properties of Hadamard fractional calculus, we refer the reader to [7, 10]. Recall that the Hadamard fractional integral of order $\nu > 0$ for a suitable function $x\colon [1,T] \to \mathbb{R}$ is defined by

$$ {}^{H}I_{1^+}^{\nu}x(t) = \frac{1}{\Gamma(\nu)} \int_1^t \left(\log\frac{t}{s}\right)^{\nu-1} \frac{x(s)}{s}\, ds, $$

and the Hadamard fractional derivative of order $\nu > 0$ is given by

$$ {}^{H}D_{1^+}^{\nu}x(t) = \delta^n\, {}^{H}I_{1^+}^{n-\nu}x(t) = \frac{1}{\Gamma(n-\nu)}\delta^n \int_1^t \left(\log\frac{t}{s}\right)^{n-\nu-1} \frac{x(s)}{s}\, ds, $$

where $\delta = t\frac{d}{dt}$ and $n = \lceil \nu \rceil$. In particular, we frequently use the composition property $\delta^k\, {}^H I_{1^+}^{\mu} = {}^H I_{1^+}^{\mu-k}$ for $\mu > k \geq 0$.

Lemma 2.1. [7] *If $a, \mu > 0$ and $n-1 < \nu < n$, where $n = [\nu] + 1$, then*

$$\left({}^H I_a^{\nu} \left(\log \frac{t}{a}\right)^{\mu-1}\right)(x) = \frac{\Gamma(\mu)}{\Gamma(\mu+\nu)} \left(\log \frac{x}{a}\right)^{\mu+\nu-1},$$

$$\left({}^H D_a^{\nu} \left(\log \frac{t}{a}\right)^{\mu-1}\right)(x) = \frac{\Gamma(\mu)}{\Gamma(\mu-\nu)} \left(\log \frac{x}{a}\right)^{\mu-\nu-1},$$

$$\left({}^H D_a^{\nu} \left(\log \frac{t}{a}\right)^{\nu-j}\right)(x) = 0, \qquad j = 1, 2, \ldots, n.$$

Lemma 2.2. [7] *Let $\nu > 0$. If $c \in L^1[1, T)$, then the equality*

$$({}^H D_{1^+}^{\nu}\, {}^H I_{1^+}^{\nu} c)(t) = c(t)$$

holds a.e. on $[1, T)$.

Lemma 2.3. [7] *Let $\nu > 0$ and assume that $c \in C[1, T) \cap L^1[1, T)$. If ${}^H D_{1^+}^{\nu} c(t) = 0$, then the general solution is given by*

$$c(t) = \sum_{i=1}^{m} c_i (\log t)^{\nu-i},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, m$, and $m - 1 < \nu < m$, with $m = [\nu] + 1$. Moreover,

$${}^H I_{1^+}^{\nu}\, {}^H D_{1^+}^{\nu} c(t) = c(t) - \sum_{i=1}^{m} d_i (\log t)^{\nu-i},$$

where $d_i \in \mathbb{R}$, $i = 1, 2, \ldots, m$.

Lemma 2.4. [5] *Let $\phi_p \colon \mathbb{R} \to \mathbb{R}$ be the p-Laplacian operator defined by $\phi_p(t) = |t|^{p-2}t$. Then we have the following basic properties:*

(1) *If $1 < p < 2$ and $t \neq 0$, then $(\phi_p(t))' = (p-1)|t|^{p-2}$.*
(2) *If $1 < p < 2$, $tz > 0$, and $|t|, |z| \geq l > 0$, then $|\phi_p(t) - \phi_p(z)| \leq (p-1)l^{p-2}|t - z|$.*
(3) *If $p > 2$ and $|t|, |z| \leq L$, then $|\phi_p(t) - \phi_p(z)| \leq (p-1)L^{p-2}|t - z|$.*
(4) *ϕ_p is invertible such that $\phi_p^{-1}(t) = \phi_q(t)$, where $\frac{1}{p} + \frac{1}{q} = 1$.*

3. Main Results

In this section, we first derive an equivalent integral representation corresponding to the boundary value problem (1.1)–(1.4). Based on this representation, sufficient conditions for the existence and uniqueness of solutions are established by applying the Banach fixed point theorem.

3.1. **Equivalent Integral Equation.**

Lemma 3.1. *Let $a \in C([1,T],\mathbb{R})$ and $f\colon [1,T]\times\mathbb{R}^2 \to \mathbb{R}$ be continuous. Then the solution $u \in E$ of the boundary value problem (1.1)–(1.4) admits the following integral representation:*

$$u(t) = \frac{1}{\Gamma(\alpha)K}\int_1^T G(t,s)\,w(s)\,\frac{ds}{s} + \frac{1}{\Gamma(\alpha)\Gamma(\alpha+\eta)K}\sum_{i=1}^{l-2}\lambda_i\int_1^{b_i(t)} H_i(t,s)\,w(s)\,\frac{ds}{s}, \tag{3.1}$$

where $w(s) = \phi_q\Big({}^{H}I^{\gamma}_{1^+}a(s)f\big(s,u(s),{}^{H}D^{\beta}_{1^+}u(s)\big)\Big)$, $b_i(t) = \max\{t,\xi_i\}$, and

$$K = 1 - \frac{1}{\Gamma(\alpha+\eta)}\sum_{i=1}^{l-2}\lambda_i(\log\xi_i)^{\alpha+\eta-1} \neq 0.$$

The kernel functions $G(t,s)$ and $H_i(t,s)$ are defined by

$$G(t,s) = \begin{cases} (\log t)^{\alpha-1} - \left(\log\frac{t}{s}\right)^{\alpha-1}, & s \le t, \\ (\log t)^{\alpha-1}, & t \le s, \end{cases}$$

and

$$H_i(t,s) = \begin{cases} (\log\xi_i)^{\alpha+\eta-1}\left(\log\frac{t}{s}\right)^{\alpha-1} - (\log t)^{\alpha-1}\left(\log\frac{\xi_i}{s}\right)^{\alpha+\eta-1}, & s \le \min\{t,\xi_i\}, \\ -(\log t)^{\alpha-1}\left(\log\frac{\xi_i}{s}\right)^{\alpha+\eta-1}, & t \le s \le \xi_i, \\ (\log\xi_i)^{\alpha+\eta-1}\left(\log\frac{t}{s}\right)^{\alpha-1}, & \xi_i < s \le t. \end{cases}$$

Proof. From (1.1), we have

$${}^{H}D^{\gamma}_{1^+}\left[\phi_p\left({}^{H}D^{\alpha}_{1^+}u(t)\right)\right] = -a(t)f\left(t,u(t),{}^{H}D^{\beta}_{1^+}u(t)\right).$$

Applying the Hadamard fractional integral operator ${}^{H}I^{\gamma}_{1^+}$ to both sides and using Lemma 2.3, we obtain

$$\begin{aligned}\phi_p\left({}^{H}D^{\alpha}_{1^+}u(t)\right) = {} & c_1(\log t)^{\gamma-1} + c_2(\log t)^{\gamma-2} + \cdots + c_{n-1}(\log t)^{\gamma-(n-1)} \\ & - {}^{H}I^{\gamma}_{1^+}\left[a(t)f\left(t,u(t),{}^{H}D^{\beta}_{1^+}u(t)\right)\right].\end{aligned}$$

Since ${}^{H}I^{\gamma}_{1^+}\left[a(t)f\left(t,u(t),{}^{H}D^{\beta}_{1^+}u(t)\right)\right]_{t=1} = 0$, applying the boundary conditions (1.4) and matching the orders of the powers of $\log t$, we sequentially obtain $c_1 = c_2 = \cdots = c_{n-1} = 0$. Therefore,

$$\phi_p\left({}^{H}D^{\alpha}_{1^+}u(t)\right) = -{}^{H}I^{\gamma}_{1^+}\left[a(t)f\left(t,u(t),{}^{H}D^{\beta}_{1^+}u(t)\right)\right].$$

Since $\phi_p^{-1} = \phi_q$ and ϕ_q is an odd function, this yields

$${}^{H}D^{\alpha}_{1^+}u(t) = -\phi_q\left({}^{H}I^{\gamma}_{1^+}\left[a(t)f\left(t,u(t),{}^{H}D^{\beta}_{1^+}u(t)\right)\right]\right).$$

Applying the Hadamard fractional integral operator ${}^{H}I^{\alpha}_{1^+}$ to both sides and invoking Lemma 2.3, we find

$$u(t) = c_1'(\log t)^{\alpha-1} + c_2'(\log t)^{\alpha-2} + \cdots + c_n'(\log t)^{\alpha-n} - {}^{H}I^{\alpha}_{1^+}\phi_q\left({}^{H}I^{\gamma}_{1^+}\left[a(t)f\left(t, u(t), {}^{H}D^{\beta}_{1^+}u(t)\right)\right]\right),$$

where $c_1', c_2', \ldots, c_n' \in \mathbb{R}$. Applying the initial conditions (1.2) along with the boundedness near $t = 1$, we obtain $c_2' = c_3' = \cdots = c_n' = 0$. Hence,

$$u(t) = c_1'(\log t)^{\alpha-1} - {}^{H}I^{\alpha}_{1^+}\phi_q\left({}^{H}I^{\gamma}_{1^+}\left[a(t)f\left(t, u(t), {}^{H}D^{\beta}_{1^+}u(t)\right)\right]\right).$$

Now, applying the nonlocal boundary condition (1.3) together with Lemma 2.1 and Lemma 2.2, the left-hand side gives

$${}^{H}D^{\alpha-1}_{1^+}u(T) = c_1'\Gamma(\alpha) - \int_1^T \phi_q\left({}^{H}I^{\gamma}_{1^+}\left[a(s)f\left(s, u(s), {}^{H}D^{\beta}_{1^+}u(s)\right)\right]\right)\frac{ds}{s}.$$

On the other hand, applying the operator ${}^{H}I^{\eta}_{1^+}$ to $u(t)$ at $t = \xi_i$, we obtain

$$\sum_{i=1}^{l-2}\lambda_i\left({}^{H}I^{\eta}_{1^+}u(\xi_i)\right) = c_1'\frac{\Gamma(\alpha)}{\Gamma(\alpha+\eta)}\sum_{i=1}^{l-2}\lambda_i(\log\xi_i)^{\alpha+\eta-1} - \frac{1}{\Gamma(\alpha+\eta)}\sum_{i=1}^{l-2}\lambda_i\int_1^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\alpha+\eta-1}w(s)\frac{ds}{s},$$

where $w(s) = \phi_q({}^{H}I^{\gamma}_{1^+}[a(s)f(s, u(s), {}^{H}D^{\beta}_{1^+}u(s))])$. Equating both sides and solving for c_1', we deduce

$$c_1' = \frac{1}{\Gamma(\alpha)K}\int_1^T w(s)\frac{ds}{s} - \frac{1}{\Gamma(\alpha)\Gamma(\alpha+\eta)K}\sum_{i=1}^{l-2}\lambda_i\int_1^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\alpha+\eta-1}w(s)\frac{ds}{s}.$$

Substituting c_1' into the expression for $u(t)$, we have

$$u(t) = \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)K}\int_1^T w(s)\frac{ds}{s} - \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}w(s)\frac{ds}{s} - \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)\Gamma(\alpha+\eta)K}\sum_{i=1}^{l-2}\lambda_i\int_1^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\alpha+\eta-1}w(s)\frac{ds}{s}.$$

Expanding K in the third term and rearranging the integrals, we arrive at

$$u(t) = \frac{1}{\Gamma(\alpha)K}\left[\int_1^T(\log t)^{\alpha-1}w(s)\frac{ds}{s} - \int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}w(s)\frac{ds}{s}\right] + \frac{1}{\Gamma(\alpha)\Gamma(\alpha+\eta)K}\sum_{i=1}^{l-2}\lambda_i\left[\int_1^t(\log\xi_i)^{\alpha+\eta-1}\left(\log\frac{t}{s}\right)^{\alpha-1}w(s)\frac{ds}{s} - \int_1^{\xi_i}(\log t)^{\alpha-1}\left(\log\frac{\xi_i}{s}\right)^{\alpha+\eta-1}w(s)\frac{ds}{s}\right].$$

Using the definitions of $G(t,s)$ and $H_i(t,s)$, this can be written compactly as

$$u(t) = \frac{1}{\Gamma(\alpha)K}\int_1^T G(t,s)\,w(s)\,\frac{ds}{s} + \frac{1}{\Gamma(\alpha)\Gamma(\alpha+\eta)K}\sum_{i=1}^{l-2}\lambda_i\int_1^{b_i(t)} H_i(t,s)\,w(s)\,\frac{ds}{s}.$$

which,by the definitions of G and H_i, is exactly (3.1).

Conversely, let $u \in E$ satisfy (3.1). Reversing the integration process and applying the definitions of the kernels $G(t,s)$ and $H_i(t,s)$, $u(t)$ can be rewritten in the expanded form

$$u(t) = c_1'(\log t)^{\alpha-1} - {}^{H}I_{1^+}^{\alpha}w(t),$$

where $w(t) = \phi_q\left({}^{H}I_{1^+}^{\gamma}h(t)\right)$ and $h(t) = a(t)f\left(t, u(t), {}^{H}D_{1^+}^{\beta}u(t)\right)$. Since $w \in C[1,T]$ and ${}^{H}D_{1^+}^{\alpha}(\log t)^{\alpha-1} = 0$, applying the Hadamard fractional derivative operator ${}^{H}D_{1^+}^{\alpha}$ to $u(t)$ and using Lemma 2.1 and Lemma 2.2 yields

$${}^{H}D_{1^+}^{\alpha}u(t) = -{}^{H}D_{1^+}^{\alpha}\,{}^{H}I_{1^+}^{\alpha}w(t) = -w(t).$$

Hence, using the oddness of ϕ_p and the inversion property $\phi_p(\phi_q(s)) = s$, we obtain

$$\phi_p\left({}^{H}D_{1^+}^{\alpha}u(t)\right) = -\phi_p\left(\phi_q\left({}^{H}I_{1^+}^{\gamma}h(t)\right)\right) = -{}^{H}I_{1^+}^{\gamma}h(t).$$

Consequently, applying ${}^{H}D_{1^+}^{\gamma}$ to both sides, we deduce

$${}^{H}D_{1^+}^{\gamma}\left[\phi_p\left({}^{H}D_{1^+}^{\alpha}u(t)\right)\right] = -{}^{H}D_{1^+}^{\gamma}\,{}^{H}I_{1^+}^{\gamma}h(t) = -h(t),$$

which proves that the fractional differential equation (1.1) holds.

Furthermore, for $0 \le k \le n-2$,

$$\delta^k\left[\phi_p\left({}^{H}D_{1^+}^{\alpha}u(t)\right)\right] = -{}^{H}I_{1^+}^{\gamma-k}h(t) \to 0 \quad \text{as } t \to 1^+,$$

since $\gamma - k \ge \gamma - (n-2) > 0$; thus the boundary conditions (1.4) are satisfied. Next, differentiating $u(t)$ via δ^k, we have

$$\delta^k u(t) = c_1'\frac{\Gamma(\alpha)}{\Gamma(\alpha-k)}(\log t)^{\alpha-1-k} - {}^{H}I_{1^+}^{\alpha-k}w(t), \quad 0 \le k \le n-2.$$

Both terms vanish as $t \to 1^+$ since $\alpha - 1 - k \ge \alpha - n + 1 > 0$, confirming that (1.2) holds. Finally, the nonlocal boundary condition (1.3) holds by the choice of the constant c_1', as established in the necessity part. Thus, the boundary value problem (1.1)–(1.4) is equivalent to the integral equation (3.1).

□

3.2. **Existence and Uniqueness of Solutions.** In this subsection, we establish the existence and uniqueness of solutions to the boundary value problem (1.1)–(1.4).

We work in the Banach space

$$E = \left\{u \in C([1,T],\mathbb{R}) : {}^{H}D_{1^+}^{\beta}u \in C([1,T],\mathbb{R})\right\}$$

endowed with the norm $\|u\| = \max\{\|u\|_1, \|u\|_2\}$, where

$$\|u\|_1 = \max_{t\in[1,T]}|u(t)| \qquad \text{and} \qquad \|u\|_2 = \max_{t\in[1,T]}\left|{}^{H}D_{1^+}^{\beta}u(t)\right|.$$

It is known that E is a Banach space with respect to this norm [4].

Throughout this subsection, we assume that the following hypotheses hold:

(H_1) There exists a positive constant M such that

$$\left|f(t,u_1,v_1)-f(t,u_2,v_2)\right| \le M\left(|u_1-u_2|+|v_1-v_2|\right)$$

for all $t\in[1,T]$ and $u_1,u_2,v_1,v_2\in\mathbb{R}$.

(H_2) There exists a positive constant B such that $f(t,u,v)\ge B$ for all $t\in[1,T]$ and $u,v\in\mathbb{R}$.

(H_3) The function $a\in C([1,T],\mathbb{R})$ satisfies $a(t)>0$ for all $t\in[1,T]$, and we set $\varphi:=\min_{t\in[1,T]}a(t)>0$.

For convenience in the subsequent analysis, we define the following positive constants:

$$\omega_1 := \frac{2M(q-1)\|a\|_\infty(\log T)^{\alpha+\gamma(q-1)}}{\Gamma(\alpha)\Gamma(\gamma+1)|K|\left[\gamma(q-1)+1\right]}\left(\frac{B\varphi}{\Gamma(\gamma+1)}\right)^{q-2} \times\left[1+\frac{1}{\Gamma(\alpha+\eta)}\sum_{i=1}^{l-2}\lambda_i(\log\xi_i)^{\alpha+\eta-1}\right], \tag{3.2}$$

$$\omega_2 := \frac{M(q-1)\|a\|_\infty(\log T)^{\alpha-\beta+\gamma(q-1)}}{\Gamma(\alpha-\beta)\Gamma(\gamma+1)\left[\gamma(q-1)+1\right]}\left(\frac{B\varphi}{\Gamma(\gamma+1)}\right)^{q-2} \times\left\{1+\frac{1}{|K|}\left[1+\frac{1}{\Gamma(\alpha+\eta)}\sum_{i=1}^{l-2}\lambda_i(\log\xi_i)^{\alpha+\eta-1}\right]\right\}, \tag{3.3}$$

where $K\neq 0$ is the constant defined in Lemma 3.1.

Theorem 3.2. *Assume that $p>2$ and that (H_1)–(H_3) hold. If*

$$2\max\{\omega_1,\omega_2\}<1,$$

then the boundary value problem (1.1)–(1.4) admits a unique solution in E.

Proof. By assumptions (H_2) and (H_3), it follows that

$$k(\tau)=a(\tau)f\left(\tau,u(\tau),{}^{H}D^{\beta}_{1^+}u(\tau)\right)\ge B\,\varphi.$$

Consequently, for every $s\in(1,T]$,

$$\frac{1}{\Gamma(\gamma)}\int_1^s\left(\log\frac{s}{\tau}\right)^{\gamma-1}k(\tau)\,\frac{d\tau}{\tau}\ge\frac{B\varphi}{\Gamma(\gamma+1)}(\log s)^{\gamma}.$$

Let $u, v \in E$. By Lemma 2.4, assumption (H_1), and the above estimate, we obtain

$$\begin{aligned}
&\left|\phi_q\left[{}^H I^{\gamma}_{1+}\left(a(s)f\left(s,u(s),{}^H D^{\beta}_{1+}u(s)\right)\right)\right]-\phi_q\left[{}^H I^{\gamma}_{1+}\left(a(s)f\left(s,v(s),{}^H D^{\beta}_{1+}v(s)\right)\right)\right]\right| \\
&\le (q-1)\left(\frac{B\varphi}{\Gamma(\gamma+1)}(\log s)^{\gamma}\right)^{q-2} \\
&\quad\times\left|{}^H I^{\gamma}_{1+}\left[a(s)\left(f\left(s,u(s),{}^H D^{\beta}_{1+}u(s)\right)-f\left(s,v(s),{}^H D^{\beta}_{1+}v(s)\right)\right)\right]\right| \\
&\le \frac{M(q-1)}{\Gamma(\gamma)}\left(\frac{B\varphi}{\Gamma(\gamma+1)}(\log s)^{\gamma}\right)^{q-2}(\|u-v\|_1+\|u-v\|_2) \\
&\quad\times\int_1^s\left(\log\frac{s}{\tau}\right)^{\gamma-1}a(\tau)\,\frac{d\tau}{\tau}. \qquad (3.4)
\end{aligned}$$

Let $A\colon E\to E$ be the operator defined by

$$(Au)(t)=\frac{1}{\Gamma(\alpha)K}\int_1^T G(t,s)w(s)\,\frac{ds}{s}+\frac{1}{\Gamma(\alpha)\Gamma(\alpha+\eta)K}\sum_{i=1}^{l-2}\lambda_i\int_1^{b_i(t)}H_i(t,s)w(s)\,\frac{ds}{s}.$$

Using the definitions of G and H_i, the operator A can equivalently be written as

$$\begin{aligned}
(Au)(t)=\frac{1}{\Gamma(\alpha)K}&\left[\int_1^T(\log t)^{\alpha-1}w(s)\,\frac{ds}{s}-\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}w(s)\,\frac{ds}{s}\right] \\
&+\frac{1}{\Gamma(\alpha)\Gamma(\alpha+\eta)K}\sum_{i=1}^{l-2}\lambda_i\left[\int_1^t(\log\xi_i)^{\alpha+\eta-1}\left(\log\frac{t}{s}\right)^{\alpha-1}w(s)\,\frac{ds}{s}\right. \\
&\qquad\left.-\int_1^{\xi_i}(\log t)^{\alpha-1}\left(\log\frac{\xi_i}{s}\right)^{\alpha+\eta-1}w(s)\,\frac{ds}{s}\right].
\end{aligned}$$

First, we show that the operator $A\colon E\to E$ is well-defined. For any $u\in E$, by the continuity of $a(t)$, $f\left(t,u(t),{}^H D^{\beta}_{1+}u(t)\right)$, and the properties of the Hadamard fractional integral operator, it follows that $w\in C[1,T]$, where

$$w(t)=\phi_q\left({}^H I^{\gamma}_{1+}\left[a(t)f\left(t,u(t),{}^H D^{\beta}_{1+}u(t)\right)\right]\right).$$

Consequently, from the explicit formula of $(Au)(t)$, we have $Au\in C([1,T],\mathbb{R})$.

Furthermore, applying the Hadamard fractional derivative of order β to $(Au)(t)$, we obtain

$$\begin{aligned}
{}^H D^{\beta}_{1+}(Au)(t)=c_1'&\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}(\log t)^{\alpha-\beta-1} \\
&-\frac{1}{\Gamma(\alpha-\beta)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-\beta-1}\phi_q\left[{}^H I^{\gamma}_{1+}\left(a(s)f\left(s,u(s),{}^H D^{\beta}_{1+}u(s)\right)\right)\right]\frac{ds}{s}.
\end{aligned}$$

Since $\alpha-\beta>1$ (as $n-1<\alpha\le n$ and $n-3<\beta\le n-2$), the power function $(\log t)^{\alpha-\beta-1}$ is continuous on $[1,T]$ with value 0 at $t=1$. Moreover,

since $w \in C[1,T]$, the fractional integral ${}^{H}I_{1+}^{\alpha-\beta}w(t)$ is continuous on $[1,T]$. Hence, ${}^{H}D_{1+}^{\beta}(Au) \in C([1,T],\mathbb{R})$. Thus, $A(E) \subseteq E$, confirming that A is well-defined.

We now prove that A is a contraction mapping. Let $u, v \in E$. By estimate (3.4), we obtain $|(Au)(t) - (Av)(t)| \leq I_1 + I_2 + I_3 + I_4$, where

$$I_k = \frac{M(q-1)}{\Gamma(\alpha)\Gamma(\gamma)|K|}\left(\frac{B\varphi}{\Gamma(\gamma+1)}\right)^{q-2}(\|u-v\|_1 + \|u-v\|_2)\,J_k(t), \quad (k=1,2,3,4).$$

Setting $\Phi(s) = (\log s)^{\gamma(q-2)}\int_1^s \left(\log\frac{s}{\tau}\right)^{\gamma-1} a(\tau)\,\frac{d\tau}{\tau}$, the terms $J_k(t)$ are given explicitly by

$$J_1(t) = (\log t)^{\alpha-1}\int_1^T \Phi(s)\,\frac{ds}{s},$$

$$J_2(t) = \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\Phi(s)\,\frac{ds}{s},$$

$$J_3(t) = \frac{\sum_{i=1}^{l-2}\lambda_i(\log\xi_i)^{\alpha+\eta-1}}{\Gamma(\alpha+\eta)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\Phi(s)\,\frac{ds}{s},$$

$$J_4(t) = \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha+\eta)}\sum_{i=1}^{l-2}\lambda_i\int_1^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\alpha+\eta-1}\Phi(s)\,\frac{ds}{s}.$$

Using the bounds $(\log t)^{\alpha-1} \leq (\log T)^{\alpha-1}$, $(\log\frac{t}{s})^{\alpha-1} \leq (\log T)^{\alpha-1}$, and $(\log\frac{\xi_i}{s})^{\alpha+\eta-1} \leq (\log\xi_i)^{\alpha+\eta-1}$, we obtain

$$\begin{aligned}\big|(Au)(t) - (Av)(t)\big| \leq{}& \frac{2M(q-1)(\log T)^{\alpha-1}}{\Gamma(\alpha)\Gamma(\gamma)|K|}\left(\frac{B\varphi}{\Gamma(\gamma+1)}\right)^{q-2}(\|u-v\|_1+\|u-v\|_2)\\ &\times\left[1+\frac{1}{\Gamma(\alpha+\eta)}\sum_{i=1}^{l-2}\lambda_i(\log\xi_i)^{\alpha+\eta-1}\right]\\ &\times\int_1^T(\log s)^{\gamma(q-2)}\int_1^s\left(\log\frac{s}{\tau}\right)^{\gamma-1}a(\tau)\,\frac{d\tau}{\tau}\frac{ds}{s}.\end{aligned}$$

Setting $\|a\|_\infty := \max_{t\in[1,T]}|a(t)|$, a direct integration yields

$$\begin{aligned}\int_1^T(\log s)^{\gamma(q-2)}\int_1^s\left(\log\frac{s}{\tau}\right)^{\gamma-1}a(\tau)\,\frac{d\tau}{\tau}\frac{ds}{s} &\leq \|a\|_\infty\int_1^T(\log s)^{\gamma(q-2)}\int_1^s\left(\log\frac{s}{\tau}\right)^{\gamma-1}\frac{d\tau}{\tau}\frac{ds}{s}\\ &= \frac{\|a\|_\infty}{\gamma\left[\gamma(q-1)+1\right]}(\log T)^{\gamma(q-1)+1}.\end{aligned}$$

Substituting this estimate and invoking $\gamma\Gamma(\gamma) = \Gamma(\gamma+1)$, we obtain

$$\|Au - Av\|_1 \leq \omega_1\left(\|u-v\|_1 + \|u-v\|_2\right),$$

where ω_1 is defined in (3.2).

Next, we evaluate $\left|{}^{H}D^{\beta}_{1^+}(Au)(t) - {}^{H}D^{\beta}_{1^+}(Av)(t)\right|$. Denoting the constant c_1' by $c_1'(u)$ to explicitly indicate its dependence on u, we have

$$\begin{aligned}
&\left|{}^{H}D^{\beta}_{1^+}(Au)(t) - {}^{H}D^{\beta}_{1^+}(Av)(t)\right| \\
&\leq \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}(\log t)^{\alpha-\beta-1}\left|c_1'(u) - c_1'(v)\right| \\
&\quad + \frac{1}{\Gamma(\alpha-\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-\beta-1} \Bigg| \phi_q\left[{}^{H}I^{\gamma}_{1^+}\left(a(s)f\left(s,u(s),{}^{H}D^{\beta}_{1^+}u(s)\right)\right)\right] \\
&\qquad\qquad - \phi_q\left[{}^{H}I^{\gamma}_{1^+}\left(a(s)f\left(s,v(s),{}^{H}D^{\beta}_{1^+}v(s)\right)\right)\right] \Bigg| \frac{ds}{s}.
\end{aligned}$$

Using the definition of $c_1'(u)$ and (3.4), the difference $|c_1'(u) - c_1'(v)|$ satisfies

$$\begin{aligned}
|c_1'(u) - c_1'(v)| \leq{}& \frac{M(q-1)\|a\|_\infty(\log T)^{\gamma(q-1)+1}}{\Gamma(\alpha)\Gamma(\gamma+1)|K|\left[\gamma(q-1)+1\right]}\left(\frac{B\varphi}{\Gamma(\gamma+1)}\right)^{q-2} \\
&\times\left[1+\frac{1}{\Gamma(\alpha+\eta)}\sum_{i=1}^{l-2}\lambda_i(\log\xi_i)^{\alpha+\eta-1}\right]\left(\|u-v\|_1 + \|u-v\|_2\right).
\end{aligned}$$

Substituting this bound into the derivative estimate, taking the supremum over $t\in[1,T]$, and recalling (3.3), we find

$$\|Au - Av\|_2 \leq \omega_2\left(\|u-v\|_1 + \|u-v\|_2\right).$$

Therefore,

$$\begin{aligned}
\|Au - Av\| &= \max\left\{\|Au-Av\|_1, \|Au-Av\|_2\right\} \\
&\leq \max\{\omega_1,\omega_2\}\left(\|u-v\|_1 + \|u-v\|_2\right) \\
&\leq 2\max\{\omega_1,\omega_2\}\|u-v\|.
\end{aligned}$$

Since $2\max\{\omega_1,\omega_2\} < 1$, the operator A is a contraction on the Banach space E. By the Banach contraction principle [3], A admits a unique fixed point in E, which corresponds to the unique solution of the boundary value problem (1.1)–(1.4). □

4. An Example

In this section, we present an illustrative example to confirm the applicability of Theorem 3.2.

Example 4.1. Consider the following boundary value problem:

$$\begin{cases}
{}^{H}D^{3/2}_{1^+}\left[\phi_3\left({}^{H}D^{5/2}_{1^+}u(t)\right)\right] + f\left(t,u(t),{}^{H}D^{1/2}_{1^+}u(t)\right) = 0, & t\in(1,2), \\
u(1) = \delta u(1) = 0, \\
{}^{H}D^{3/2}_{1^+}u(2) = {}^{H}I^{1/2}_{1^+}u\left(\dfrac{3}{2}\right), \\
\phi_3\left({}^{H}D^{5/2}_{1^+}u(t)\right)\Big|_{t=1} = \delta\left[\phi_3\left({}^{H}D^{5/2}_{1^+}u(t)\right)\right]\Big|_{t=1} = 0,
\end{cases}$$

where $a(t) = 1$, $\delta = t\frac{d}{dt}$, and $f(t, u, v) = 1 + |u| + |v|$. We choose the parameters as

$$n = 3, \quad \alpha = \frac{5}{2}, \quad \beta = \frac{1}{2}, \quad \gamma = \frac{3}{2}, \quad \eta = \frac{1}{2}, \quad p = 3, \quad q = \frac{3}{2}, \quad T = 2, \quad \lambda_1 = 1, \quad \xi_1 = \frac{3}{2},$$

which satisfy all structural conditions.

Furthermore, assumptions (H_1)–(H_3) clearly hold with $M = 1$, $B = 1$, and $\varphi = \|a\|_\infty = 1$. Using these parameters, direct computation yields

$$K = 1 - \frac{1}{2}\left(\log\frac{3}{2}\right)^2 \approx 0.917799 \neq 0.$$

Substituting these values into (3.2) and (3.3), we obtain

$$\omega_1 \approx 0.133562 \quad \text{and} \quad \omega_2 \approx 0.197101.$$

Consequently,

$$2\max\{\omega_1, \omega_2\} = 2\omega_2 \approx 0.394202 < 1.$$

Since all hypotheses of Theorem 3.2 are satisfied, the boundary value problem admits a unique solution in E.

5. Conclusions

In this paper, we have investigated a class of Hadamard fractional p-Laplacian boundary value problems subject to multi-point fractional integral boundary conditions. First, the considered boundary value problem was transformed into an equivalent integral equation. Based on this representation, an appropriate operator was defined on a suitable Banach space. Under the imposed continuity and Lipschitz-type assumptions, sufficient conditions for the existence and uniqueness of solutions were established by means of the Banach fixed point theorem. In particular, explicit estimates for the associated operator were derived, and a contraction condition expressed in terms of the constants ω_1 and ω_2 was obtained. Finally, an illustrative example was presented to demonstrate the applicability of the theoretical results. The results obtained in this study contribute to the qualitative analysis of nonlinear fractional boundary value problems involving Hadamard fractional derivatives and p-Laplacian operators with $p > 2$. It is worth noting that the case $1 < p \leq 2$ requires different analytical inequalities for the p-Laplacian operator, which remains an interesting topic for future investigation.

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