Gulf Journal of Mathematics
Vol 24, Issue 2 (2026) 1-17

ISSN: 2309-4966
https://doi.org/10.56947/gfnpgn66

FIXED POINT THEOREMS FOR PAIRED MAPPINGS IN b-METRIC SPACES

HEZEKIAH SEUN ADEWINBI [1*]

Abstract. In this paper, we introduce the notion of a Paired–Chatterjea contraction for single-valued mappings and establish corresponding fixed point theorems under this contractive condition. We show that every Chatterjea-type contraction is a Paired–Chatterjea contraction, whereas the converse does not hold, demonstrating that the latter class properly extends the former. We further establish fixed point results for Paired and Paired–Kannan contractions, including a necessary and sufficient condition for the existence of fixed points in the Paired–Kannan setting. We present several examples to illustrate the applicability and sharpness of the results. In addition, our results extend and provide converses to several existing paired contractive theorems.

Keywords. Paired Contraction, Paired-Kannan, Paired-Chatterjea, Fixed Point
2020 Mathematics Subject Classification. Primary 47H10; Secondary 54H25

1. Introduction and Preliminaries

The celebrated Banach contraction principle [5] is a cornerstone of nonlinear analysis and fixed point theory. It guarantees the existence and uniqueness of fixed points for contraction mappings on complete metric spaces and provides an iterative method for their approximation. Its elegance and broad applicability [1, 3, 20] have motivated numerous extensions and generalizations [19, 21, 34, 37, 38, 39, 40]. Bakhtin [4] and Czerwik [18] introduced b-metric spaces as a natural generalization of metric spaces. This framework extends the applicability of fixed point methods beyond classical metric spaces and has prompted extensive research on generalized fixed point theorems [2, 9, 11, 22, 23, 24, 26, 27, 28, 30, 31]. In 2024, Chand and Rohen [12] introduced the concept of paired contractions in complete metric spaces, providing a new framework for studying self-mappings satisfying a joint contractive condition involving three points. For recent developments in this direction, we refer the reader to [6, 7, 8, 15, 16, 17, 32, 33, 35, 36].

Motivated by these developments, we introduce Paired–Chatterjea contractions in b-metric spaces and establish corresponding fixed point results. Furthermore, we extend the admissible contraction range in the paired framework from $[0, 1/s)$

Date: Received: Sep 3, 2026; Revised: Sep 30, 2026; Accepted: Oct 7, 2026.
* Corresponding author
© The Author(s) 2026. This article is licensed under a Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License. To view a copy of the licence, visit https://creativecommons.org/licenses/by-nc-nd/4.0/.

to $[0, 1)$. Additionally, we show that the coefficient condition in [10, Theorem 3] is insufficient and then establish a fixed point theorem under the revised condition $\alpha < \min\{\frac{1}{2}, \frac{1}{s}\}$. Several examples are provided to illustrate the validity of our results and distinguish the proposed paired contractive mappings from existing classes.

Definition 1.1. ([4], [18]) Let X be a nonempty set, and let $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}_+$ is called a *b-metric* if for all $x, y, z \in X$,

(a) $d(x, y) = 0$ if and only if $x = y$;
(b) $d(x, y) = d(y, x)$;
(c) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

A pair (X, d) is called a b-metric space.

Example 1.2. Let (X, d_0) be a metric space, and let $n > 1$. Define $d(x, y) = d_0(x, y)^n$. Thus, (X, d) is a b-metric space with coefficient $s = 2^{n-1}$.

Theorem 1.3. *([25]) Let (X, d) be a complete b-metric space with coefficient $s \geq 1$. Suppose $T : X \to X$ is a mapping satisfying the Chatterjea-type contraction condition*

$$d(Tx, Ty) \leq \alpha\,[d(x, Ty) + d(y, Tx)] \tag{1.1}$$

for all $x, y \in X$, where $\alpha \in \left[0, \frac{1}{2s}\right)$. Then T has a unique fixed point in X.

Lemma 1.4. *([29]) Let (X, d) be a b-metric space with coefficient $s \geq 1$. Suppose that $\{x_n\}$ is a sequence in X. If there exists $\gamma \in [0, 1)$ satisfying*

$$d(x_{n+1}, x_n) \leq \gamma d(x_n, x_{n-1}),$$

for every $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.

2. Fixed Point Theorem for Paired-Chatterjea Contractions

In this section, we introduce Paired–Chatterjea contractions in b-metric spaces and establish corresponding fixed point theorems. We also show that every Chatterjea-type contraction is a Paired–Chatterjea contraction.

Definition 2.1. Let (X, d) be a b-metric space with $|X| \geq 3$. A mapping $T : X \to X$ is called a *Paired-Chatterjea contraction* on X if there exists a constant $\alpha \in \left[0, \frac{1}{2s}\right)$ such that

$$d(Tx, Ty) + d(Ty, Tz) \leq \alpha(d(x, Ty) + d(y, Tx) + d(y, Tz) + d(z, Ty)) \tag{2.1}$$

holds for all three pairwise distinct $x, y, z \in X$.

Remark 2.2. Let (X, d) be a b-metric space with $|X| \geq 3$, $T : X \to X$ be a Chatterjea-type contraction on X and let $x, y, z \in X$ be pairwise distinct. Consider the inequality (1.1) for the pair y, z:

$$d(Ty, Tz) \leq \alpha(d(y, Tz) + d(z, Ty)) \tag{2.2}$$

Adding the left and the right parts of the inequalities (1.1) and (2.2), we obtain (2.1). Hence, every Chatterjea-type contraction is a Paired-Chatterjea contraction.

Lemma 2.3. *Let (X,d) be a b-metric space with $|X| \geq 3$ and let $T : X \to X$ be a Paired-Chatterjea contraction. Suppose that $\{x_n\}$ is a sequence of pairwise distinct points in X satisfying $x_{n+1} = Tx_n,\ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Then $\{x_n\}$ is a Cauchy sequence.*

Proof. Let $\delta_n = d(x_n, x_{n+1})$ for all $n \in \mathbb{N}_0$. Since x_{n-1}, x_n, x_{n+1} are pairwise distinct, setting $x = x_{n-1}$, $y = x_n$, and $z = x_{n+1}$ in (2.1) and applying the b-metric inequality, we obtain

$$\begin{aligned}\delta_n + \delta_{n+1} &\leq \alpha(d(x_{n-1}, x_{n+1}) + d(x_n, x_{n+2})) \\ &\leq s\alpha(d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})).\end{aligned}$$

Thus, for all $n \geq 1$,

$$\delta_n + \delta_{n+1} \leq \frac{\alpha s}{1 - \alpha s}(\delta_{n-1} + \delta_n) = \lambda(\delta_{n-1} + \delta_n), \tag{2.3}$$

where $\lambda = \frac{\alpha s}{1-\alpha s} < 1$ since $\alpha < \frac{1}{2s}$. It follows inductively that

$$\delta_n + \delta_{n+1} \leq \lambda^n(\delta_0 + \delta_1),\ n \in \mathbb{N}_0.$$

Consequently,

$$\lim_{n\to\infty} \delta_n = 0. \tag{2.4}$$

We now prove that $\{x_n\}$ is Cauchy. Suppose, to the contrary, that it is not Cauchy. Then there exist $\varepsilon > 0$ and sequences of integers $\{m_k\}$ and $\{n_k\}$ such that $m_k > n_k,\ n_k \to \infty$, and

$$d(x_{m_k}, x_{n_k}) \geq \varepsilon \tag{2.5}$$

for every k. In view of (2.4), we may assume, after passing to a subsequence if necessary, that $m_k \geq n_k + 2$ (since if $m_k = n_k + 1$, then $x_{m_k-1} = x_{n_k}$). Since the orbit points are pairwise distinct, the three points $x_{m_k-1},\ x_{n_k-1},\ x_{n_k}$ are pairwise distinct. Hence, by (2.1), we have

$$d(x_{m_k}, x_{n_k}) + \delta_{n_k} \leq \alpha\left(d(x_{m_k-1}, x_{n_k}) + d(x_{n_k-1}, x_{m_k}) + d(x_{n_k-1}, x_{n_k+1})\right). \tag{2.6}$$

By the b-metric inequality, we have the following

$$d(x_{m_k-1}, x_{n_k}) \leq s\left(d(x_{m_k-1}, x_{m_k}) + d(x_{m_k}, x_{n_k})\right), \tag{2.7}$$

$$d(x_{n_k-1}, x_{m_k}) \leq s\left(d(x_{n_k-1}, x_{n_k}) + d(x_{n_k}, x_{m_k})\right), \tag{2.8}$$

$$d(x_{n_k-1}, x_{n_k+1}) \leq s\left(d(x_{n_k-1}, x_{n_k}) + d(x_{n_k}, x_{n_k+1})\right). \tag{2.9}$$

Putting (2.7), (2.8) and (2.9) in (2.6), we get

$$d(x_{m_k}, x_{n_k}) + \delta_{n_k} \leq \alpha s\Big[2d(x_{m_k}, x_{n_k}) + \delta_{m_k-1} + 2\delta_{n_k-1} + \delta_{n_k}\Big].$$

Consequently,

$$(1 - 2s\alpha)d(x_{m_k}, x_{n_k}) \leq s\alpha\left(\delta_{m_k-1} + 2\delta_{n_k-1} + \delta_{n_k}\right). \tag{2.10}$$

Since $m_k, n_k \to \infty$, (2.4) implies $\delta_{m_k-1} \to 0,\ \delta_{n_k-1} \to 0,\ \delta_{n_k} \to 0$. Also, since $1 - 2s\alpha > 0$, it follows from (2.10) that

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = 0.$$

This contradicts (2.5). Hence $\{x_n\}$ is a Cauchy sequence. □

Theorem 2.4. *Let (X, d) be a complete b-metric space with $|X| \geq 3$ and let $T : X \to X$ be a Paired-Chatterjea contraction. Then T has a fixed point in X if and only if T has no periodic points of prime period 2. Moreover, the number of fixed points of T is at most two.*

Proof. Let T be a mapping with no periodic point of prime period 2. Let $x_0 \in X$ such that $x_1 = Tx_0,\ x_2 = Tx_1,\ \ldots,\ x_n = Tx_{n-1} = T^n x_0$. If x_n is a fixed point of T for some $n \in \mathbb{N}_0$, then the proof is complete. Otherwise, $x_n \neq Tx_n = x_{n+1}$ for every $n \in \mathbb{N}_0$. Moreover, since T has no periodic points of prime period 2, $x_{n+1} = T(T(x_{n-1})) \neq x_{n-1}$. Thus, x_{n-1}, x_n, x_{n+1} are pairwise distinct. Putting $x = x_{n-1}, y = x_n, z = x_{n+1}$ in (2.1), we get (2.3). Let

$$p_n = d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) = \delta_n + \delta_{n+1}$$

for all $n \in \mathbb{N}$. Since x_n, x_{n+1} and x_{n+2} are pairwise distinct, it follows from (2.3) that

$$p_1 \leq \lambda p_0,\ p_2 \leq \lambda p_1,\ p_n \leq \lambda p_{n-1} \text{ and } p_0 > p_1 > \cdots > p_n > \ldots. \tag{2.11}$$

Next, we show that all points in the orbit $\{x_n : n \in \mathbb{N}_0\}$ are distinct. Suppose, to the contrary, that $x_i = x_j$ for some $0 \leq i < j$. If $j = i + 1$, then $x_i = Tx_i$, contradicting the assumption that the orbit contains no fixed point. Hence $j \geq i + 2$, and consequently $x_{j+1} = x_{i+1},\ x_{j+2} = x_{i+2}$. Thus,

$$p_i = d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2}) = d(x_j, x_{j+1}) + d(x_{j+1}, x_{j+2}) = p_j,$$

which contradicts $p_j < p_i$ for $i < j$ in (2.11). Hence, the orbit points are pairwise distinct. By Lemma 2.3, $\{x_n\}$ is a Cauchy sequence. The completeness of (X, d) implies $x_n \to x^* \in X$.

Next, we show that x^* is a fixed point of T. Since the orbit points are pairwise distinct, the point x^* can coincide with at most one point of the orbit. Hence there exists $N \in \mathbb{N}$ such that $x_n \neq x^*, x_{n+1} \neq x^*$ for all $n \geq N$. Thus, for every $n \geq N$, the three points x^*, x_n, x_{n+1} are pairwise distinct. By (2.1) and b-metric inequality, we obtain

$$\begin{aligned} d(x_{n+1}, Tx^*) = d(Tx_n, Tx^*) &\leq d(Tx_n, Tx^*) + d(Tx^*, Tx_{n+1}) \\ &\leq \alpha(d(x_n, Tx^*) + d(x^*, Tx_n) + d(x^*, Tx_{n+1}) + d(x_{n+1}, Tx^*)) \\ &\leq s\alpha d(x_n, x_{n+1}) + s\alpha d(x_{n+1}, Tx^*) + \alpha d(x^*, x_{n+1}) + \alpha d(x^*, x_{n+2}) \\ &\quad + \alpha d(x_{n+1}, Tx^*) \\ &\leq s\alpha d(x_n, x_{n+1}) + 2s\alpha d(x_{n+1}, Tx^*) + \alpha d(x^*, x_{n+1}) + \alpha d(x^*, x_{n+2}). \end{aligned}$$

Thus, we have

$$(1 - 2s\alpha)d(x_{n+1}, Tx^*) \leq s\alpha d(x_n, x_{n+1}) + \alpha d(x^*, x_{n+1}) + \alpha d(x^*, x_{n+2}).$$

Therefore, as $n \to \infty$, we obtain

$$\lim_{n\to\infty} d(x_{n+1}, Tx^*) = 0$$

since $\alpha < \frac{1}{2s}$. By the b-metric inequality, we have

$$d(x^*, Tx^*) \leq s(d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*)).$$

Taking the limit as $n \to \infty$, we obtain $d(x^*, Tx^*) = 0$. Therefore, x^* is a fixed point of T.

On the other hand, suppose that T has a fixed point $x \in X$. i.e. $Tx = x$ and assume also that T has a periodic point y of prime period 2. i.e. $T(Ty) = y$. If we let $z = Ty$, then $Tz = y$. If $x = y$ or $y = z$, then $Ty = y$, which contradicts the assumption that y is a periodic point of prime period 2. Also, if $x = z$, then $Ty = z = x = Tx = Tz = y$, which is a contradiction with the same reason as above. Therefore, we may assume that x, y and z are pairwise distinct. By (2.1) and b-metric inequality, we obtain

$$\begin{aligned} d(x,z) + d(z,y) &\leq \alpha(d(x,z) + d(y,x)) \\ &\leq \alpha(d(x,z) + sd(y,z) + sd(z,x)) \\ &\leq \alpha(1+s)(d(x,z) + d(y,z)). \end{aligned}$$

Thus $\alpha \geq \frac{1}{s+1} \geq \frac{1}{2s}$, contradicting the assumption that $\alpha < \frac{1}{2s}$. Hence, T has no periodic point of prime period 2.

Finally, suppose T has three distinct fixed points, say $Tx = x$, $Ty = y$, and $Tz = z$. Then, (2.1) yields $\alpha \geq \frac{1}{2}$, contradicting the assumption that $\alpha \in [0, \frac{1}{2s})$. Therefore, T has at most two fixed points. □

The following example shows that the class of paired-Chatterjea contractions properly extends the corresponding class of Chatterjea-type contractions.

Example 2.5. Let $X = \{a, b, c\}$ and define $d : X \times X \to \mathbb{R}^+$ by

$$d(a,b) = 1,\ d(b,c) = 1,\ d(a,c) = 6,\ d(x,x) = 0 \text{ for all } x \in X.$$

Since $d(a,c) \leq 3(d(a,b) + d(b,c))$ and X is finite, (X,d) is a complete b-metric space with $s = 3$. Define $T : X \to X$ by

$$Tx = \begin{cases} b, & \text{if } x = a; \\ c, & \text{if } x \in \{b, c\}; \end{cases}$$

Clearly T has no periodic points of prime period 2. By (2.1), the six permutations of (a, b, c) yield the following contraction constants: (a, b, c) and (c, b, a) give $1/7$; (a, c, b) and (b, c, a) give $1/8$; and (b, a, c) and (c, a, b) give $2/13$. Therefore, all the hypotheses of Theorem 2.4 are satisfied with any $\alpha \in [\frac{2}{13}, \frac{1}{6})$ and $Tc = c$. Next, we show that T is not a Chatterjea-type contraction. Indeed, taking $x = a$ and $y = b$, we obtain

$$d(Ta, Tb) = d(b,c) = 1 \text{ and } d(a,Tb) + d(b,Ta) = d(a,c) + d(b,b) = 6.$$

Thus, by (1.1), we have $\alpha \geq \frac{1}{6}$, which contradicts the hypothesis of Theorem 1.3, namely, $\alpha < \frac{1}{2s} = \frac{1}{6}$. Therefore, T is not a Chatterjea-type contraction.

Remark 2.6. Putting $s = 1$ in Theorem 2.4, we get [14, Theorem 3].

3. Fixed Point Theorem for Paired Contractions

In this section, we introduce Paired contractions in b-metric spaces with $\alpha \in [0, 1)$ and establish a corresponding fixed point theorem.

Definition 3.1. Let (X, d) be a b-metric space with $|X| \geq 3$. A mapping $T : X \to X$ is called *Paired contraction* if there exists a constant $\alpha \in [0, 1)$ such that

$$d(Tx, Ty) + d(Ty, Tz) \leq \alpha(d(x, y) + d(y, z)) \tag{3.1}$$

holds for all pairwise distinct $x, y, z \in X$.

Remark 3.2. The pairwise distinctness of the points $x, y, z \in X$ is essential, since without it, the inequality (3.1) reduces to the standard contraction condition.

Lemma 3.3. *Let (X, d) be a b-metric space with $|X| \geq 3$ and let $T : X \to X$ be a Paired contraction with $\alpha \in [0, 1)$. If the orbit $\{x_n : n \geq 0\}$, defined by $x_{n+1} = Tx_n$, is pairwise distinct, then $\{x_n\}$ is a Cauchy sequence.*

Proof. If $\alpha = 0$, then (3.1) becomes $d(Tx, Ty) + d(Ty, Tz) = 0$ for all pairwise distinct $x, y, z \in X$. Hence $Tx = Ty = Tz$. Since $|X| \geq 3$, T is constant, and therefore the orbit $\{x_n\}$ is eventually constant and hence Cauchy. Now suppose that $\alpha \in (0, 1)$. Let $\delta_n = d(x_n, x_{n+1})$ for all $n \in \mathbb{N}_0$. Since the points x_{n-1}, x_n, x_{n+1} are pairwise distinct, substituting them in (3.1) gives

$$d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) \leq \alpha \left(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\right). \tag{3.2}$$

Hence, for $n \geq 1$,

$$\delta_n + \delta_{n+1} \leq \alpha(\delta_{n-1} + \delta_n).$$

Consequently, by induction,

$$\delta_n + \delta_{n+1} \leq \alpha^n(\delta_0 + \delta_1), \ n \in \mathbb{N}_0.$$

In particular,

$$\delta_n \leq \alpha^n(\delta_0 + \delta_1), \ n \in \mathbb{N}_0. \tag{3.3}$$

Since $\alpha \in (0, 1)$, there exists $n_0 \geq 2$ such that $s^2\alpha^{n_0} < 1$. We first establish an estimate for shifted orbit points. Let $m > n$ and put

$$A_j = d(x_{m+j}, x_{n+j}), \ j \geq 0.$$

For $j \geq 1$ and $m \neq n + 1$, the points $x_{m+j-1}, x_{n+j-1}, x_{n+j}$ are pairwise distinct (if $m = n + 1$, we have $d(x_m, x_n) = \delta_n \to 0$ as $n \to \infty$). Hence, Substituting these points in (3.1), we obtain

$$A_j + \delta_{n+j} \leq \alpha \left(A_{j-1} + \delta_{n+j-1}\right).$$

Therefore, $A_j \leq \alpha A_{j-1} + \alpha\delta_{n+j-1}$. Using (3.3), we obtain

$$A_j \leq \alpha A_{j-1} + \alpha^{n+j}(\delta_0 + \delta_1).$$

Iterating this inequality for $j = 1, \ldots, n_0$ yields

$$A_{n_0} \leq \alpha^{n_0} A_0 + n_0\alpha^{n+n_0}(\delta_0 + \delta_1).$$

Thus,

$$d(x_{m+n_0}, x_{n+n_0}) \leq \alpha^{n_0} d(x_m, x_n) + n_0\alpha^{n+n_0}(\delta_0 + \delta_1). \tag{3.4}$$

Next, using the b-metric inequality twice, we obtain

$$\begin{aligned} d(x_m, x_n) &\leq s \left(d(x_m, x_{m+n_0}) + d(x_{m+n_0}, x_n)\right) \\ &\leq s \left(d(x_m, x_{m+n_0}) + s[d(x_{m+n_0}, x_{n+n_0}) + d(x_{n+n_0}, x_n)]\right). \end{aligned}$$

Hence,

$$d(x_m, x_n) \leq s\, d(x_m, x_{m+n_0}) + s^2 d(x_{m+n_0}, x_{n+n_0}) + s^2 d(x_{n+n_0}, x_n). \tag{3.5}$$

We next estimate the two end terms in (3.5). Define $P_n = d(x_{n+n_0}, x_n),\ n \in \mathbb{N}_0$. For $n \geq 1$, applying (3.1) to the three points $x_{n+n_0-1}, x_{n-1}, x_n$ gives

$$d(x_{n+n_0}, x_n) + d(x_n, x_{n+1}) \leq \alpha \left(d(x_{n+n_0-1}, x_{n-1}) + d(x_{n-1}, x_n) \right).$$

Therefore, $P_n + \delta_n \leq \alpha(P_{n-1} + \delta_{n-1})$, and hence $P_n \leq \alpha P_{n-1} + \alpha\delta_{n-1}$. Iterating this inequality, we obtain

$$P_n \leq \alpha^n P_0 + \sum_{k=1}^{n} \alpha^{n-k+1} \delta_{k-1}.$$

By (3.3), we have

$$d(x_{n+n_0}, x_n) \leq \alpha^n d(x_{n_0}, x_0) + n\alpha^n(\delta_0 + \delta_1). \tag{3.6}$$

Similarly,

$$d(x_{m+n_0}, x_m) \leq \alpha^m d(x_{n_0}, x_0) + m\alpha^m(\delta_0 + \delta_1). \tag{3.7}$$

Substituting (3.4), (3.6), and (3.7) into (3.5), we obtain

$$\begin{aligned} d(x_m, x_n) \leq\ & s\alpha^m d(x_{n_0}, x_0) + sm\alpha^m(\delta_0 + \delta_1) + s^2\alpha^{n_0} d(x_m, x_n) + s^2 n_0 \alpha^{n+n_0}(\delta_0 + \delta_1) \\ & + s^2\alpha^n d(x_{n_0}, x_0) + s^2 n\alpha^n(\delta_0 + \delta_1). \end{aligned}$$

Therefore,

$$\begin{aligned} (1 - s^2\alpha^{n_0}) d(x_m, x_n) \leq\ & s\alpha^m d(x_{n_0}, x_0) + s^2\alpha^n d(x_{n_0}, x_0) + sm\alpha^m(\delta_0 + \delta_1) \\ & + s^2 n\alpha^n(\delta_0 + \delta_1) + s^2 n_0 \alpha^{n+n_0}(\delta_0 + \delta_1). \end{aligned} \tag{3.8}$$

Set $r_n = \sup_{k \geq n} k\alpha^k$. Since $\alpha \in (0,1)$, so $k\alpha^k \to 0$ as $k \to \infty$, and thus $r_n \to 0$ as $n \to \infty$. Since $m > n$ and $\alpha \in (0,1)$, we have $\alpha^m \leq \alpha^n,\ m\alpha^m \leq r_n,\ n\alpha^n \leq r_n$. It follows from (3.8) that

$$d(x_m, x_n) \leq \frac{(s + s^2)\alpha^n}{1 - s^2\alpha^{n_0}} d(x_{n_0}, x_0) + \frac{(s + s^2) r_n + s^2 n_0 \alpha^{n+n_0}}{1 - s^2\alpha^{n_0}} (\delta_0 + \delta_1).$$

The right-hand side is independent of m. Moreover, $\alpha^n \to 0,\ r_n \to 0,\ \alpha^{n+n_0} \to 0$ as $n \to \infty$. Hence

$$\lim_{n \to \infty} \sup_{m > n} d(x_m, x_n) = 0,$$

which proves that $\{x_n\}$ is a Cauchy sequence. □

Theorem 3.4. *Let (X, d) be a complete b-metric space with $|X| \geq 3$, and let $T : X \to X$ be a Paired contraction. Then T has a fixed point in X if and only if T has no periodic points of prime period 2. Moreover, the number of fixed points of T is at most two.*

Proof. Let T be a mapping with no periodic point of prime period 2. Let $x_0 \in X$ such that $x_1 = Tx_0,\ x_2 = Tx_1,\ \ldots,\ x_n = Tx_{n-1} = T^n x_0$. If x_n is a fixed point of T for some $n \in \mathbb{N}_0$, then the proof is complete. Otherwise, $x_n \neq Tx_n = x_{n+1}$ for every $n \in \mathbb{N}_0$. Moreover, since T has no periodic points of prime period 2, $x_{n+1} = T(T(x_{n-1})) \neq x_{n-1}$. Thus, x_{n-1}, x_n, x_{n+1} are pairwise distinct. Substituting

$x = x_{n-1}, y = x_n, z = x_{n+1}$ in (3.1), we obtain (3.2). Let $p_n = d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})$ for all $n \in \mathbb{N}_0$. Then

$$p_1 \leq \alpha p_0,\ p_2 \leq \alpha p_1,\ p_n \leq \alpha p_{n-1} \leq \alpha^n p_0 \text{ and } p_0 > p_1 > \cdots > p_n > \ldots. \quad (3.9)$$

Next, we show that all points in the orbit $\{x_n : n \in \mathbb{N}_0\}$ are distinct. Suppose, to the contrary, that $x_i = x_j$ for some $0 \leq i < j$. If $j = i + 1$, then $x_i = Tx_i$, contradicting the assumption that the orbit contains no fixed point. Hence $j \geq i + 2$, and consequently $x_{j+1} = x_{i+1},\ x_{j+2} = x_{i+2}$. Thus,

$$p_i = d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2}) = d(x_j, x_{j+1}) + d(x_{j+1}, x_{j+2}) = p_j,$$

which contradicts $p_j < p_i$ for $i < j$ in (3.9). Therefore, by Lemma 3.3, $\{x_n\}$ is a Cauchy sequence.

Let us prove that $Tx^* = x^*$. Consider the set $\mathcal{B} = \{n \in \mathbb{N}_0 : x_n = x^*\}$.

Case 1. If $\mathcal{B}$ is an infinite set, then there exists a subsequence $x_{n_k} \to x^*$ and hence $x_{n_k+1} = Tx_{n_k} = Tx^*$. Since x_{n_k+1} is also a subsequence of a convergent sequence. Thus $x_{n_k+1} \to x^*$. Hence $Tx^* = x^*$.

Case 2. If $\mathcal{B}$ is a finite set, then there exists $N \in \mathbb{N}$ such that $x_n \neq x^*$ for all $n \geq N$. Since the orbit points are distinct, the points x^*, x_n and x_{n+1} are pairwise distinct for every $n \geq N + 1$. Thus, by b-metric inequality and inequality (3.1) for each $n \geq N$, we have

$$\begin{aligned} d(x^*, Tx^*) &\leq s(d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*)) = s(d(x^*, x_{n+1}) + d(Tx_n, Tx^*)) \\ &\leq sd(x^*, x_{n+1}) + s(d(Tx_n, Tx^*) + d(Tx^*, Tx_{n+1})) \\ &\leq sd(x^*, x_{n+1}) + s\alpha(d(x_n, x^*) + d(x^*, x_{n+1})). \end{aligned}$$

Taking the limit as $n \to \infty$, we obtain $d(x^*, Tx^*) = 0$. Hence, $Tx^* = x^*$ as desired. In both cases, we can conclude that T has a fixed point in X.

On the other hand, suppose that T has a fixed point $x \in X$ and a periodic point $y \in X$ of prime period 2, so that $T(Ty) = y$. Let $z = Ty$. Then $Tz = y$. If $x = y$ or $y = z$, then $Ty = y$, contradicting the fact that y has prime period 2. Similarly, if $x = z$, then $Ty = z = x = Tx = Tz = y$, which again implies $Ty = y$, a contradiction. Hence, x, y, z are pairwise distinct. Applying (3.1) to the ordered triple (x, y, z), we obtain

$$d(x, z) + d(y, z) \leq \alpha\left(d(x, y) + d(y, z)\right). \quad (3.10)$$

Similarly, applying (3.1) to the ordered triple (x, z, y) yields

$$d(x, y) + d(y, z) \leq \alpha\left(d(x, z) + d(y, z)\right). \quad (3.11)$$

Set $A = d(x, z) + d(y, z) > 0$ and $B = d(x, y) + d(y, z) > 0$. Then (3.10) and (3.11) imply $A \leq \alpha B$ and $B \leq \alpha A$. Therefore, $A \leq \alpha B \leq \alpha^2 A$. Since $A > 0$, we obtain $\alpha \geq 1$. This contradicts the assumption $\alpha \in [0, 1)$. Hence, T has no periodic points of prime period 2.

Finally, let x, y and z be three distinct fixed points, say $Tx = x$, $Ty = y$, and $Tz = z$. Then, the inequality (3.1) yields $\alpha \geq 1$, contradicting the assumption that $\alpha \in [0, 1)$. Therefore, T has at most two fixed points. □

Remark 3.5. Theorem 3.4 extends [10, Theorem 2] by enlarging the admissible range of the contraction coefficient from $\alpha \in \left[0, \frac{1}{s}\right)$ to $\alpha \in [0, 1)$. Moreover, we establish the converse of [10, Theorem 2]. Furthermore, by setting $s = 1$ in Theorem 3.4, we get [12, Theorem 2.1].

The following example illustrates that the class of Paired contractions covered by Theorem 3.4 is strictly larger than the class considered in [10, Theorem 2].

Example 3.6. Let $X = \{a, b, c\}$ and define $d : X \times X \to \mathbb{R}_+$ by

$$d(a, b) = d(b, c) = 1,\ d(a, c) = 4,\ \text{and } d(x, x) = 0 \text{ for all } x \in X.$$

Clearly, (X, d) is a complete b-metric space with coefficient $s = 2$. Now, define $T : X \to X$ by

$$Ta = b,\ Tb = c,\ Tc = c.$$

Clearly, T has no periodic points of prime period 2 and $Tc = c$. By (3.1), the six permutations of (a, b, c) yield the following lower bounds for α: (a, b, c) and (c, b, a) give $\alpha \geq 1/2$; (a, c, b) and (b, c, a) give $\alpha \geq 1/5$; and (b, a, c) and (c, a, b) give $\alpha \geq 2/5$. Hence, T satisfies all the hypotheses of Theorem 3.4 for every $\alpha \in \left[\frac{1}{2}, 1\right)$. On the other hand, since $s = 2$, the contraction constant in [10, Theorem 2] is required to satisfy $\alpha \in \left[0, \frac{1}{2}\right)$.

The following example shows that the assumption in Theorem 3.4 that T has no periodic point of prime period 2 is essential. Indeed, we construct a Paired contraction T on X that has periodic points of prime period 2 but no fixed point.

Example 3.7. Let $X = \{a, b, c\}$, and define $d : X \times X \to [0, \infty)$ by

$$d(a, b) = 1,\ d(a, c) = 2,\ d(b, c) = 6,\ \text{and } d(x, x) = 0\ \forall\ x \in X.$$

Then (X, d) is a b-metric space with coefficient $s = 2$. Define $T : X \to X$ by

$$Ta = b,\ Tb = a,\ Tc = a.$$

Clearly, $T^2a = a$, $Ta \neq a$, and $T^2b = b$, $Tb \neq b$. Thus, a and b are periodic points of T of prime period 2 and T has no fixed point. For the six permutations (a, b, c), namely $(a, b, c), (a, c, b), (b, a, c), (b, c, a), (c, a, b), (c, b, a)$, the corresponding contraction constants are, respectively, $\frac{1}{7}, \frac{1}{8}, \frac{2}{3}, \frac{1}{8}, \frac{2}{3}, \frac{1}{7}$. Hence, the maximum is $\frac{2}{3}$, and therefore T satisfies the Paired contraction condition for $\alpha \in \left[\frac{2}{3}, 1\right)$.

The next example demonstrates the applicability of Theorem 3.4 and illustrates the significance of its hypotheses.

Example 3.8. Let $X = \{0\} \cup [1, 2] \cup \{3\}$, and let $d : X \times X \to [0, \infty)$ be defined by

$$d(x, y) = |x - y|^2 \text{ for all } x, y \in X.$$

Then (X, d) is a complete b-metric space with $s = 2$. Define a mapping $T : X \to X$ by

$$Tx = \begin{cases} \frac{x}{2} + 1 & \text{if } x \in [1, 2], \\ 1 & \text{if } x = 0, \\ 2 & \text{if } x = 3. \end{cases}$$

We claim that T is a Paired contraction with $\alpha = \frac{1}{4}$. We verify this in three cases.

Case 1: All $x, y, z \in [1, 2]$. Then

$$d(Tx, Ty) + d(Ty, Tz) = \frac{1}{4}\left((x-y)^2 + (y-z)^2\right) = \frac{1}{4}\left(d(x, y) + d(y, z)\right).$$

Case 2: Two points in $[1, 2]$ and one isolated point. The four cases are:

$(z = 0, x, y \in [1, 2]), (z = 3, x, y \in [1, 2]), (y = 0, x, z \in [1, 2]), (y = 3, x, z \in [0, 2]),$

the required contraction inequality is satisfied with $\alpha = \frac{1}{4}$.

Case 3: One point in $[1, 2]$ and two isolated points. Since x, y, z are pairwise distinct, the two isolated points must be 0 and 3, while the remaining point belongs to $[1, 2]$. Up to the interchange of x and z, the two possibilities are $x \in [1, 2], y = 3, z = 0$ and $x \in [1, 2], y = 0, z = 3$.

In all cases, T is a Paired contraction with $\alpha = \frac{1}{4}$. Moreover, T has no periodic points of prime period 2. Indeed, for $x \in [1, 2]$,

$$T^2x = T\left(\frac{x}{2} + 1\right) = \frac{x}{4} + \frac{3}{2} \neq x,$$

unless $x = 2$, but $T(2) = 2$ is a fixed point. Thus, by Theorem 3.4, $T(2) = 2$.

As a consequence of Theorem 3.4, we obtain the Banach fixed point theorem in the setting of b-metric spaces.

Corollary 3.9. *Let (X, d) be a nonempty complete b-metric space with a mapping $T : X \to X$ such that for all $x, y \in X$*

$$d(Tx, Ty) \leq \alpha d(x, y) \tag{3.12}$$

with $\alpha \in [0, 1)$. Then T has a unique fixed point.

Proof. If $|X| = 1$, the unique point of X is trivially a fixed point. If $|X| = 2$, then there are four possible cases as follows: $Tx = y$ and $Ty = y$, $Tx = x$ and $Ty = x$, $Tx = y$ and $Ty = x$, $Tx = x$ and $Ty = y$. The first two cases yield a unique fixed point, while the remaining two cases would force $\alpha \geq 1$, thereby contradicting the assumption that $\alpha \in [0, 1)$. Now, suppose that $|X| \geq 3$. Assume, for contradiction, that there exists $x \in X$ such that $T(Tx) = x$. Then

$$d(Tx, x) = d(Tx, T(Tx)) \leq \alpha d(Tx, x),$$

which contradicts that $\alpha \in [0, 1)$. Hence, T has no periodic points of prime period 2. Let x, y, z be distinct points in X. Applying (3.12) to the pairs (x, y) (y, z) and summing the resulting inequalities yields (3.1). Thus, by Theorem 3.4, T must have a fixed point. Finally, suppose $Tx = x$ and $Ty = y$ with $x \neq y$. Then

$$d(x, y) = d(Tx, Ty) \leq \alpha\, d(x, y),$$

which implies $\alpha \geq 1$, a contradiction. Thus $x = y$. Therefore T has a unique fixed point. □

4. Fixed Point Theorem for Paired-Kannan Contractions

We investigate Paired–Kannan contractions in b-metric spaces. We observe that the corresponding result in [10, Theorem 3] assumes that the contraction constant belongs to $\left[0, \frac{1}{2}\right)$. However, when $s > 2$, this condition is insufficient for the b-metric limit argument used to establish the fixed point property.

Definition 4.1. Let (X, d) be a b-metric space with $|X| \geq 3$. A mapping $T : X \to X$ is called *Paired-Kannan contraction* if there exists a nonnegative constant

$$\alpha < \begin{cases} \frac{1}{2}, & 1 \leq s \leq 2; \\ \frac{1}{s}, & s > 2. \end{cases} \tag{4.1}$$

such that

$$d(Tx, Ty) + d(Ty, Tz) \leq \alpha \left(d(x, Tx) + d(y, Ty) + d(z, Tz) \right), \tag{4.2}$$

holds for all pairwise distinct $x, y, z \in X$.

Remark 4.2. The contraction constant in (4.1) is equivalent to $0 \leq \alpha < \min\left\{\frac{1}{2}, \frac{1}{s}\right\}$. Moreover, the Kannan-type contraction and the Paired-Kannan contraction are distinct (see [13, Example 2.6]).

Proposition 4.3. *The hypotheses of* [10, Theorem 3] *do not imply the existence of a fixed point when* $s > 2$.

We present an example showing that the stated hypotheses of [10, Theorem 3] do not guarantee the asserted conclusion. By constructing a complete b-metric space (X, d) with coefficient $s = 5$ and a mapping $T : X \to X$ satisfying all the hypotheses of [10, Theorem 3] with $\alpha = \frac{2}{5} < \frac{1}{2}$, but having no fixed points.

Example 4.4. Let $X = \mathbb{N}_0 \cup \{p, q\}$, where $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. Define $d : X \times X \to [0, \infty)$ by

$$\begin{cases} d(x, x) = 0, \\ d(m, n) = 10^{-m} + 10^{-n}, \ m, n \in \mathbb{N}_0, \\ d(q, n) = 10^{-n}, \ n \in \mathbb{N}_0, \\ d(p, n) = \frac{1}{5}, \ n \in \mathbb{N}_0, \\ d(p, q) = 1. \end{cases}$$

To show that (X, d) is a b-metric space. Consider the following cases, $(x, z \in \mathbb{N}_0, y = p)$, $(x, z \in \mathbb{N}_0, y = q)$, $(x = p, z = q, y \in \mathbb{N}_0)$, $(x = p, z \in \mathbb{N}_0, y = q)$, $(x = q, z \in \mathbb{N}_0, y = p)$. The remaining cases are either trivial or follow by symmetry. Hence, (X, d) is a b-metric space with coefficient $s = 5$.

Show that the sequence is Cauchy. Let $\{x_n\}$ be the sequence defined by $x_n = n$. For any $m, n \in \mathbb{N}_0$, we have $d(m, n) = 10^{-m} + 10^{-n}$. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $10^{-N} < \frac{\varepsilon}{2}$. Then, for all $m, n \geq N$,

$$d(m, n) = 10^{-m} + 10^{-n} \leq 10^{-N} + 10^{-N} = 2 \cdot 10^{-N} < \varepsilon.$$

Therefore, $\{x_n\}_{n \in \mathbb{N}_0}$ is a Cauchy sequence in (X, d). In fact, the sequence converges to q, since $d(x_n, q) = d(n, q) = 10^{-n} \to 0$. Hence, $x_n \to q$.

We now show that (X,d) ***is complete.*** Let (x_j) be a Cauchy sequence in X. If the range of (x_j) is finite, then

$$\delta := \min\{d(u,v) : u,v \in \mathrm{Ran}\{x_j\},\ u \neq v\} > 0.$$

Since (x_j) is Cauchy, there exists N such that $d(x_j,x_k) < \delta,\ j,k \geq N$. Hence $x_j = x_k$ for all $j,k \geq N$, so the sequence is eventually constant and therefore convergent. Suppose now that the range of (x_j) is infinite. Choose N such that $d(x_j,x_k) < \frac{1}{5},\ j,k \geq N$. Since $d(p,n) = 1/5$ for every $n \in \mathbb{N}_0$, the tail $\{x_j : j \geq N\}$ cannot contain both p and a natural number. If p occurred infinitely often, the Cauchy property would therefore imply that only finitely many distinct natural numbers could occur in the tail, contradicting the assumption that the range is infinite. Thus, p occurs only finitely many times. Consequently, for all sufficiently large j, $x_j \in \mathbb{N}_0 \cup \{q\}$. Since the range is infinite, infinitely many distinct natural numbers occur. We claim that the values of these natural-number terms tend to infinity. Indeed, if some $m \in \mathbb{N}_0$ occurred infinitely often while arbitrarily large natural numbers also occurred, then for a sequence (n_k) of such natural numbers with $n_k \to \infty$,

$$d(m,n_k) = 10^{-m} + 10^{-n_k} \to 10^{-m} > 0,$$

contradicting the Cauchy property. Hence, the natural-number values occurring in the sequence tend to infinity. Since $d(n,q) = 10^{-n}$, it follows that $d(x_j,q) \to 0$. Thus $x_j \to q$. Therefore every Cauchy sequence in X converges in X, and hence (X,d) is complete.

Define the Mapping. Let $T : X \to X$ be defined by

$$Tq = p,\ Tp = 2,\ Tn = n+1 \text{ for all } n \in \mathbb{N}_0.$$

Clearly, $Tq = p \neq q,\ Tp = 2 \neq p,\ Tn = n+1 \neq n$, so T has no fixed point. Moreover, $T^2q = Tp = 2 \neq q,\ T^2p = T2 = 3 \neq p,\ T^2n = n+2 \neq n$. Nevertheless, T has no periodic points of prime period 2. The mapping T satisfies (4.2) for every pairwise distinct $x,y,z \in X$.

Verify the paired-Kannan Hypotheses. Since both sides of are unchanged when x and z are interchanged, it is enough to consider one representative from each pair obtained by the interchange $x \leftrightarrow z$.
Case A: $x,y,z \in \mathbb{N}_0$. Then, (4.2) holds for $\alpha \geq \frac{2}{11}$.
Case B: Two points belong to $\mathbb{N}_0$ and the third is p. Consider the two possible arrangements $(x,y,z) = (m,p,n),(m,n,p)$, where $m,n \in \mathbb{N}_0$ and $m \neq n$. Then (4.2) holds for $\alpha \geq \frac{1}{10}$ and $\alpha \geq \frac{21}{130}$ respectively.
Case C: Two points belong to $\mathbb{N}_0$ and the third is q. Consider the two possible arrangements $(x,y,z) = (m,q,n),(m,n,q)$, where $m,n \in \mathbb{N}_0$ and $m \neq n$. Then (4.2) holds for $\alpha \geq \frac{2}{5}$ and $\alpha \geq \frac{1}{5}$ respectively.
Case D: One point belongs to $\mathbb{N}_0$ and the other two are p and q. Consider the three possible arrangements $(x,y,z) = (p,q,n),(q,p,n),(p,n,q)$, where $n \in \mathbb{N}_0$. Then (4.2) holds for $\alpha \geq \frac{1}{3}, \alpha \geq \frac{31}{230}, \alpha \geq \frac{31}{230}$, respectively.

Thus, T satisfies (4.2) with $\alpha = \max\{\frac{31}{230}, \frac{1}{10}, \frac{2}{5}, \frac{1}{5}, \frac{21}{130}, \frac{2}{11}, \frac{1}{3}\} = \frac{2}{5} < \frac{1}{2}$. Hence, all the hypotheses of [10, Theorem 3] are satisfied, but T has no fixed point.

Theorem 4.5. *Let (X, d) be a complete b-metric space with $|X| \geq 3$, and let $T : X \to X$ be a Paired-Kannan contraction. Then T has a fixed point in X if and only if T has no periodic points of prime period 2. Moreover, the number of fixed points of T is at most two.*

Proof. Let T be a mapping with no periodic point of prime period 2. Let $x_0 \in X$ such that $x_1 = Tx_0,\ x_2 = Tx_1,\ \ldots,\ x_n = Tx_{n-1} = T^n x_0$. If x_n is a fixed point of T for some $n \in \mathbb{N}_0$, then the proof is complete. Otherwise, $x_n \neq Tx_n = x_{n+1}$ for every $n \in \mathbb{N}_0$. Moreover, since T has no periodic points of prime period 2, $x_{n+1} = T(T(x_{n-1})) \neq x_{n-1}$. Thus, x_{n-1}, x_n, x_{n+1} are pairwise distinct. Putting $x = x_{n-1}, y = x_n, z = x_{n+1}$ in (4.2), we have

$$d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) \leq \alpha(d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}))$$

Thus, by simple rearrangement, we obtain

$$d(x_n, x_{n+1}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) \leq \frac{\alpha}{1-\alpha} d(x_{n-1}, x_n) \leq \gamma d(x_{n-1}, x_n),$$

where $\gamma = \dfrac{\alpha}{1-\alpha}$. If $\alpha \in [0, \frac{1}{2})$, then $\gamma \in [0, 1)$. Also, if $\alpha < \frac{1}{s}$ for $s > 2$, then $\gamma \in [0, 1)$. In either case, $\gamma \in [0, 1)$. Thus,

$$d(x_n, x_{n+1}) \leq \gamma d(x_{n-1}, x_n),\ \gamma \in [0, 1). \tag{4.3}$$

By Lemma 1.4, $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete, $x_n \to x^* \in X$ as $n \to \infty$.

Consider the set $\mathcal{D} = \{n \in \mathbb{N} : x_n = x^*\}$ in the following two cases:

Case 1. If $\mathcal{D}$ is an infinite set, then there exists a subsequence $x_{n_k} \to x^*$ and hence $x_{n_k+1} = Tx_{n_k} = Tx^*$. Since x_{n_k+1} is also a subsequence of a convergent sequence. Thus $x_{n_k+1} \to x^*$. Hence $Tx^* = x^*$.

Case 2. If $\mathcal{D}$ is a finite set, then there exists $N \in \mathbb{N}$ such that $x_n \neq x^*$ for all $n \geq N$. We first show that all points of the orbit $\{x_n : n \in \mathbb{N}_0\}$ are distinct. Suppose, to the contrary, that $x_i = x_j$ for some $0 \leq i < j$. If $j = i + 1$, then $x_i = Tx_i$, contradicting the assumption that no point of the orbit is fixed. Hence $j \geq i + 2$. Since $x_i = x_j$, we have $x_{i+1} = Tx_i = Tx_j = x_{j+1}$. Therefore, $d(x_i, x_{i+1}) = d(x_j, x_{j+1})$. On the other hand, from the estimate (4.3), we obtain

$$d(x_j, x_{j+1}) \leq \gamma^{j-i} d(x_i, x_{i+1}) < d(x_i, x_{i+1}),$$

which is a contradiction. Hence, $x_i \neq x_j$ for all $i \neq j$. Thus, the orbit $\{x_n : n \in \mathbb{N}_0\}$ consists of pairwise distinct points. Since the orbit points are distinct, the points x^*, x_n and x_{n+1} are pairwise distinct for every $n \geq N + 1$. By the b-metric inequality and (4.2), we have

$$\begin{aligned} d(x_{n+1}, Tx^*) &\leq d(Tx_n, Tx^*) + d(Tx^*, Tx_{n+1}) \\ &\leq \alpha(d(x_n, x_{n+1}) + d(x^*, Tx^*) + d(x_{n+1}, x_{n+2})) \\ &\leq \alpha(d(x_n, x_{n+1}) + sd(x^*, x_{n+1}) + sd(x_{n+1}, Tx^*) + d(x_{n+1}, x_{n+2})). \end{aligned}$$

Thus,

$$(1 - \alpha s) d(x_{n+1}, Tx^*) \leq \alpha(d(x_n, x_{n+1}) + sd(x^*, x_{n+1}) + d(x_{n+1}, x_{n+2})).$$

Since $1-\alpha s>0$ by the hypothesis, we have

$$0\leq d(x_{n+1},Tx^*)\leq \frac{\alpha}{(1-\alpha s)}(d(x_n,x_{n+1})+sd(x^*,x_{n+1})+d(x_{n+1},x_{n+2})).$$

The right-hand side tends to zero as $n\to\infty$. Thus $\lim_{n\to\infty} d(x_{n+1},Tx^*)=0$. By the b-metric inequality, we have

$$d(x^*,Tx^*)\leq s(d(x^*,x_{n+1})+d(x_{n+1},Tx^*)).$$

Taking the limit as $n\to\infty$, we obtain $d(x^*,Tx^*)=0$. Therefore, x^* is a fixed point of T.

On the other hand, suppose that T has a fixed point $x\in X$ and a periodic point $y\in X$ of prime period 2, so that $T(Ty)=y$. Let $z=Ty$. Then $Tz=y$. If $x=y$ or $y=z$, then $Ty=y$, contradicting the fact that y has prime period 2. Similarly, if $x=z$, then $Ty=z=x=Tx=Tz=y$, which again implies $Ty=y$, a contradiction. Hence, x,y,z are pairwise distinct. In this case, applying inequality (4.2), we obtain

$$d(x,z)+d(z,y)\leq \alpha(d(y,z)+d(z,y)).$$

Since $d(z,y)\leq d(x,z)+d(z,y)$, we have $\alpha\geq\frac{1}{2}$, contradicting the assumption in (4.1). Hence, T has no periodic point of prime period 2.

Finally, suppose T has three distinct fixed points, say $Tx=x$, $Ty=y$, and $Tz=z$. Then, (4.2) yields $d(x,y)+d(y,z)=0$, which contradicts the distinctness of x,y and z. Hence, T has at most two fixed points. □

Remark 4.6. Theorem 4.5 extends [10, Theorem 3] by enlarging the admissible range of the contraction constant from $\left[0,\frac{1}{s}\right)$ to $[0,1)$. Moreover, we establish a converse result to [10, Theorem 3]. By setting $s=1$ in Theorem 4.5, we get [13, Theorem 3.2].

The following example demonstrates the applicability of Theorem 4.5.

Example 4.7. Let $X=\left\{0,\frac{1}{3},\frac{2}{3},1\right\}$, and define $d:X\times X\to[0,\infty)$ by

$$d(0,\tfrac{1}{3})=5,\ d(\tfrac{1}{3},\tfrac{2}{3})=2,\ d(\tfrac{2}{3},1)=5,$$
$$d(0,\tfrac{2}{3})=2,\ d(\tfrac{1}{3},1)=2,\ d(0,1)=1,$$

with $d(x,x)=0$ and $d(x,y)=d(y,x)$ for all $x,y\in X$. Then (X,d) is a complete b-metric space with $s=\frac{5}{3}$. Indeed, the only nontrivial b-metric inequalities are

$$d(0,\tfrac{2}{3})=2\leq\frac{5}{3}\left(d(0,\tfrac{1}{3})+d(\tfrac{1}{3},\tfrac{2}{3})\right)\ \text{and}\ d(0,\tfrac{1}{3})=5\leq\frac{5}{3}\left(d(0,1)+d(1,\tfrac{1}{3})\right).$$

Define $T:X\to X$ by

$$Tx=\begin{cases}0, & x\in\left\{0,\frac{1}{3}\right\},\\ 1, & x\in\left\{\frac{2}{3},1\right\}.\end{cases}$$

Clearly, $\mathrm{Fix}(T)=\{0,1\}$. Moreover, T has no periodic point of prime period 2, since

$$T(T\left(\frac{1}{3}\right))=T(0)=0\neq\frac{1}{3}\ \text{and}\ T(T\left(\frac{2}{3}\right))=T(1)=1\neq\frac{2}{3}.$$

We now verify the inequality (4.2) for all pairwise distinct $x, y, z \in X$. Since X contains four points, there are 24 ordered triples of pairwise distinct points. However, (4.2) is invariant under the interchange of x and z, so it suffices to consider the corresponding 12 triples. Direct computation shows that the inequality holds for $\alpha \geq \frac{1}{10}, \frac{1}{7}, \frac{2}{7}, \frac{2}{5}$, depending on the triple. The maximum is attained precisely for $\left(0, 1, \frac{1}{3}\right), \left(\frac{1}{3}, 1, 0\right), \left(\frac{2}{3}, 0, 1\right), \left(1, 0, \frac{2}{3}\right)$. Thus, the sharp contraction constant for this example is $\alpha = \frac{2}{5} < \frac{3}{5} = \frac{1}{s}$. Consequently, the inequality (4.2) holds for every $\alpha \in \left[\frac{2}{5}, \frac{3}{5}\right)$. Therefore, all the hypotheses of Theorem 4.5 are satisfied.

5. Conclusion

In this paper, we introduced Paired–Chatterjea contractions and established fixed point results in b-metric spaces. We extended the admissible contraction range in the Paired contraction as in [10, Theorem 2] from $\left[0, \frac{1}{s}\right)$ to $[0, 1)$. We showed that the condition for the Paired–Kannan framework in [10, Theorem 3] is insufficient and established the revised condition $\alpha < \min\left\{\frac{1}{2}, \frac{1}{s}\right\}$. The examples demonstrate the distinctions among the paired contractive classes and the significance of the coefficient restrictions.

References

1. Alansari, M., & Shagari, M. S. (2022). Analysis of fractional differential inclusion models for COVID-19 via fixed point results in metric space. Journal of Function Spaces, **2022** (1). https://doi.org/10.1155/2022/8311587
2. Ali, B., Butt, H. A., & De la Sen, M. (2022). Existence of fixed points of generalized set-valued F-contractions of b-metric spaces. *AIMS Mathematics*, **7** (10), 17967–17988. https://doi.org/10.3934/math.2022990
3. Alshehry, A. S., Mukhtar, S., Khan, H. S., & Shah, R. (2023). Fixed-point theory and numerical analysis of an epidemic model with fractional calculus: Exploring dynamical behavior. Open Physics, **21** (1). https://doi.org/10.1515/phys-2023-0121
4. Bakhtin, I. (1989). The contraction mapping principle in quasimetric spaces. *Funkt. Anal. Ulianowsk Gos. Ped. Inst.*, **30**, 26–37. https://www.sciepub.com/reference/250034
5. Banach, S. (1922). Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundamenta Mathematicae*, **3**, 133–181. https://doi.org/10.4064/fm-3-1-133-181
6. Banerjee, A., Mondal, P., Dey, L. K., & Sintunavarat, W. (2026). A perspective on Banach and Kannan mappings contracting the perimeter of triangles in interpolative metric spaces. *Nonlinear Analysis: Modelling and Control*, **31** (5), 1021–1041. https://doi.org/10.15388/namc.2026.31.47663
7. Bey, C., Petrov, E., & Salimov, R. (2025). On three-point generalizations of Banach and Edelstein fixed point theorems. *Filomat*, **39** (1), 185–195. https://doi.org/10.2298/FIL.2501185B
8. Bisht, R. K., & Petrov, E. (2024). Three point analogue of Ćirić–Reich–Rus type mappings with non-unique fixed points. *Journal of Analysis*, **32**, 2609–2627. https://doi.org/10.1007/s41478-024-00743-2
9. Bota, M.-F., Guran, L., & Petruşel, A. (2020). New fixed point theorems on b-metric spaces with applications to coupled fixed point theory. *Journal of Fixed Point Theory and Applications*, **22**, 74. https://doi.org/10.1007/s11784-020-00808-2
10. Bouker, Z. I., Hamaizia, T., Chand, D., & Rohen, Y. (2026). Fixed point theorems for paired contractions in b-metric spaces. *Mathematica Moravica*, **30** (1), 19–30. https://doi.org/10.5937/MatMor2601019I

11. Chaib, R., Merghadi, F., & Mouhoubi, Z. (2023). Improvement of fixed point theorems for Hardy–Rogers contraction type in b-metric spaces without F-contraction assumption. *Rendiconti del Circolo Matematico di Palermo Series 2*, **72** (8), 4209–4237. https://doi.org/10.1007/s12215-023-00892-6
12. Chand, D., & Rohen, Y. (2024). Paired contractive mappings and fixed point results. *AIMS Mathematics*, **9** (1), 1959–1968. https://doi.org/10.3934/math.2024097
13. Chand, D., Rohen, Y., & Fabiano, N. (2024). Paired-Kannan contraction mappings and fixed point results. *Gulf Journal of Mathematics*, **17** (2), 136–154. https://doi.org/10.56947/gjom.v17i2.2185
14. Chand, D., Rohen, Y., Singh, S. S., & Fabiano, N. (2025). Paired-Chatterjea type contractions: Novel fixed point results and continuity properties in metric spaces. *Vojnotehnički glasnik*, **73** (2), 423–449. https://doi.org/10.5937/vojtehg73-54620
15. Chiroma, R., Shagari, M. S., Alansari, M., Alotaibi, T., & Saliu, A. (2026). On Hammerstein integral equations and engineering models via paired contractions in partial metric spaces. *Journal of Mathematics*, **2026** (1). https://doi.org/10.1155/jom/8223084
16. Cvetković, M. (2026). Iterates of mappings contracting perimeters of triangles. *Bulletin of the Iranian Mathematical Society*, **52** (4). https://doi.org/10.1007/s41980-026-01055-w
17. Cvetković, M. (2026). Remetrization techniques for mappings contracting perimeters of triangles. *Nonlinear Analysis: Modelling and Control*, **31** (5), 1089–1108. https://doi.org/10.15388/namc.2026.31.47703
18. Czerwik, S. (1993). Contraction mappings in b-metric spaces. *Acta Mathematica et Informatica Universitatis Ostraviensis*, **1** (1), 5–11. https://dml.cz/handle/10338.dmlcz/120469
19. Fatai, M. O., Aribike, E. E., Adewinbi, H. S., Mohammed, M. T., & Rauf, K. (2026). On π-asymmetric Geraghty type contractive mappings in metric spaces. *Nigerian Journal of Technological Development*, **23** (2), 67–74. https://journal.njtd.com.ng/index.php/njtd/article/view/4125
20. Gdawiec, K., & Adewinbi, H. (2022). Procedural generation of artistic patterns using a modified orbit trap method. Applied Sciences, 12, 2923. https://doi.org/10.3390/app12062923
21. Jleli, M., Păcurar, C. M., & Samet, B. (2025). Fixed point results for contractions of polynomial type. *Demonstratio Mathematica*, **58** (1). https://doi.org/10.1515/dema-2025-0098
22. Joonaghany, G. H., Mewomo, O. T., & Khojasteh, F. (2021). Weak KKH-Caristi type expansion on complete b-metric and b-ordered metric spaces and its consequences. *Afrika Matematika*, **32** (7–8), 1281–1294. https://doi.org/10.1007/s13370-021-00899-1
23. Kadelburg, Z., Paunović, L., & Radenović, S. (2015). A note on fixed point theorems for weakly T-Kannan and weakly T-Chatterjea contractions in b-metric spaces. *Gulf Journal of Mathematics*, **3** (3), 57–67. https://doi.org/10.56947/gjom.v3i3.148
24. Khan, A. A., Ali, B., & George, R. (2023). On semi best proximity points for multivalued mappings in quasi metric spaces. *AIMS Mathematics*, **8** (10), 23835–23849. https://doi.org/10.3934/math.20231215
25. Kir, M., & Kiziltune, H. (2013). On some well-known fixed-point theorems in b-metric spaces. *Turkish Journal of Analysis and Number Theory*, **1** (1), 13–16. https://doi.org/10.12691/tjant-1-1-4
26. Leyew, B. T., & Mewomo, O. T. (2022). Common fixed point results for generalized orthogonal F-Suzuki contraction for a family of multivalued mappings in orthogonal b-metric spaces. *Communications of the Korean Mathematical Society*, **37** (4), 1147–1170. https://doi.org/10.4134/CKMS.c210410
27. Mani, G., Gnanaprakasam, A. J., George, R., & Mitrović, Z. D. (2023). The existence of a solution of a nonlinear Fredholm integral equations over bicomplex b-metric spaces. Gulf Journal of Mathematics, 14 (1), 69–83. https://doi.org/10.56947/gjom.v14i1.984

28. Mebawondu, A. A., Izuchukwu, C., Ogwo, G. N., & Mewomo, O. T. (2022). Existence of solutions for boundary value problems and nonlinear matrix equations via F-contraction mappings in b-metric spaces. *Asian-European Journal of Mathematics*, **15** (11). https://doi.org/10.1142/S1793557122502011
29. Miculescu, R., & Mihail, A. (2017). New fixed point theorems for set-valued contractions in b-metric spaces. *Journal of Fixed Point Theory and Applications*, **19** (3), 2153–2163. https://doi.org/10.1007/s11784-016-0400-2
30. Mitrović, Z. D. (2019). Fixed point results in b-metric space. *Fixed Point Theory*, **20** (2), 559–566. https://doi.org/10.24193/fpt-ro.2019.2.36
31. Mitrović, Z. D., George, R., & Hosseinzadeh, H. (2025). Some results on fixed points on the product of b-metric spaces with applications. *Fixed Point Theory*, **26** (1), 225–234. https://doi.org/10.24193/fpt-ro.2025.1.14
32. Özkan, M., & Khan, S. H. (2026). Fixed point theorems for multipoint Chatterjea-type mappings. *Contemporary Mathematics*, **7** (3), 3066–3085. https://doi.org/10.37256/cm.7320268485
33. Păcurar, C. M., & Popescu, O. (2024). Fixed point theorem for generalized Chatterjea type mappings. *Acta Mathematica Hungarica*, **173** (2), 500–509. https://doi.org/10.1007/s10474-024-01455-6
34. Patel, D., Younis, M., Singh, D., & Chauhan, O. P. (2026). Banach contraction induced by Hardy-Rogers contractions: Computation and applications. *Computational and Applied Mathematics*, **45** (9), 358. https://doi.org/10.1007/s40314-026-03738-w
35. Petrov, E. (2023). Fixed point theorem for mappings contracting perimeters of triangles. *Journal of Fixed Point Theory and Applications*, **25**, 74. https://doi.org/10.1007/s11784-023-01078-4
36. Petrov, E., & Bisht, R. K. (2024). Fixed point theorem for generalized Kannan type mappings. *Rendiconti del Circolo Matematico di Palermo Series 2*, **73** (8), 2895–2912. https://doi.org/10.1007/s12215-024-01079-3
37. Popescu, O. (2021). Some remarks on the paper "Fixed point theorems for generalized contractive mappings in metric spaces." *Journal of Fixed Point Theory and Applications*, **23**, 72. https://doi.org/10.1007/s11784-021-00908-7
38. Proinov, P. D. (2020). Fixed point theorems for generalized contractive mappings in metric spaces. *Journal of Fixed Point Theory and Applications*, **22**, 21. https://doi.org/10.1007/s11784-020-0756-1
39. Shatnawi, M., & Bataihah, A. (2025). Fixed point theorems for contractions of integral type through generalized Omega distance. *Gulf Journal of Mathematics*, **20** (2), 429–442. https://doi.org/10.56947/gjom.v20i2.3080
40. Wahab, O. T., Usamot, I. F., & Tijani, K. R. (2025). On the existence of fixed points via condensed Reich–Rus–Ćirić-type contractions. *Gulf Journal of Mathematics*, **20** (2), 344–353. https://doi.org/10.56947/gjom.v20i2.3212

[1]Department of Mathematical Sciences, Kent State University, Kent, OH, 44242, United States.

Email address: hadewinb@kent.edu