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FITTED DIFFERENCE SCHEME FOR REACTION-DIFFUSION PROBLEMS

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Abstract. In this paper, we consider the numerical solution of a semilinear singularly perturbed reaction-diffusion boundary-value problem exhibiting boundary layers. A new difference scheme is constructed, and its second-order convergence on a modified Shishkin mesh is established. Numerical experiments are presented to confirm the theoretical results.

Keywords. Singular perturbation, reaction-diffusion, boundary layer, Shishkin mesh, layer-adapted mesh, uniform convergence
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1. Introduction

Let us consider the following boundary value problem

$$-\varepsilon^2 y'' + f(x,y) = 0, \quad x \in (0,1), \tag{1.1}$$

$$y(0) = 0,\ y(1) = 0, \tag{1.2}$$

where $0 < \varepsilon < 1$. We assume that the nonlinear function f is continuously differentiable; more precisely, for $k \geqslant 2$, $f \in C^k([0,1] \times \mathbb{R})$, and that it has a strictly positive partial derivative with respect to y,

$$\frac{\partial f}{\partial y} := f_y \geqslant m > 0, \quad (x,y) \in [0,1] \times \mathbb{R}, \tag{1.3}$$

where m is a constant. Under condition (1.3), the problem (1.1)–(1.2) has a unique solution (see Lorenz [15]).

It is well known that the exact solution to the problem (1.1)–(1.3) changes rapidly near the endpoints $x = 0$ and $x = 1$. Furthermore, when classical numerical methods, which do not account for these rapid changes, are applied to the problem (1.1)–(1.3), a large number of mesh points must be used to obtain a satisfactory numerical solution. This is computationally expensive, which is one of the primary reasons for developing numerical methods that take into account the properties of singularly perturbed boundary-value problems.

Many authors have investigated the numerical treatment of the problem (1.1)–(1.2) under various assumptions regarding the function f, as well as of more

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general nonlinear problems. To avoid listing all authors individually, we refer the reader to the monographs [10, 16, 5, 18] and the references therein.

In this paper, we employ a method introduced by Boglaev [1] to construct a new difference scheme on a layer-adapted mesh introduced by Vulanović [21]. A similar difference scheme for this problem was constructed in [8], where the authors used the same layer-adapted mesh. For additional technical details regarding the implementation of the numerical scheme, see [6, 7]. The results obtained in this study can be further extended in several directions, with related ideas for future research outlined in [2, 19, 22].

It is a well-known fact that the Shishkin mesh is the simplest layer-adapted mesh. It is a piecewise-uniform mesh consisting of two glued uniform meshes: a finer one in the layer region and a coarser one outside of it. The simplicity of this construction facilitates a much more straightforward convergence analysis compared to smoothly graded meshes (such as Bakhvalov or Liseikin grids), which is the primary benefit of employing the Shishkin mesh. However, the cost of this simplicity, and the lack of a smooth transition between the mesh subdomains, is a degradation in accuracy. Specifically, the error bounds for Shishkin meshes typically feature a logarithmic factor, which increases the overall error.

The remainder of the paper is organized as follows. In Section 2, we introduce the layer-adapted meshes and provide their generating functions. A new difference scheme is constructed in Section 3, and its stability is analyzed in Section 4. The uniform convergence is proved in Section 5. Finally, numerical experiments are presented in Section 6, followed by concluding remarks in Section 7.

2. Layer-adapted meshes

As mentioned previously, classical numerical methods are not suitable for boundary value problems exhibiting layer phenomena, such as (1.1)–(1.2). These methods suffer from issues regarding stability, high computational costs, and poor accuracy. These difficulties have been overcome by constructing specialized numerical methods that account for the presence of boundary layers. One such approach is the fitted mesh method. Meshes constructed via this method feature a non-uniform distribution of mesh points, which is dictated by the behavior of the exact solution and its derivatives within the layer regions. The estimates provided in the theorem below are crucial for understanding this behavior. To analyze this behavior, we decompose the solution y of problem (1.1)–(1.2) into a regular component r and a layer component s, as specified in the following theorem.

Theorem 2.1. [20] *The solution y to the problem (1.1)−(1.2) can be represented as:*

$$y = r + s,$$

where for $j = 0, 1, ..., k+2$, and $x \in [0,1]$ we have that

$$\left|r^{(j)}(x)\right| \leq C, \tag{2.1}$$

and

$$\left|s^{(j)}(x)\right| \leq C\varepsilon^{-j}\left(e^{-\frac{x}{\varepsilon}\sqrt{m}} + e^{-\frac{1-x}{\varepsilon}\sqrt{m}}\right). \tag{2.2}$$

Theorem 2.1 implies that the derivatives of the layer component s grow significantly as $\varepsilon \to 0$, meaning that the exact solution to problem (1.1)–(1.3) changes rapidly near the endpoints $x = 0$ and $x = 1$. Thus, the problem possesses two boundary layers of exponential type near these endpoints.

Numerous meshes have been developed for the numerical solution of problems exhibiting exponential layers. In this paper, we employ three different meshes. Let $N + 1$ denote the number of mesh points. The meshes used in the remainder of the paper, denoted by

$$0 = x_0 < x_1 < \ldots < x_N = 1,$$

are obtained via appropriate generating functions, i.e., $x_i = \psi(i/N)$ for $i = 0, 1, \ldots, N$. These generating functions are defined as follows.

The first mesh is the modified Shishkin mesh proposed by Vulanović [21]; its generating function is given by

$$\psi(t) = \begin{cases} 4\lambda t, & t \in [0, 1/4], \\ p(t - 1/4)^3 + 4\lambda t, & t \in [1/4, 1/2], \\ 1 - \psi(1 - t), & t \in [1/2, 1], \end{cases} \tag{2.3}$$

where p is chosen so that $\psi(1/2) = 1/2$, and λ is the Shishkin mesh transition point, defined by

$$\lambda := \min\left\{\frac{2\varepsilon \ln N}{\sqrt{m}}, \frac{1}{4}\right\}. \tag{2.4}$$

Note that $\psi \in C^1[0, 1]$ with $\|\psi'\|_\infty \leqslant C$ and $\|\psi''\|_\infty \leqslant C$. Therefore, the mesh sizes $h_i = x_{i+1} - x_i$, $i = 0, \ldots, N - 1$, satisfy (see [11])

$$\begin{aligned} h_i &= \int_{i/N}^{(i+1)/N} \psi'(t)\,\mathrm{d}\,t \leqslant CN^{-1}, \\ |h_{i+1} - h_i| &= \left| \int_{i/N}^{(i+1)/N} \int_t^{t+1/N} \psi''(s)\,\mathrm{d}\,s \right| \leqslant CN^{-2}. \end{aligned} \tag{2.5}$$

The second mesh is the modified Bakhvalov mesh, also proposed by Vulanović [20]; its generating function is defined as

$$\psi(t) = \begin{cases} \mu(t) := \frac{a\varepsilon t}{q - t}, & t \in [0, \alpha], \\ \mu(\alpha) + \mu'(\alpha)(t - \alpha), & t \in [\alpha, 1/2], \\ 1 - \psi(1 - t), & t \in [1/2, 1], \end{cases} \tag{2.6}$$

where a and q are constants independent of ε such that $q \in (0, 1/2)$, $a \in (0, q/\varepsilon)$, and additionally $a\sqrt{m} \geqslant 2$. The parameter α is the abscissa of the contact point

of the tangent line from $(1/2, 1/2)$ to $\mu(t)$, and its value is

$$\alpha = \frac{q - \sqrt{aq\varepsilon(1-2q+2a\varepsilon)}}{1+2a\varepsilon}.$$

Finally, we consider the mesh proposed by Liseikin [14, 13], in its modified form from [12]. The generating function for this mesh is given by

$$\psi(t,\varepsilon,a,k) = \begin{cases} c_1\varepsilon^k((1-dt)^{-1/a}-1), & 0 \leqslant t \leqslant 1/4, \\ c_1\Big[\varepsilon^{kan/(1+na)} - \varepsilon^k + d\dfrac{1}{a}\varepsilon^{ka(n-1)/(1+na)}(t-1/4) + \\ \dfrac{1}{2}d^2\dfrac{1}{a}\Big(\dfrac{1}{a}+1\Big)\varepsilon^{ka(n-2)/(1+na)}(t-1/4)^2 + c_0(t-1/4)^3\Big], & 1/4 \leqslant t \leqslant 1/2, \\ 1-\psi(1-t,\varepsilon,a,k), & 1/2 \leqslant t \leqslant 1, \end{cases} \tag{2.7}$$

where $d = 4(1-\varepsilon^{ka/(1+na)})$, a is a positive constant subject to $a \geqslant m_1 > 0$, and $c_0 \geqslant 0$. Here we choose $a = 1$, $n = 2$, $k = 1$ and set

$$\frac{1}{c_1} = 2\Big[\varepsilon^{kan/(1+na)} - \varepsilon^k + \frac{d}{4a}\varepsilon^{ka(n-1)/(1+na)} + \frac{d^2}{2}\frac{1}{a}\Big(\frac{1}{a}+1\Big)\varepsilon^{ka(n-2)/(1+na)}(1/4)^2 + c_0(1/4)^3\Big]$$

3. The difference scheme

The first step in the numerical solution of problem (1.1)–(1.3) is the construction of a difference scheme, which generates a nonlinear system of equations. The solution to this nonlinear system yields the discrete numerical solution to problem (1.1)–(1.3).

3.1. **Construction of the difference scheme.** In his paper [1], Boglaev constructed a difference scheme for the numerical solution of problem (1.1)–(1.3). To construct this scheme, Boglaev utilized the Green's function to represent the exact solution to the problem. From [1], we have the following identity:

$$\frac{\beta}{\sinh(\beta h_{i-1})}y_{i-1} - \left(\frac{\beta}{\tanh(\beta h_{i-1})} + \frac{\beta}{\tanh(\beta h_i)}\right)y_i + \frac{\beta}{\sinh(\beta h_i)}y_{i+1} = \frac{1}{\varepsilon^2}\left[\int_{x_{i-1}}^{x_i} u_{i-1}^{II}(s)\psi(s,y(s))\,\mathrm{d}\,s + \int_{x_i}^{x_{i+1}} u_i^{I}(s)\psi(s,y(s))\,\mathrm{d}\,s\right], \tag{3.1}$$

$$y_0 = 0, \;\; y_N = 0, \; i = 1, 2, \ldots, N-1,$$

where $\psi(s,y(s)) = f(s,y(s)) - \gamma y(s)$, $u_i^I(x) = \frac{\sinh(\beta(x_{i+1}-x))}{\sinh(\beta h_i)}$, $u_i^{II}(x) = \frac{\sinh(\beta(x-x_i))}{\sinh(\beta h_i)}$, $x \in [x_i, x_{i+1}]$, $\beta = \frac{\sqrt{\gamma}}{\varepsilon}$ and

$$\gamma \geqslant f_y. \tag{3.2}$$

Identity (3.1) serves as an excellent foundation for constructing a difference scheme; it is only necessary to evaluate the integrals that appear on the right-hand side. In general, we cannot explicitly evaluate these integrals because they both involve the unknown function f (via ψ). Therefore, they must be approximated. Since the function f appears within ψ, the simplest approach is to approximate ψ by a constant over each subinterval. Consequently, the remaining integrals $\int_{x_{i-1}}^{x_i} u_{i-1}^{II}(s)\,\mathrm{d}\,s$ and $\int_{x_i}^{x_{i+1}} u_i^{I}(s)\,\mathrm{d}\,s$ can be evaluated exactly. Depending on how these integrals are approximated, various difference schemes can be derived.

Boglaev introduced this approach in [1], where he approximated the function ψ on the interval $[x_{i-1}, x_{i+1}]$ by $\psi(x_i, y(x_i))$, meaning that both integrals were evaluated using the same constant value $\psi(x_i, y(x_i))$. In [4, 8, 9], the authors considered a similar approach for the numerical solution of problem (1.1)–(1.3). However, the approximations introduced in those works are somewhat cumbersome and unnecessarily complex for a problem of this nature. This underlying complexity was subsequently reflected both in the structure of the resulting difference schemes and in their convergence analysis.

To circumvent these difficulties and streamline the analysis as much as possible, we propose an alternative approximation for the integrals in (3.1). Specifically, on each subinterval $[x_{i-1}, x_i]$ for $i = 1, 2, \ldots, N$, we approximate the function ψ by the average:

$$\overline{\psi}_i = \frac{\psi(x_{i-1}, \overline{y}_{i-1}) + \psi(x_i, \overline{y}_i)}{2}. \tag{3.3}$$

Substituting (3.3) into (3.1) yields, after some algebraic manipulation, the following difference scheme:

$$\begin{aligned}\frac{\cosh(\beta h_{i-1}) + 1}{\sinh(\beta h_{i-1})}\overline{y}_{i-1} - \left(\frac{\cosh(\beta h_{i-1}) + 1}{\sinh(\beta h_{i-1})} + \frac{\cosh(\beta h_i) + 1}{\sinh(\beta h_i)}\right)\overline{y}_i + \frac{\cosh(\beta h_i) + 1}{\sinh(\beta h_i)}\overline{y}_{i+1} \\ - \frac{f_{i-1} + f_i}{\gamma} \cdot \frac{\cosh(\beta h_{i-1}) - 1}{\sinh(\beta h_{i-1})} - \frac{f_i + f_{i+1}}{\gamma} \cdot \frac{\cosh(\beta h_i) - 1}{\sinh(\beta h_i)} = 0,\end{aligned} \tag{3.4}$$

where $f_i = f(x_i, \overline{y}_i)$, $\overline{y}_0 = y(0)$, $\overline{y}_N = y(1)$, and $i = 1, 2, \ldots, N-1$.

4. Stability

The difference scheme (3.4) generates a system of nonlinear algebraic equations, the solution of which represents the discrete numerical solution to problem (1.1)–(1.3). Our next task is to establish the existence and uniqueness of this discrete solution, as well as the stability of the difference scheme. To this end, we define the discrete operator

$$Tu = (Tu_0, Tu_1, \ldots, Tu_N)^T, \tag{4.1}$$

where

$$
\begin{aligned}
Tu_0 &= -u_0 \\
Tu_i &= \frac{\gamma}{\frac{\cosh(\beta h_{i-1})-1}{\sinh(\beta h_{i-1})}+\frac{\cosh(\beta h_i)-1}{\sinh(\beta h_i)}} \\
&\quad\cdot\left[\frac{\cosh(\beta h_{i-1})+1}{\sinh(\beta h_{i-1})}u_{i-1}-\left(\frac{\cosh(\beta h_{i-1})+1}{\sinh(\beta h_{i-1})}+\frac{\cosh(\beta h_i)+1}{\sinh(\beta h_i)}\right)u_i\right. \\
&\qquad+\frac{\cosh(\beta h_i)+1}{\sinh(\beta h_i)}u_{i+1}-\frac{f(x_{i-1},u_{i-1})+f(x_i,u_i)}{\gamma}\cdot\frac{\cosh(\beta h_{i-1})-1}{\sinh(\beta h_{i-1})} \\
&\qquad\left.-\frac{f(x_i,u_i)+f(x_{i+1},u_{i+1})}{\gamma}\cdot\frac{\cosh(\beta h_i)-1}{\sinh(\beta h_i)}\right], \qquad (4.2)\\
&\qquad\qquad i=1,\dots,N-1 \\
Tu_N &= -u_N.
\end{aligned}
$$

Clearly, it holds that

$$T\overline{y}=0, \tag{4.3}$$

where $\overline{y}=(\overline{y}_0,\overline{y}_1,\dots,\overline{y}_N)^T$, $\overline{y}_0=y(0)$, $\overline{y}_N=y(1)$ is the discrete numerical solution to problem (1.1)–(1.3) obtained via the difference scheme (3.4).

Let us formulate and prove the following theorem on stability.

Theorem 4.1. *The discrete problem (4.1)–(4.3) possesses a unique solution $\overline{y}$ provided that $\gamma \geqslant f_y$. Moreover, for any $v, w \in \mathbb{R}^{N+1}$, the following stability inequality holds:*

$$\|v-w\| \leqslant C\|Tv-Tw\|. \tag{4.4}$$

Proof. Let H denote the Fréchet derivative of the discrete operator T, i.e., $H := (T\overline{y})'$, where $H=(h_{ij})$. The non-zero elements of the matrix H are given by:

$$
\begin{aligned}
h_{1,1} &= -1,\ h_{N+1,N+1}=-1, \\
h_{i,i-1} &= \frac{\gamma}{\frac{\cosh(\beta h_{i-1})-1}{\sinh(\beta h_{i-1})}+\frac{\cosh(\beta h_i)-1}{\sinh(\beta h_i)}}\left(\frac{\cosh(\beta h_{i-1})+1}{\sinh(\beta h_{i-1})}-\frac{\frac{\partial f}{\partial \overline{y}_{i-1}}}{\gamma}\frac{\cosh(\beta h_{i-1})-1}{\sinh(\beta h_{i-1})}\right), \\
h_{i,i+1} &= \frac{\gamma}{\frac{\cosh(\beta h_{i-1})-1}{\sinh(\beta h_{i-1})}+\frac{\cosh(\beta h_i)-1}{\sinh(\beta h_i)}}\left(\frac{\cosh(\beta h_i)+1}{\sinh(\beta h_i)}-\frac{\frac{\partial f}{\partial \overline{y}_{i+1}}}{\gamma}\cdot\frac{\cosh(\beta h_i)-1}{\sinh(\beta h_i)}\right), \\
h_{i,i} &= -\frac{\gamma}{\frac{\cosh(\beta h_{i-1})-1}{\sinh(\beta h_{i-1})}+\frac{\cosh(\beta h_i)-1}{\sinh(\beta h_i)}}\left[\frac{\cosh(\beta h_{i-1})+1}{\sinh(\beta h_{i-1})}+\frac{\cosh(\beta h_i)+1}{\sinh(\beta h_i)}\right. \\
&\qquad\left.+\frac{\cosh(\beta h_{i-1})-1}{\sinh(\beta h_{i-1})}\cdot\frac{\frac{\partial f}{\partial \overline{y}_i}}{\gamma}+\frac{\cosh(\beta h_i)-1}{\sinh(\beta h_i)}\cdot\frac{\frac{\partial f}{\partial \overline{y}_i}}{\gamma}\right],\ i=2,\dots,N.
\end{aligned}
$$

Due to (3.2), we have

$$h_{1,1}<0,\ h_{N+1,N+1}<0,\ h_{i,i-1}\geqslant 0,\ h_{i,i+1}\geqslant 0,\ h_{i,i}<0, \tag{4.5}$$

which implies that H is an L-matrix.

Next, we show that H is diagonally dominant. Taking into account that $f_y \geqslant m$, we obtain:

$$
\begin{aligned}
&|h_{i,i}| - |h_{i,i-1}| - |h_{i,i+1}| \\
&\quad = \frac{\gamma}{\frac{\cosh(\beta h_{i-1})-1}{\sinh(\beta h_{i-1})} + \frac{\cosh(\beta h_i)-1}{\sinh(\beta h_i)}} \left(\frac{\cosh(\beta h_{i-1})+1}{\sinh(\beta h_{i-1})} + \frac{\cosh(\beta h_i)+1}{\sinh(\beta h_i)} \right. \\
&\qquad \left. + \frac{\cosh(\beta h_{i-1})-1}{\sinh(\beta h_{i-1})} \cdot \frac{\frac{\partial f}{\partial \overline{y}_i}}{\gamma} + \frac{\cosh(\beta h_i)-1}{\sinh(\beta h_i)} \cdot \frac{\frac{\partial f}{\partial \overline{y}_i}}{\gamma} \right) \\
&\quad - \frac{\gamma}{\frac{\cosh(\beta h_{i-1})-1}{\sinh(\beta h_{i-1})} + \frac{\cosh(\beta h_i)-1}{\sinh(\beta h_i)}} \left(\frac{\cosh(\beta h_{i-1})+1}{\sinh(\beta h_{i-1})} - \frac{\frac{\partial f}{\partial \overline{y}_{i-1}}}{\gamma} \frac{\cosh(\beta h_{i-1})-1}{\sinh(\beta h_{i-1})} \right) \\
&\quad - \frac{\gamma}{\frac{\cosh(\beta h_{i-1})-1}{\sinh(\beta h_{i-1})} + \frac{\cosh(\beta h_i)-1}{\sinh(\beta h_i)}} \left(\frac{\cosh(\beta h_i)+1}{\sinh(\beta h_i)} - \frac{\frac{\partial f}{\partial \overline{y}_{i+1}}}{\gamma} \cdot \frac{\cosh(\beta h_i)-1}{\sinh(\beta h_i)} \right) \\
&\quad = \frac{\gamma}{\frac{\cosh(\beta h_{i-1})-1}{\sinh(\beta h_{i-1})} + \frac{\cosh(\beta h_i)-1}{\sinh(\beta h_i)}} \left(\frac{\cosh(\beta h_{i-1})-1}{\sinh(\beta h_{i-1})} \cdot \frac{\frac{\partial f}{\partial \overline{y}_i}}{\gamma} + \frac{\cosh(\beta h_i)-1}{\sinh(\beta h_i)} \cdot \frac{\frac{\partial f}{\partial \overline{y}_i}}{\gamma} \right. \\
&\qquad \left. + \frac{\frac{\partial f}{\partial \overline{y}_{i-1}}}{\gamma} \frac{\cosh(\beta h_{i-1})-1}{\sinh(\beta h_{i-1})} + \frac{\frac{\partial f}{\partial \overline{y}_{i+1}}}{\gamma} \cdot \frac{\cosh(\beta h_i)-1}{\sinh(\beta h_i)} \right) \\
&\quad \geqslant 2m > 0. \qquad (4.6)
\end{aligned}
$$

Thus, we conclude that H is an M-matrix, which guarantees that

$$\|H^{-1}\| \leqslant C. \qquad (4.7)$$

By Hadamard's theorem [17, Th. 5.3.10], the discrete operator T is a homeomorphism, ensuring that (4.3) has a unique solution.

The second assertion of the theorem follows from the mean value theorem, yielding

$$Tv - Tw = (T\xi)'(v - w)$$

for some $\xi = (\xi_0, \xi_1, \ldots, \xi_N)^T \in \mathbb{R}^{N+1}$. Consequently, based on (4.7), we finally obtain

$$\|v - w\| \leqslant C\|Tv - Tw\|.$$

□

Remark 4.2. If we multiply the difference scheme (3.4) by -1, we obtain $h_{1,1} > 0$, $h_{N+1,N+1} > 0$, $h_{i,i-1} \leqslant 0$, $h_{i,i+1} \leqslant 0$, $h_{i,i} > 0$.

5. Uniform convergence

In this section, we address the discretization error, which is a fundamental aspect of the numerical solution to problem (1.1)–(1.3).

For the subsequent analysis, it is sufficient to consider the interval $[0, 1/2]$ — specifically, the subintervals $[0, \lambda]$ and $[\lambda, 1/2]$. On the interval $[0, 1/2]$, the layer component s from the estimate (2.2) can be omitted since the term $\varepsilon^{-j} e^{-(1-x)\sqrt{m}/\varepsilon}$ is negligible. Throughout this section, we also assume that $\varepsilon \leqslant \frac{C}{N}$.

To prove the convergence theorem, we require the following auxiliary lemmas.

Lemma 5.1. *Assume that* $\varepsilon \leqslant \frac{C}{N}$. *On the portion of the modified mesh* (2.3) *where* x_i, $x_{i\pm1} \in [x_{N/4-1}, \lambda] \cup [\lambda, 1/2]$, *the following estimate holds:*

$$\frac{\gamma}{\frac{\cosh(\beta h_{i-1})-1}{\sinh(\beta h_{i-1})} + \frac{\cosh(\beta h_i)-1}{\sinh(\beta h_i)}} \left| \frac{f(x_{i-1}, y(x_{i-1})) + f(x_i, y(x_i))}{\gamma} \cdot \frac{\cosh(\beta h_{i-1}) - 1}{\sinh(\beta h_{i-1})} \right.$$
$$\left. + \frac{f(x_i, y(x_i)) + f(x_{i+1}, y(x_{i+1}))}{\gamma} \cdot \frac{\cosh(\beta h_i) - 1}{\sinh(\beta h_i)} \right| \leqslant \frac{C}{N^2},\ i = N/4, \ldots, N/2-1.$$

Proof. By virtue of the relation $\varepsilon^2 y''(x_i) = f(x_i, y(x_i))$, the solution decomposition into the regular and layer components r and s on the mesh subset corresponding to $[x_{N/4-1}, \lambda] \cup [\lambda, 1/2]$, and the assumption $\varepsilon \leqslant \frac{C}{N}$, it follows that

$$\frac{\gamma}{\frac{\cosh(\beta h_{i-1})-1}{\sinh(\beta h_{i-1})} + \frac{\cosh(\beta h_i)-1}{\sinh(\beta h_i)}} \cdot \left| \frac{f(x_{i-1}, y(x_{i-1})) + f(x_i, y(x_i))}{\gamma} \cdot \frac{\cosh(\beta h_{i-1}) - 1}{\sinh(\beta h_{i-1})} \right.$$
$$\left. + \frac{f(x_i, y(x_i)) + f(x_{i+1}, y(x_{i+1}))}{\gamma} \cdot \frac{\cosh(\beta h_i) - 1}{\sinh(\beta h_i)} \right|$$
$$\leqslant |f(x_{i-1}, y(x_{i-1})) + f(x_i, y(x_i))| + |f(x_i, y(x_i)) + f(x_{i+1}, y(x_{i+1}))| \leqslant \frac{C}{N^2}.$$

□

Lemma 5.2. [8] *Assume that* $\varepsilon \leqslant \frac{C}{N}$. *On the portion of the modified mesh* (2.3) *from Section 2 where* x_i, $x_{i\pm1} \in [x_{N/4-1}, \lambda] \cup [\lambda, 1/2]$, *the following estimate holds:*

$$\left| \frac{\frac{\cosh(\beta h_{i-1})-1}{\sinh(\beta h_{i-1})}(y(x_{i-1}) - y(x_i)) - \frac{\cosh(\beta h_i)-1}{\sinh(\beta h_i)}(y(x_i) - y(x_{i+1}))}{\frac{\cosh(\beta h_{i-1})-1}{\sinh(\beta h_{i-1})} + \frac{\cosh(\beta h_i)-1}{\sinh(\beta h_i)}} \right| \leqslant \frac{C}{N^2},$$

$i = N/4, \ldots, N/2-1$.

Lemma 5.3. [8] *Assume that* $\varepsilon \leqslant \frac{C}{N}$. *On the portion of the modified mesh* (2.3) *from Section 2 where* x_i, $x_{i\pm1} \in [x_{N/4-1}, \lambda] \cup [\lambda, 1/2]$, *the following estimate holds:*

$$\left| \frac{\frac{y(x_{i-1})-y(x_i)}{\sinh(\beta h_{i-1})} - \frac{y(x_i)-y(x_{i+1})}{\sinh(\beta h_i)}}{\frac{\cosh(\beta h_{i-1})-1}{\sinh(\beta h_{i-1})} + \frac{\cosh(\beta h_i)-1}{\sinh(\beta h_i)}} \right| \leqslant \frac{C}{N^2},\ i = N/4, \ldots, N/2-1.$$

We are now in a position to state and prove the main convergence theorem.

Theorem 5.4. *The discrete problem* (4.1)–(4.3) *on the modified Shishkin mesh* (2.3) *from Section 2 is uniformly convergent with respect to* ε, *and the following error estimate holds:*

$$\max_i |y(x_i) - \overline{y}_i| \leqslant C \begin{cases} \left(\ln^2 N\right)/N^2, & i = 0, \ldots, N/4-1, \\ 1/N^2, & i = N/4, \ldots, 3N/4, \\ \left(\ln^2 N\right)/N^2, & i = 3N/4+1, \ldots, N, \end{cases}$$

where $y(x_i)$ and $\overline{y}_i$ denote the values of the exact and numerical solutions to problem (1.1)–(1.3) at the mesh point x_i, respectively, and $C > 0$ is a constant independent of N and ε.

Proof. We divide the proof of the theorem into several cases: the first part corresponds to the fine region of the mesh, while the second deals with the coarse region.

Case $1 \leqslant i \leqslant N/4 - 1$**:** In this region, we have $h_{i-1} = h_i = \mathcal{O}(\varepsilon \ln N/N)$ and $\beta h_i = \sqrt{\gamma} \ln N/N$.

By utilizing the relation $\varepsilon^2 y''(x_k) = f(x_k, y(x_k))$ along with the Taylor expansions for $\cosh(\beta h_i)$, $y(x_{i-1})$, and $y(x_{i+1})$, a straightforward calculation yields:

$$Ty(x_i) = \frac{\gamma}{\frac{\beta^2 h_i^2}{2} + \mathcal{O}(\beta^4 h_i^4)} \left[\left(2 + \frac{\beta^2 h_i^2}{2} + \mathcal{O}(\beta^4 h_i^4)\right) \left(y''(x_i) h_i^2 + \frac{y^{(iv)}(\xi_i^-) + y^{(iv)}(\xi_i^+)}{24} h_i^4\right) - \frac{\varepsilon^2}{\gamma} \left(y''(x_{i-1}) + 2y''(x_i) + y''(x_{i+1})\right) \left(\frac{\beta^2 h_i^2}{2} + \mathcal{O}(\beta^4 h_i^4)\right) \right], \tag{5.1}$$

where $\xi_i^- \in (x_{i-1}, x_i)$ and $\xi_i^+ \in (x_i, x_{i+1})$. Applying the Taylor expansion to $y''(x_{i-1})$ and $y''(x_{i+1})$, we obtain:

$$Ty(x_i) = \frac{\gamma}{\frac{\beta^2 h_i^2}{2} + \mathcal{O}(\beta^4 h_i^4)} \left[\left(\frac{\beta^2 h_i^2}{2} + \mathcal{O}(\beta^4 h_i^4)\right) \left(y''(x_i) h_i^2 + \frac{y^{(iv)}(\xi_i^-) + y^{(iv)}(\xi_i^+)}{24} h_i^4\right) + \frac{y^{(iv)}(\xi_i^-) + y^{(iv)}(\xi_i^+)}{12} h_i^4 - \varepsilon^2 \frac{\left(y^{(iv)}(\mu_i^-) + y^{(iv)}(\mu_i^+)\right) h_i^2}{2\gamma} \left(\frac{\beta^2 h_i^2}{2} + \mathcal{O}(\beta^4 h_i^4)\right) - \frac{4\varepsilon^2 y''(x_i) \mathcal{O}(\beta^4 h_i^4)}{\gamma} \right], \tag{5.2}$$

where μ_i^-, $\xi_i^- \in (x_{i-1}, x_i)$, μ_i^+, $\xi_i^+ \in (x_i, x_{i+1})$, which finally gives:

$$|Ty(x_i)| \leqslant \frac{C \ln^2 N}{N^2}, \; i = 1, \ldots, N/4 - 1. \tag{5.3}$$

Case $N/4 \leqslant i \leqslant N/2 - 1$**:** The discrete operator (4.1) can be rewritten as follows:

$$Ty(x_i) = \frac{\gamma}{\frac{\cosh(\beta h_{i-1}) - 1}{\sinh(\beta h_{i-1})} + \frac{\cosh(\beta h_i) - 1}{\sinh(\beta h_i)}} \left[\frac{\cosh(\beta h_{i-1}) - 1}{\sinh(\beta h_{i-1})} (y(x_{i-1}) - y(x_i)) - \frac{\cosh(\beta h_i) - 1}{\sinh(\beta h_i)} (y(x_i) - y(x_{i+1})) + \frac{2(y(x_{i-1}) - y(x_i))}{\sinh(\beta h_{i-1})} - \frac{2(y(x_i) - y(x_{i+1}))}{\sinh(\beta h_i)} - \frac{f(x_{i-1}, y(x_{i-1})) + f(x_i, y(x_i))}{\gamma} \cdot \frac{\cosh(\beta h_{i-1}) - 1}{\sinh(\beta h_{i-1})} - \frac{f(x_i, y(x_i)) + f(x_{i+1}, y(x_{i+1}))}{\gamma} \cdot \frac{\cosh(\beta h_i) - 1}{\sinh(\beta h_i)} \right].$$

Consequently, by applying Lemma 5.1, Lemma 5.2, and Lemma 5.3, we obtain the estimate:

$$|Ty(x_i)| \leqslant \frac{C}{N^2}, \; i = N/4, \ldots, N/2 - 1. \tag{5.4}$$

Case $i = N/2$**:** This case follows directly since $h_{N/2-1} = h_{N/2}$ and the influence of the layer component s is negligible, yielding:

$$\left|Ty(x_{N/2})\right| \leqslant \frac{C}{N^2}. \tag{5.5}$$

The case $N/2 < i \leqslant N - 1$ follows by symmetry. Finally, combining the stability inequality (4.4) with the estimates (5.3), (5.4), and (5.5), the proof is complete. □

6. Numerical experiments

In this section, we present numerical experiments to test the proposed method and validate the theoretical findings. The numerical results are obtained by solving the discrete problem (3.4) on three distinct meshes: modified Shishkin (2.3), modified Bakhvalov (2.6), and Liseikin (2.7) meshes. Although the theoretical analysis guarantees second-order convergence on the modified Shishkin mesh, numerical experiments were also performed on the two additional grids to provide a comprehensive comparison.

The test problem under consideration is nonlinear and lacks a known exact solution. Computations are carried out for various values of the perturbation parameter ε and the number of mesh intervals N. Specifically, ε ranges from 2^{-3} to 2^{-40}, and N varies from 2^4 up to 2^{11}.

To demonstrate the robustness and efficiency of the proposed method, we compute the maximum nodal error E_N and the numerical rate of convergence Ord. For the modified Bakhvalov and Liseikin meshes, these quantities are computed as follows.

Because the exact solution is unknown, we use the double-mesh principle to estimate the error:

$$E_N = \max_{0 \leqslant i \leqslant N} |y_{2N,2i} - y_{N,i}|,$$

where $y_{N,i}$ is the numerical solution obtained on a mesh with N intervals at the mesh point x_i, and $y_{2N,2i}$ is the numerical solution evaluated at the corresponding points on a nested mesh with $2N$ intervals. The rate of convergence is then calculated by:

$$\mathrm{Ord} = \frac{\ln(E_N) - \ln(E_{2N})}{\ln 2}.$$

When using the modified Shishkin mesh, both the error E_N and the numerical rate of convergence Ord are computed via the double-mesh method (see [3]):

$$E_N = \max_{0 \leqslant i \leqslant N} |\tilde{y}_{2N,2i} - y_{N,i}|,$$

where $y_{N,i}$ denotes the numerical solution obtained on a mesh with N subintervals at the grid point x_i. The term $\tilde{y}_{2N,2i}$ represents the numerical solution on a mesh with $2N$ subintervals evaluated at the same point, where the transition parameter has been slightly modified to

$$\tilde{\alpha}_{Sh} = \min\left\{1/4,\, 4\gamma_0^{-1}\varepsilon \ln(N/2)\right\},$$

i.e., it is computed with N instead of $2N$. This specific adjustment ensures that the mesh points of the coarse grid perfectly coincide with those of the fine grid. In scenarios involving the Shishkin-type meshes (regardless of whether the exact solution is known), the numerical rate of convergence is determined by:

$$\mathrm{Ord} = \frac{\ln E_N - \ln E_{2N}}{\ln \frac{2k}{k+1}}, \quad k = 4, ..., 11.$$

Example 6.1. Let us consider the following boundary value problem:

$$-\varepsilon^2 y'' + y^3 + y - 2 = 0, \quad x \in (0,1),$$
$$y(0) = 0,\ y(1) = 0.$$

Here $f(x,y) = y^3 + y - 2$, so

$$\frac{\partial f}{\partial y} = 3y^2 + 1, \quad \forall x \in [0,1],$$

and condition (1.3) holds with $\gamma = 4$. The exact solution of this problem is not known.

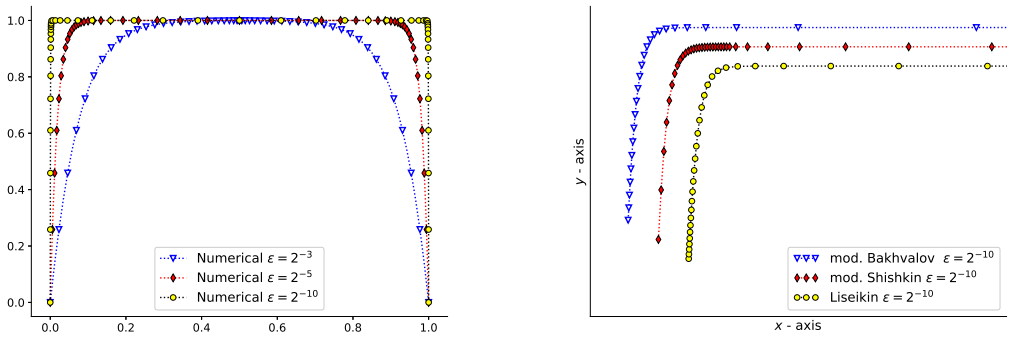


Fig. 1. Graphs of the numerical solutions for various values of the parameter ε (left), and enlarged views near the endpoint $x = 0$ (right).

The left side of Figure 1 shows the graphs of the numerical solutions calculated for the parameter values $\varepsilon = 2^{-3}$, 2^{-5}, 2^{-10} on the modified Shishkin mesh, while the right side of Figure 1 displays the graphs of the numerical solutions near the endpoint $x = 0$. These plots illustrate the behavior of the numerical solutions in the boundary layer near $x = 0$. They were calculated for the same parameter value $\varepsilon = 2^{-10}$ and the same number of mesh points, but on different meshes.

Table 1 presents the values of E_N and Ord. These values were calculated for all three meshes, for parameter values ε ranging from 2^{-3} to 2^{-40}, while the number of mesh points N varied from 16 to 2048.

ε	2^{-3}		2^{-5}		2^{-10}		2^{-40}	
N	E_N	Ord	E_N	Ord	E_N	Ord	E_N	Ord
modified Shishkin mesh (2.3)								
2^4	8.822e-3	1.87	3.486e-2	1.91	3.486e-2	1.91	3.486e-2	1.91
2^5	3.648e-3	1.96	1.414e-2	1.91	1.414e-2	1.91	1.414e-2	1.91
2^6	1.337e-3	1.99	5.311e-3	1.99	5.311e-3	1.99	5.311e-3	1.99
2^7	4.553e-4	1.99	1.810e-3	1.98	1.810e-3	1.98	1.810e-3	1.98
2^8	1.490e-4	1.99	5.954e-4	1.99	5.954e-4	1.99	5.955e-4	1.99
2^9	4.720e-5	1.99	1.887e-4	1.99	1.887e-4	1.99	1.887e-4	1.99
2^{10}	1.457e-5	1.99	5.827e-5	1.99	5.827e-5	1.99	5.828e-5	1.99
2^{11}	4.408e-6	-	1.763e-5	-	1.763e-5	-	1.763e-5	-
modified Bakhvalov mesh (2.6)								
2^4	2.253e-2	1.93	2.917e-2	1.73	2.917e-2	1.73	2.917e-2	1.73
2^5	5.891e-3	1.97	8.791e-3	1.96	8.791e-3	1.96	8.792e-3	1.96
2^6	1.495e-3	1.98	2.257e-3	1.99	2.257e-3	1.99	2.257e-3	1.99
2^7	3.777e-4	2.00	5.655e-4	1.99	5.655e-4	1.99	5.655e-4	1.99
2^8	9.423e-5	2.01	1.416e-4	2.01	1.415e-4	2.01	1.415e-4	2.01
2^9	2.327e-5	2.07	3.503e-5	2.07	3.498e-5	2.07	3.498e-5	2.07
2^{10}	5.541e-6	2.32	8.329e-6	2.32	8.300e-6	2.32	8.330e-6	2.32
2^{11}	1.108e-6	-	1.665e-6	-	1.666e-6	-	1.666e-6	-
Liseikin mesh (2.7)								
2^4	5.852e-3	1.92	1.127e-2	1.81	2.422e-2	1.85	3.089e-2	1.56
2^5	1.544e-3	1.98	3.199e-3	1.98	6.696e-3	1.96	1.045e-2	1.89
2^6	3.895e-4	1.99	8.071e-4	1.99	1.710e-3	1.96	2.804e-3	1.98
2^7	9.757e-5	2.00	2.020e-4	2.00	4.394e-4	2.00	7.084e-4	2.00
2^8	2.438e-5	2.02	5.050e-5	2.01	1.096e-4	2.01	1.077e-4	2.01
2^9	6.011e-6	2.07	1.252e-5	2.07	2.710e-5	2.07	4.388e-5	2.07
2^{10}	1.431e-6	2.32	2.968e-6	2.32	6.455e-6	2.32	1.044e-5	2.32
2^{11}	2.862e-7	-	5.936e-7	-	1.291e-6	-	2.089e-6	-

Table 1. The values of E_N and Ord for the difference scheme (3.4) on the different meshes.

6.1. **Analysis of the numerical results.** As mentioned above, numerical experiments were conducted on one nonlinear test problem.

The expected values of E_N obtained by the computations are $\mathcal{O}(\ln^2 N/N^2)$, with a convergence rate Ord close to 2. The results presented in Table 1, obtained using the modified Shishkin mesh, agree with the theoretical findings. Furthermore, the results obtained using the modified Bakhvalov and Liseikin meshes fall within the expected range for the application of these meshes to the proposed difference scheme.

The most critical and computationally challenging part of a numerical solution for problems featuring layers is the region within the layer itself. Therefore, it is crucial that the grid points are appropriately distributed inside the layer. As shown in Figure 1, the points representing the numerical solutions are fairly evenly distributed across all graphs, even within the layer regions. The Liseikin mesh, which was not primarily designed for problems with exponential layers, also proved to be excellent for numerically solving the problem under consideration. It is evident from the graphs that this mesh provides the best distribution of numerical solution points.

7. Conclusion

In this paper, we present a discretization of a one-dimensional semilinear reaction-diffusion boundary value problem. A difference scheme is constructed, and ε-uniform convergence of order 2 on a modified Shishkin mesh is established. Similar results were obtained in a previous paper by the first author and his coauthors [8]. However, in that work, the approximation of the function ψ, which appears in the integrals, was chosen in a cumbersome manner, making the analysis unnecessarily difficult. In the present paper, we have chosen a simpler approximation, leading to a much more straightforward analysis. One limitation from the aforementioned paper remains unresolved: our analysis still relies on the modified Shishkin mesh introduced by Vulanović. Replacing the modified Shishkin mesh with the standard Shishkin mesh remains a task for future research. In the numerical experiments, in addition to the modified Shishkin mesh, we utilized the modified Bakhvalov and Liseikin meshes. All considered meshes yielded the expected results.

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