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ELEMENT ϕ-PRIMARY TO ANOTHER IN MULTIPLICATIVE LATTICES

ASHOK V. BINGI*

Abstract. In this paper, we study the properties of "an element ϕ-primary to another element" in a multiplicative lattice L, extending the concept of "an element primary to another element" through a reduction function ϕ on L, and exploring the interplay among these concepts.

Keywords. reduction function ϕ on L, meet preserving property, element n-potent primary to another element, element almost primary to another element
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Dedicated to the memory of my guide, Late Professor C. S. Manjarekar

1. Introduction and Preliminaries

The main intention of introducing multiplicative lattices by Dilworth R.P. in [5] is to provide a concrete abstract formulation of the ideal theory of commutative rings with identity. Since that pivotal work, multiplicative lattices have been widely researched. One of the main branches of ring theory is the concept of prime ideals. This paper is motivated by [2], in which generalizations of prime ideals are introduced, a revolutionary concept in commutative rings with identity.

The notion of an element primary to another element in a multiplicative lattice L is introduced by Manjarekar C. S. et al. in [6]. Further, the concept of an element weakly primary to another element in a multiplicative lattice L is introduced by Manjarekar C. S. et al. in [7]. To generalise these concepts, in this paper we introduce and study the notion of "an element ϕ-primary to another element" in a multiplicative lattice L, where ϕ is a reduction function on L. The main aim of this paper is to obtain characterizations and many properties of the generalized notions of "an element primary to another element" in L, and then investigate the relations among these generalized notions; afterwards, we link the concepts introduced in this paper with the corresponding concepts introduced in [3].

Let us revisit some key terms within the context of a multiplicative lattice. A multiplicative lattice is a complete poset $(L, \leqslant)$ bounded by 0 and 1 featuring

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* Corresponding author
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a commutative, associative, and fully join distributive binary multiplication $(\cdot)$ where the greatest element 1 serves as the multiplicative identity and, for brevity, uv denotes the product of u and v in L. The set of positive integers is denoted by $\mathbb{Z}_+$. Given any subset $B \subseteq L$, the notations $\vee B$ and $\wedge B$ represent its supremum (least upper bound) and infimum (greatest lower bound), respectively. To streamline notation, the join of a two-element set $\{u, v\}$ is written as $u \vee v$, while their meet is written as $u \wedge v$. The expression u^n signifies the product of an element $u \in L$ iteratively multiplied by itself n times, where $n \in \mathbb{Z}_+$. For $u, v \in L$, $(u : v) = \vee\{c \in L \mid cv \leqslant u\}$, where $(u : v)$ is read as the residuation of u by v. An element $c \in L$ is called compact if for $S \subseteq L$, $c \leqslant \vee S$, there exists a finite subset $F \subseteq S$ such that $c \leqslant \vee F$. The set of all compact elements of L is denoted by L_*. An element $p \in L$ is said to be proper if $p < 1$. The radical of $a \in L$ is denoted by $\sqrt{a}$ and is defined as $\vee\{x \in L_* \mid x^n \leqslant a, \text{ for some } n \in \mathbb{Z}_+\}$.

Henceforth, L denotes a compactly generated multiplicative lattice with a compact top element 1, such that the set of compact elements is closed under finite products. For further foundational definitions and background, the reader can cite [1], [8] and [5].

This paper is organized as follows: In Section 2, we obtain several properties of "an element (weakly) prime to another element" and "an element (weakly) primary to another element" in L. The concept of "element q-primary to another element" in L is introduced and analyzed.

In Section 3, we define an element ϕ-primary to another element in a multiplicative lattice L to obtain its characterization (see Theorem 3.12), where ϕ is a reduction function on L. We then, in L, establish the concept of ϕ_β-primary to another element and investigate the relations among these concepts (see Theorem 3.14). We link these notions with respective notions introduced in [3], which is generalization of an element prime to another element in L. By a counter example, it is shown that if $b \in L$ is ϕ-primary to a proper element $p \in L$, then b is not necessarily primary to p (see Example 3.20). We establish various sufficient conditions under which b is primary to p if b is ϕ-primary to p (see Theorems 3.21, 3.23, 3.24, 3.26). One approach involves introducing the concept of an element n-potent primary to another element of L where $n \geqslant 2$ and $n \in \mathbb{Z}_+$. The paper is rich in counter examples (see Examples 3.20, 3.6, 3.16, 3.17). We conclude this paper by summarizing the notions introduced with an implications diagram.

2. Some Results on an Element Primary To Another Element

Initially, we will familiarize ourselves with the terminology of an element "(weakly) prime to" and "(weakly) primary to" another element in a multiplicative lattice L.

Definition 2.1. An element $b \in L$ is said to be prime to a proper element $p \in L$ if for all $x \in L$, $xb \leqslant p$ imply $x \leqslant p$ (see [1]).

Definition 2.2. An element $b \in L$ is said to be weakly prime to a proper element $p \in L$ if for all $x \in L$, $0 \neq xb \leqslant p$ imply $x \leqslant p$ (see [7]).

Definition 2.3. An element $b \in L$ is said to be primary to a proper element $p \in L$ if for all $x \in L$, $xb \leqslant p$ imply $x \leqslant \sqrt{p}$ (see [6]).

Definition 2.4. An element $b \in L$ is said to be weakly primary to a proper element $p \in L$ if for all $x \in L$, $0 \neq xb \leqslant p$ imply $x \leqslant \sqrt{p}$ (see [7]).

Clearly, if $b \in L$ is (weakly) prime to a proper element $p \in L$, then b is (weakly) primary to p, since $a \leqslant \sqrt{a}$. Its converse need not be true [refer to examples in ([7])[6]].

Now we will investigate some elementary results related to these notions.

Theorem 2.5. *Let p be a proper element of L and $b \in L$. Then the following statements hold.*

(1) *If b is prime to p, then b^s is prime to p $\forall s \in \mathbb{Z}_+$. Also, if b is prime to $\sqrt{p}$, then b^s is prime to $\sqrt{p}$ $\forall s \in \mathbb{Z}_+$. Moreover, if b is prime to p, then b^s is prime to $\sqrt{p}$ $\forall s \in \mathbb{Z}_+$.*
(2) *If b is primary to p, then b^s is primary to p $\forall s \in \mathbb{Z}_+$. Also, if b is primary to $\sqrt{p}$, then b^s is primary to $\sqrt{p}$ $\forall s \in \mathbb{Z}_+$. Moreover, if b is primary to p, then b^s is primary to $\sqrt{p}$ $\forall s \in \mathbb{Z}_+$.*
(3) *If b is prime to p, then b^s is primary to p $\forall s \in \mathbb{Z}_+$. Also, if b is prime to $\sqrt{p}$, then b^s is primary to $\sqrt{p}$ $\forall s \in \mathbb{Z}_+$. Moreover, if b is prime to p, then b^s is primary to $\sqrt{p}$ $\forall s \in \mathbb{Z}_+$.*
(4) *If b is primary to $\sqrt{p}$, then b is prime to $\sqrt{p}$.*
(5) *If b is primary to p, then b is prime to $\sqrt{p}$ and hence b is primary to $\sqrt{p}$.*
(6) *If b is prime to p, then b is prime to $\sqrt{p}$ and hence b is primary to $\sqrt{p}$.*
(7) *If b^s is primary to p $\forall s \in \mathbb{Z}_+$, then b is prime to $\sqrt{p}$ and hence b^s is prime to $\sqrt{p}$ $\forall$ $s \in \mathbb{Z}_+$. Moreover, b^s is primary to $\sqrt{p}$ $\forall$ $s \in \mathbb{Z}_+$.*
(8) *If b^s is prime to p $\forall s \in \mathbb{Z}_+$, then b is prime to $\sqrt{p}$ and hence b^s is prime to $\sqrt{p}$ $\forall$ $s \in \mathbb{Z}_+$. Moreover, b^s is primary to $\sqrt{p}$ $\forall$ $s \in \mathbb{Z}_+$.*

Proof. We prove only (5). The proofs of (1)-(4) and (6)-(8) can be worked out on similar lines by the reader.

(5) Let $x \in L$ be such that $xb \leqslant \sqrt{p}$. Then there exists a least positive integer k such that $x^k b^k \leqslant p$. If $k = 1$, then $x \leqslant \sqrt{p}$ and we are done. If $k > 1$, then $(x^k b^{k-1})b \leqslant p$ and b is primary to p gives $(x^k b^{k-1}) \leqslant \sqrt{p}$. Continuing this argument, after a finite number of times, we arrive at $x^k b \leqslant \sqrt{p}$. Then again using b is primary to p, we get $x^k \leqslant \sqrt{p}$ and hence $x \leqslant \sqrt{\sqrt{p}} = \sqrt{p}$. Thus, b is prime to $\sqrt{p}$ and we are done.

□

According to [1], a proper element $p \in L$ is called a q-primary element if p is primary and $q = \sqrt{p}$ is prime. Similarly, a proper element $p \in L$ is called a q-prime element if p is prime and $q = \sqrt{p}$ is prime. Now, we define such terminologies in the study of "an element prime to another element" and "an element primary to another element" in a multiplicative lattice L.

Definition 2.6. Let p be a proper element of L. If an element $b \in L$ is

(1) primary to p, then by Theorem 2.5-*(5)*, b is prime to $q = \sqrt{p}$. In this case, we say that b is q-primary to p.
(2) prime to p, then by Theorem 2.5-*(6)*, b is prime to $q = \sqrt{p}$. In this case, we say that b is q-prime to p.

Theorem 2.7. *Let $b \in L$. If b is q-primary to proper elements $p_1, \cdots, p_n$ in L where $q = \sqrt{p_i}\ \forall\ i = 1, \cdots, n$, then b is q-primary to $\bigwedge_{i=1}^{n} p_i$.*

Proof. The proof follows directly from the property: $\sqrt{\bigwedge_{i=1}^{n} p_i} = \bigwedge_{i=1}^{n} \sqrt{p_i} = q$. □

Theorem 2.8. *Let $b \in L$.*

(1) *If b^s is q-primary to proper elements $p_1, \cdots, p_n$ in $L\ \forall\ s \in \mathbb{Z}_+$ where $q = \sqrt{p_i}\ \forall\ i = 1, \cdots, n$, then b^s is q-primary to $\bigwedge_{i=1}^{n} p_i\ \forall\ s \in \mathbb{Z}_+$.*
(2) *If b is q-prime to proper elements $p_1, \cdots, p_n$ in L where $q = \sqrt{p_i}\ \forall\ i = 1, \cdots, n$, then b is q-prime to $\bigwedge_{i=1}^{n} p_i$.*
(3) *If b^s is q-prime to proper elements $p_1, \cdots, p_n$ in $L\ \forall\ s \in \mathbb{Z}_+$ where $q = \sqrt{p_i}\ \forall\ i = 1, \cdots, n$, then b^s is q-prime to $\bigwedge_{i=1}^{n} p_i\ \forall\ s \in \mathbb{Z}_+$.*

Proof. The proof is similar to that of Theorem 2.7 and is therefore omitted. □

Now we state the lemmas required to prove that if an element in L is (weakly) primary to a family of elements in L, then that element is (weakly) primary to the meet and join of elements in that family.

Lemma 2.9. *Let $\{p_i \mid i \in \mathbb{Z}_+\}$ be an ascending chain of elements in L. Then*

(1) $\underset{i\in\mathbb{Z}_+}{\wedge} \sqrt{p_i} = \sqrt{\underset{i\in\mathbb{Z}_+}{\wedge} p_i}$. (2) $\sqrt{\underset{i\in\mathbb{Z}_+}{\vee} \sqrt{p_i}} \leqslant \sqrt{\underset{i\in\mathbb{Z}_+}{\vee} p_i}$.

Proof. The proof is obvious. □

Theorem 2.10. *If $\{p_i \mid i \in \mathbb{Z}_+\}$ is an ascending chain of proper elements in L such that $b \in L$ is (weakly) primary to each p_i, then b is (weakly) primary to*

(1) $\underset{i\in\mathbb{Z}_+}{\wedge} p_i$. (2) $\underset{i\in\mathbb{Z}_+}{\vee} p_i$.

Proof. The proof follows from Lemma 2.9. □

3. An Element ϕ-Primary To Another Element

. We begin by recalling the notion of a reduction function $\phi_{\mathrm{L}} : L \longrightarrow L \cup \{\emptyset\}$, where $\emptyset$ is the empty set.

Definition 3.1. A reduction function on L is a function $\phi_{\mathrm{L}} : L \longrightarrow L \cup \{\emptyset\}$ which satisfies the following two conditions:
①. $\phi_{\mathrm{L}}(a) \leqslant a\ \forall\ a \in L$.
②. $a \leqslant b$ imply $\phi_{\mathrm{L}}(a) \leqslant \phi_{\mathrm{L}}(b)\ \forall\ a,\ b \in L$ (see [3]).

From now on, for convenience, we denote ϕ_{L} by ϕ, and throughout this paper, ϕ always denotes a reduction function on L. Also, given any two reduction functions $\gamma, \psi : L \longrightarrow L \cup \{\emptyset\}$, we write, $\gamma \leqslant \psi$ if and only if $\gamma(a) \leqslant \psi(a)\ \forall\ a \in L$.

According to [3], we have the following definitions:

(1) An element $b \in L$ is said to be almost prime to a proper element $p \in L$ if for all $x \in L$, $xb \leqslant p$ and $xb \nleqslant p^2$ imply $x \leqslant p$.
(2) An element $b \in L$ is said to be n-almost prime to a proper element $p \in L$ if for all $x \in L$, $xb \leqslant p$ and $xb \nleqslant p^n$ imply $x \leqslant p$ where $n \geqslant 2$ and $n \in \mathbb{Z}_+$.
Note that the notions of "almost prime" and "2-almost prime" coincide.
(3) An element $b \in L$ is said to be ω-prime to a proper element $p \in L$ if for all $x \in L$, $xb \leqslant p$ and $xb \nleqslant \bigwedge_{i=1}^{\infty} p^i$ imply $x \leqslant p$.
(4) Given a reduction function $\phi : L \longrightarrow L \cup \{\emptyset\}$, an element $b \in L$ is said to be ϕ-prime to a proper element $p \in L$ if for all $x \in L$, $xb \leqslant p$ and $xb \nleqslant \phi(p)$ imply $x \leqslant p$.

Now, on same lines we define the notions of an element almost primary/n-almost primary ($n \geqslant 2$)/ω-primary to another element in a multiplicative lattice L.

Definition 3.2. Let p be a proper element of L. An element $b \in L$ is called almost primary to p if, for every $x \in L$, the relation $x \leqslant \sqrt{p}$ holds whenever $xb \leqslant p$ and $xb \nleqslant p^2$.

Definition 3.3. Fix an integer $n \geqslant 2$ where $n \in \mathbb{Z}_+$. Let p be a proper element of L. An element $b \in L$ is called n-almost primary to p if, for every $x \in L$, the relation $x \leqslant \sqrt{p}$ holds whenever $xb \leqslant p$ and $xb \nleqslant p^n$.

Definition 3.4. Let p be a proper element of L. An element $b \in L$ is called ω-primary to p if, for every $x \in L$, the relation $x \leqslant \sqrt{p}$ holds whenever $xb \leqslant p$ and $xb \nleqslant \bigwedge_{i=1}^{\infty} p^i$.

To unify these various concepts within a single framework for a multiplicative lattice L, we introduce the notion of an element being ϕ-primary to another element. This definition generalizes the classical concept of an element being primary to another element in multiplicative lattices.

Definition 3.5. Let p be a proper element of L. Given a reduction function $\phi : L \longrightarrow L \cup \{\emptyset\}$, an element $b \in L$ is called ϕ-primary to p if, for every $x \in L$, the relation $x \leqslant \sqrt{p}$ holds whenever $xb \leqslant p$ and $xb \nleqslant \phi(p)$.

By specifying the reduction function $\phi_\beta : L \longrightarrow L \cup \{\emptyset\}$ where $\beta \in \{\emptyset\} \cup \{0, 1, 2, \cdots\} \cup \{\omega\}$, the general definition of "an element ϕ-primary to another element" seamlessly specializes to each of the previously defined notions 2.3, 2.4, 3.2, 3.3, 3.4:

- $\phi_\emptyset(p) = \emptyset$. Then b is primary (or $\phi_\emptyset$-primary) to p.
- $\phi_0(p) = 0$. Then b is weakly primary (or ϕ_0-primary) to p.
- $\phi_2(p) = p^2$. Then b is almost primary (or ϕ_2-primary) to p.
- $\phi_n(p) = p^n$ ($n \geqslant 2$). Then b is n-almost primary (or ϕ_n-primary) to p.
- $\phi_\omega(p) = \bigwedge_{i=1}^{\infty} p^i$. Then b is ω-primary (or ϕ_ω-primary) to p.

For completeness, we define $\phi_1(p) = p$ for any element $p \in L$.

Note that the notions of "almost primary" and "2-almost primary" coincide.

It is straightforward to verify that all these functions $\phi_\emptyset$, ϕ_0, ϕ_1, ϕ_2, ϕ_n ($n \geqslant 2$), ϕ_ω are examples of the reduction function on L.

These functions satisfy the following natural ordering:

$$\phi_\emptyset \leqslant \phi_0 \leqslant \phi_\omega \leqslant \cdots \leqslant \phi_{n+1} \leqslant \phi_n \leqslant \cdots \leqslant \phi_2 \leqslant \phi_1$$

Note that, as $a \leqslant \sqrt{a}\ \forall a \in L$, for a proper element $p \in L$ and $b \in L$, we have, if b is ϕ-prime to p, then b is ϕ-primary to p. However, the converse does not hold in general, as demonstrated by the following counter example taking ϕ as ϕ_2.

Example 3.6. Consider the ring $R =< \mathbb{Z}_{24}\ ,\ +_{24}\ ,\ \times_{24} >$ with lattice of ideals $L = \{(0),(2),(3),(4),(6),(8),(12),(1)\}$. L forms a CG multiplicative lattice. Observe that the element $(2) \in L$ is ϕ_2-primary to $(4) \in L$, but (2) is not ϕ_2-prime to (4). This is because $(6)(2) \leqslant (4)$ and $(6)(2) \nleqslant (4)^2$, but $(6) \nleqslant (4)$.

The following result is generalization of Theorem 2.5.

Theorem 3.7. *Let p be a proper element of L and $b \in L$. Then the following statements hold.*

(1) *If b is ϕ-prime to p, then b^s is ϕ-prime to $p\ \forall s \in \mathbb{Z}_+$, and hence b^s is ϕ-primary to $p\ \forall s \in \mathbb{Z}_+$.*
(2) *If b is ϕ-prime to $\sqrt{p}$, then b^s is ϕ-prime to $\sqrt{p}\ \forall s \in \mathbb{Z}_+$, and hence b^s is ϕ-primary to $\sqrt{p}\ \forall s \in \mathbb{Z}_+$.*
(3) *If b is ϕ-primary to $\sqrt{p}$, then b^s is ϕ-primary to $\sqrt{p}\ \forall s \in \mathbb{Z}_+$.*
(4) *If b is ϕ-primary to $\sqrt{p}$, then b is ϕ-prime to $\sqrt{p}$.*
(5) *If $\sqrt{\phi(p)} = \phi(\sqrt{p})$ and*
   (i) *b is ϕ-prime to p, then b is ϕ-prime to $\sqrt{p}$ and hence b^s is ϕ-prime to $\sqrt{p}\ \forall s \in \mathbb{Z}_+$.*
   (ii) *b is ϕ-prime to p, then b is ϕ-primary to $\sqrt{p}$ and hence b^s is ϕ-primary to $\sqrt{p}\ \forall s \in \mathbb{Z}_+$.*
   (iii) *b^s is ϕ-primary to $p\ \forall s \in \mathbb{Z}_+$, then b is ϕ-prime to $\sqrt{p}$. and hence b^s is ϕ-prime to $\sqrt{p}\ \forall s \in \mathbb{Z}_+$. Moreover, b^s is ϕ-primary to $\sqrt{p}\ \forall s \in \mathbb{Z}_+$.*
   (iv) *b^s is ϕ-prime to $p\ \forall s \in \mathbb{Z}_+$, then b is ϕ-prime to $\sqrt{p}$. and hence b^s is ϕ-prime to $\sqrt{p}\ \forall s \in \mathbb{Z}_+$. Moreover, b^s is ϕ-primary to $\sqrt{p}\ \forall s \in \mathbb{Z}_+$.*

Proof. Note that, by property (p3) of radicals in [9], $\sqrt{\sqrt{p}} = \sqrt{p}\ \forall\ p \in L$.

(1) Let $s \in \mathbb{Z}_+$. Let $x \in L$ be such that $xb^s \leqslant p$ and $xb^s \nleqslant \phi(p)$. If $s = 1$, then we are done. If $s > 1$, then as $(xb^{s-1})b \leqslant p$, $(xb^{s-1})b \nleqslant \phi(p)$, and b is ϕ-prime to p, we have $(xb^{s-1}) \leqslant p$. Also, then $(xb^{s-1}) \nleqslant \phi(p)$ because if $(xb^{s-1}) \leqslant \phi(p)$, then $(xb^{s-1})b \leqslant (xb^{s-1}) \leqslant \phi(p)$ which contradicts $xb^s \nleqslant \phi(p)$. Now $(xb^{s-2})b \leqslant p$, $(xb^{s-2})b \nleqslant \phi(p)$, and b is ϕ-prime to p gives $(xb^{s-2}) \leqslant p$. Continuing this argument, after a finite number of times, we arrive at $xb \leqslant p$ and $xb \nleqslant \phi(p)$. Then since b is ϕ-prime to p, we get $x \leqslant p$ and thus b^s is ϕ-prime to p. Hence, b^s is ϕ-primary to p because if an element is ϕ-prime to q, then that element is ϕ-primary to q.
(2) In the proof (1), replace p by $\sqrt{p}$.
(3) Let $s \in \mathbb{Z}_+$. Let $x \in L$ be such that $xb^s \leqslant \sqrt{p}$ and $xb^s \nleqslant \phi(\sqrt{p})$. If $s = 1$, then we are done. If $s > 1$, then as $(xb^{s-1})b \leqslant \sqrt{p}$, $(xb^{s-1})b \nleqslant \phi(\sqrt{p})$ and b is ϕ-primary to $\sqrt{p}$, we have $(xb^{s-1}) \leqslant \sqrt{\sqrt{p}} = \sqrt{p}$. Also, then $(xb^{s-1}) \nleqslant \phi(\sqrt{p})$ because if $(xb^{s-1}) \leqslant \phi(\sqrt{p})$, then $(xb^{s-1})b \leqslant (xb^{s-1}) \leqslant \phi(\sqrt{p})$

which contradicts $xb^s \nleqslant \phi(\sqrt{p})$. Now $(xb^{s-2})b \leqslant \sqrt{p}$, $(xb^{s-2})b \nleqslant \phi(\sqrt{p})$ and b is ϕ-primary to $\sqrt{p}$ gives $(xb^{s-2}) \leqslant \sqrt{\sqrt{p}} = \sqrt{p}$. Continuing this argument, after a finite number of times, we arrive at $xb \leqslant \sqrt{\sqrt{p}} = \sqrt{p}$ and $xb \nleqslant \phi(\sqrt{p})$. Then again using b is ϕ-primary to $\sqrt{p}$, we get $x \leqslant \sqrt{\sqrt{p}} = \sqrt{p}$ and thus b^s is ϕ-primary to $\sqrt{p}$.

(4) The proof is obvious..

(5) Let $\sqrt{\phi(p)} = \phi(\sqrt{p})$.

(i) Let b be ϕ-prime to p. Then by (1), b^s is ϕ-prime to p $\forall$ $s \in \mathbb{Z}_+$. Let $x \in L$ be such that $xb \leqslant \sqrt{p}$ and $xb \nleqslant \phi(\sqrt{p})$. Then there exist $k \in \mathbb{Z}_+$ such that $x^k b^k \leqslant p$. If $x^k b^k \leqslant \phi(p)$, then $xb \leqslant \sqrt{\phi(p)} = \phi(\sqrt{p})$ which contradicts $xb \nleqslant \phi(\sqrt{p})$. So we must have $x^k b^k \nleqslant \phi(p)$. As b^k is ϕ-prime to p, we have $x^k \leqslant p$, which implies $x \leqslant \sqrt{p}$. Thus b is ϕ-prime to $\sqrt{p}$, which implies b^s is ϕ-prime to $\sqrt{p}$ $\forall s \in \mathbb{Z}_+$, by (2), and we are done.

(ii) The proof is obvious.

(iii) Let $x \in L$ be such that $xb \leqslant \sqrt{p}$ and $xb \nleqslant \phi(\sqrt{p})$. Then there exists $n \in \mathbb{Z}_+$ such that $x^n b^n = (xb)^n \leqslant p$. If $(xb)^n \leqslant \phi(p)$, then by hypothesis $xb \leqslant \sqrt{\phi(p)} = \phi(\sqrt{p})$, a contradiction. So we must have $x^n b^n = (xb)^n \nleqslant \phi(p)$. Since b^n is ϕ-primary to p, we have $x^n \leqslant \sqrt{p}$ and hence $x \leqslant \sqrt{\sqrt{p}} = \sqrt{p}$. Hence, b is ϕ-prime to $\sqrt{p}$ and we are done.

(iv) The proof is obvious.

□

Now, we generalize Definition 2.6.

Definition 3.8. If an element $b^s \in L$ is ϕ-primary to a proper element $p \in L$ $\forall s \in \mathbb{Z}_+$, then by Theorem 3.7 (5)-(iii), b^s is ϕ-prime to $q = \sqrt{p}$ $\forall s \in \mathbb{Z}_+$. In this case, we say that b^s is q-ϕ-primary to p $\forall s \in \mathbb{Z}_+$.

During the attempt to generalize Theorems 2.7 to 2.10, we found that the reduction function ϕ on L must preserve meets, where this property is defined as follows:

Definition 3.9. Let the function $\phi : L \longrightarrow L \cup \{\emptyset\}$ be a reduction function on a multiplicative lattice L. A reduction function ϕ on L is said to have the meet preserving property if $\phi(a \wedge b) = \phi(a) \wedge \phi(b)$ $\forall$ $a, b \in L$.

Theorem 3.10. *Let a reduction function ϕ on L have the meet preserving property. Let $b \in L$. If b^s is q-ϕ-primary to proper elements $p_1, \cdots, p_n$ in L $\forall s \in \mathbb{Z}_+$ such that $\sqrt{\phi(p_i)} = \phi(\sqrt{p_i})$ and $q = \sqrt{p_i}$ $\forall i = 1 \cdots n$, then b^s is q-ϕ-primary to $\bigwedge_{i=1}^n p_i$.*

Proof. Let $s \in \mathbb{Z}_+$. Given that b^s is ϕ-primary to each p_i $(i = 1, \cdots, n)$ and b^s is ϕ-prime to each $\sqrt{p_i} = q$ $(i = 1, \cdots, n)$ $\forall s \in \mathbb{Z}_+$. Clearly, $\bigwedge_{i=1}^n p_i = p \neq 1$ and $\sqrt{p} = \bigwedge_{i=1}^n \sqrt{p_i} = q$ by Lemma 2.9. As the reduction function ϕ on L has the meet preserving property, we have $\phi(p) = \phi(\bigwedge_{i=1}^n p_i) = \bigwedge_{i=1}^n \phi(p_i)$. Let $x \in L$ be such that $xb^s \leqslant p$ and $xb^s \nleqslant \phi(p) = \bigwedge_{i=1}^n \phi(p_i)$. Then $xb^s \nleqslant \phi(p_j)$

for some $j = 1, \cdots, n$. So as $xb^s \leqslant p_j$ and b^s is ϕ-primary to p_j, we have, $x \leqslant \sqrt{p_j} = q = \sqrt{p}$ which implies b^s is ϕ-primary to p. Now we are left to show that b^s is ϕ-prime to $\sqrt{p} = q$. Let $x \in L$ be such that $xb^s \leqslant \sqrt{p}$ and $xb^s \nleqslant \phi(\sqrt{p})$. Then $xb^s \leqslant q = \sqrt{p_i}\ \forall\ i = 1, \cdots, n$ and $xb^s \nleqslant \phi(\sqrt{\bigwedge_{i=1}^n p_i}) = \phi(\bigwedge_{i=1}^n \sqrt{p_i}) = \bigwedge_{i=1}^n \phi(\sqrt{p_i})$ by the meet preserving property of ϕ. Then $xb^s \nleqslant \phi(\sqrt{p_j})$ for some $j = 1, \cdots, n$. So, as b^s is ϕ-prime to $\sqrt{p_j}$, we have, $x \leqslant \sqrt{p_j} = q = \sqrt{p}$. Thus, b^s is q-ϕ-primary to $p = \bigwedge_{i=1}^n p_i$. □

Theorem 3.11. *Let $\{p_i \mid i \in \mathbb{Z}_+\}$ be an ascending chain of elements in L such that $b^s \in L$ is ϕ-primary to each p_i $\forall s \in \mathbb{Z}_+$. Then $\forall s \in \mathbb{Z}_+$, b^s is ϕ-primary to*

(1) $\underset{i\in\mathbb{Z}_+}{\vee} p_i$.
(2) $\underset{i\in\mathbb{Z}_+}{\wedge} p_i$, *if the reduction function ϕ on L has the meet preserving property.*

Proof. Let $s \in \mathbb{Z}_+$.

(1) Since $1 \in L$ is compact, $\underset{i\in\mathbb{Z}_+}{\vee} p_i = p \neq 1$. Let $x \in L$ be such that $xb^s \leqslant p$ and $xb^s \nleqslant \phi(p)$. Then, as $\{p_i\}$ is a chain, we have $xb^s \leqslant p_j$ and $xb^s \nleqslant \phi(p_j)$ for some $j \in \mathbb{Z}_+$ because if $xb^s \leqslant \phi(p_j)$ then, as $p_j \leqslant p$ implies $\phi(p_j) \leqslant \phi(p)$, it contradicts $xb^s \nleqslant \phi(p)$. As $b^s \in L$ is ϕ-primary to p_j, we have $x \leqslant \sqrt{p_j} \leqslant \underset{i\in\mathbb{Z}_+}{\vee} \sqrt{p_i} \leqslant \sqrt{\underset{i\in\mathbb{Z}_+}{\vee} \sqrt{p_i}} \leqslant \sqrt{\underset{i\in\mathbb{Z}_+}{\vee} p_i} = \sqrt{p}$ by Lemma 2.9 -*(2)* and we are done.
(2) As the reduction function ϕ on L has the meet preserving property, we have $\phi(\underset{i\in\mathbb{Z}_+}{\wedge} p_i) = \underset{i\in\mathbb{Z}_+}{\wedge} \phi(p_i)$. Clearly, $\underset{i\in\mathbb{Z}_+}{\wedge} p_i = p \neq 1$. Let $x \in L$ be such that $xb^s \leqslant p$ and $xb^s \nleqslant \phi(p) = \underset{i\in\mathbb{Z}_+}{\wedge} \phi(p_i)$. Then $xb^s \nleqslant \phi(p_j)$ for some $j \in \mathbb{Z}_+$. So as $b^s \in L$ is ϕ-primary to p_j and $xb^s \leqslant p_j$, we have, $x \leqslant \sqrt{p_j}$. Now, let $p_i \neq p_j$. Then as $\{p_i\}$ is a chain, we have either $p_i < p_j$ or $p_j < p_i$. If $p_j < p_i$, then $\sqrt{p_j} < \sqrt{p_i}$ and so $x \leqslant \sqrt{p_j} \leqslant \sqrt{p_i}$. If $p_i < p_j$, then $\phi(p_i) < \phi(p_j)$ and so $xb^s \leqslant p_i$ and $xb^s \nleqslant \phi(p_i)$ which implies $x \leqslant \sqrt{p_i}$, since b^s is ϕ-primary to p_i. Therefore, $x \leqslant \underset{i\in\mathbb{Z}_+}{\wedge} \sqrt{p_i} = \sqrt{\underset{i\in\mathbb{Z}_+}{\wedge} p_i} = \sqrt{p}$ by Lemma 2.9 -*(1)* and we are done.

□

Now, we obtain the characterization of an element ϕ-primary to another element of L.

Theorem 3.12. *Let p be a proper element of L and $b \in L$. Then the following statements are equivalent:*

①. b is ϕ-primary to p.
②. either $(p : b) \leqslant \sqrt{p}$ or $(p : b) = (\phi(p) : b)$.
③. for every $s \in L_*$, $sb \leqslant p$ and $sb \nleqslant \phi(p)$ implies $r \leqslant \sqrt{p}$.

Proof. ①$\Longrightarrow$②. Suppose ① holds. Let $u \in L_*$ be such that $u \leqslant (p : b)$. Then $ub \leqslant p$. If $ub \leqslant \phi(p)$, then $u \leqslant (\phi(p) : b)$. If $ub \nleqslant \phi(p)$, then since b is ϕ-primary to p, $ub \leqslant p$ and $ub \nleqslant \phi(p)$, it follows that $u \leqslant \sqrt{p}$. Hence by Lemma

2.3.13 from [4], either $(p : b) \leqslant (\phi(p) : b)$ or $(p : b) \leqslant \sqrt{p}$. Consequently, either $(p : b) = (\phi(p) : b)$ or $(p : b) \leqslant \sqrt{p}$.

②$\Longrightarrow$③. Suppose ② holds. Let $s \in L_*$ be a compact element such that $sb \leqslant p$ and $sb \nleqslant \phi(p)$. If $(p : b) = (\phi(p) : b)$ holds, then we have $s \leqslant (\phi(p) : b)$, which contradicts $sb \nleqslant \phi(p)$. Consequently, we must have $(p : b) \leqslant \sqrt{p}$. Since $s \leqslant (p : b)$, it follows that $s \leqslant \sqrt{p}$.

③$\Longrightarrow$①. Suppose ③ holds. Let $z \in L$ be such that $zb \leqslant p$ and $zb \nleqslant \phi(p)$. Since L is compactly generated, there exists $a' \in L_*$ such that $a' \leqslant z$ and $a'b \nleqslant \phi(p)$. Now, let $a \leqslant z$ be any arbitrary compact element of L. Then $(a \vee a') \in L_*$ such that $(a \vee a')b \leqslant p$ and $(a \vee a')b \nleqslant \phi(p)$. Consequently, we must have $(a \vee a') \leqslant \sqrt{p}$. Hence by Lemma 2.3.13 from [4], $z \leqslant \sqrt{p}$ and we are done. □

To obtain the relation among an element ϕ_β-primary to another element in L, we prove the following lemma.

Lemma 3.13. *Let $\gamma_1, \gamma_2 : L \longrightarrow L \cup \{\emptyset\}$ be reduction functions on L such that $\gamma_1 \leqslant \gamma_2$ and p be a proper element of L. If an element $b \in L$ is γ_1-primary to p, then b is γ_2-primary to p. Moreover, if an element $b \in L$ is γ_1-prime to p, then b is γ_2-primary to p.*

Proof. The proof is obvious. □

Theorem 3.14. *For an element $b \in L$ and a proper element $p \in L$, consider the following statements:*

(a) b is $\phi_\emptyset$-primary to p.
(b) b is ϕ_0-primary to p.
(c) b is ϕ_ω-primary to p.
(d) b is $\phi_{(n+1)}$-primary to p where $n \geqslant 2$ and $n \in \mathbb{Z}_+$.
(e) b is ϕ_n-primary to p where $n \geqslant 2$ and $n \in \mathbb{Z}_+$.
(f) b is ϕ_2-primary to p.

Then $(a) \Longrightarrow (b) \Longrightarrow (c) \Longrightarrow (d) \Longrightarrow (e) \Longrightarrow (f)$.

Proof. Obviously, if b is primary to p, then b is weakly primary to p and hence $(a) \Longrightarrow (b)$. The remaining implications follow by using Lemma 3.13 on the fact that $\phi_\emptyset \leqslant \phi_0 \leqslant \phi_\omega \leqslant \cdots \leqslant \phi_{n+1} \leqslant \phi_n \leqslant \cdots \leqslant \phi_2 \leqslant \phi_1$. □

For an element $b \in L$ and a proper element $p \in L$, according to Theorem 2.23 of [3], we have b is $\phi_\emptyset$-prime to p implies b is ϕ_0-prime to p implies b is ϕ_ω-prime to p implies b is $\phi_{(n+1)}$-prime to p (where $n \geqslant 2$) implies b is ϕ_n-prime to p (where $n \geqslant 2$) implies b is ϕ_2-prime to p.

According to Definition 2.26 of [3], we have that an element $b \in L$ is said to be n-potent prime to a proper element $p \in L$ if for all $x \in L$, $xb \leqslant p^n$ imply $x \leqslant p$, where $n \geqslant 2$ and $n \in \mathbb{Z}_+$. On the same lines, we define the following term:

Definition 3.15. Let $n \geqslant 2$ and $n \in \mathbb{Z}_+$. An element $b \in L$ is said to be n-potent primary to a proper element $p \in L$ if for all $x \in L$, $xb \leqslant p^n$ imply $x \leqslant \sqrt{p}$.

Note that, as $a \leqslant \sqrt{a}$ $\forall a \in L$, for a proper element $p \in L$ and $b \in L$, we have, if b is n-potent prime to p, then b is n-potent primary to p where $n \geqslant 2$

and $n \in \mathbb{Z}_+$. The following example shows that its converse need not be true (by taking $n = 2$ for convenience).

Example 3.16. Let L be the lattice of ideals of $\mathbb{Z}_{24}$ defined in Example 3.6. It is easy to see that the element $(2) \in L$ is 2-potent primary to $(0) \in L$, but (2) is not 2-potent prime to (0) because $(12)(2) \leqslant (0)^2$, yet $(12) \nleqslant (0)$.

Let $n \geqslant 2$ and $n \in \mathbb{Z}_+$. If $b \in L$ is prime to a proper element $p \in L$, then b is n-potent prime to p. The following example shows that its converse need not be true (by taking $n = 2$ for convenience).

Example 3.17. Let L be the lattice of ideals of $\mathbb{Z}_{24}$ defined in Example 3.6. It is easy to see that the element $(2) \in L$ is 2-potent prime to $(4) \in L$, but (2) is not prime to (4) because $(2)(2) \leqslant (4)$, yet $(2) \nleqslant (4)$.

From Theorem 2.23 of [3], and above Theorem 3.14 and as $a \leqslant \sqrt{a}\ \forall a \in L$ and $a^n \leqslant a\ \forall a \in L,\ \ \forall n \in \mathbb{Z}_+$, for a proper element $p \in L$ and $b \in L$, we have the following implications:

(1) If b is prime to p, then b is almost prime to p. Its converse need not be true [refer to Example 2.24 in [3]].
(2) If b is weakly prime to p, then b is almost primary to p. Its converse need not be true [refer to Example 3.17. Clearly, (2) is almost primary to (4), but (2) is not weakly prime to (4) because $(0) \neq (2)(2) \leqslant (4)$, but $(2) \nleqslant (4)$].
(3) If b is weakly prime to p, then b is almost prime to p. Its converse need not be true [left to the reader].
(4) If b is weakly primary to p, then b is almost primary to p. Its converse need not be true [left to the reader].
(5) If b is primary to p, then b is almost primary to p. Its converse need not be true [refer to Example 3.20].
(6) If b is almost prime to p, then b is almost primary to p. Its converse need not be true [refer Example 3.6].
(7) If b is ω-prime to p, then b is ω-primary to p. Its converse need not be true [left to the reader].
(8) Let $n \geqslant 2$ and $n \in \mathbb{Z}_+$. If b is n-potent prime to p, then b is n-potent primary to p. Its converse need not be true [refer to Example 3.16].
(9) Let $n \geqslant 2$ and $n \in \mathbb{Z}_+$. If b is prime to p, then b is n-potent prime to p. Its converse need not be true [refer to Example 3.17].
(10) Let $n \geqslant 2$ and $n \in \mathbb{Z}_+$. If b is prime to p, then b is n-potent primary to p. Its converse need not be true [refer to Example 3.17. As (2) is 2-potent prime to (4), we have (2) is 2-potent primary to (4)].
(11) Let $n \geqslant 2$ and $n \in \mathbb{Z}_+$. If b is primary to p, then b is n-potent primary to p. Its converse need not be true [left to the reader].

The consequences of Theorem 3.14 are presented in the following corollaries.

Corollary 3.18. *Let $b \in L$ and a proper element $p \in L$ be an idempotent element. Then b is ω-primary to p and hence b is n-almost primary to $p\ \forall n \geqslant 2$. Moreover, b is almost primary to p*

Proof. The proof is obvious. □

Corollary 3.19. *Let p be a proper element of L and $b \in L$. Then b is n-almost primary to p $\forall n \geqslant 2$ if and only if b is ω-primary to p.*

Proof. Let b be n-almost primary to p $\forall n \geqslant 2$. Let $x \in L$ be such that $xb \leqslant p$ and $xb \nleqslant \phi_\omega(p) = \bigwedge_{n=1}^{\infty} p^n$. Then $xb \leqslant p$ and $xb \nleqslant p^k$ for some $k \in \mathbb{Z}_+$. If $k = 1$, then we are done. If $k \geqslant 2$, then as b is k-almost primary to p, we have $x \leqslant \sqrt{p}$. Hence, b is ω-primary to p. The converse follows from Theorem 3.14. □

From Theorem 3.14, it is clear if an element $b \in L$ is primary to a proper element $p \in L$, then b is ϕ-primary to p. However, the converse does not hold in general, as illustrated by the following counter example taking ϕ as ϕ_2.

Example 3.20. Consider the ring $R =< \mathbb{Z}_{30}\ ,\ +_{30}\ ,\ \times_{30} >$ with lattice of ideals $L = \{(0),(2),(3),(5),(6),(10),(15),(1)\}$. L forms a CG multiplicative lattice. Observe that the element $(2) \in L$ is ϕ_2-primary to $(6) \in L$, but (2) is not primary to (6). This is because $(3)(2) \leqslant (6)$, but $(3)^n \nleqslant (6)$ $\forall n \in \mathbb{Z}_+$.

In the next four theorems, we establish conditions under which, if an element $b \in L$ is ϕ-primary to a proper element p, then b is necessarily primary to p.

Theorem 3.21. *Let L be a Noether lattice and let $0 \neq p \in L$ be a non-nilpotent proper element satisfying the restricted cancellation law (RCL). Let $b \in L$ be such that $p < b$. Then b is ϕ-primary to p for some reduction function ϕ satisfying $\phi \leqslant \phi_n$ and for all $n \geqslant 2$ if and only if b is primary to p.*

Proof. If b is primary to p, then b is trivially ϕ-primary to p for every reduction function ϕ and hence for some $\phi \leqslant \phi_n$, for all $n \geqslant 2$. Conversely, assume that b is ϕ-primary to p for some $\phi \leqslant \phi_n$ and for all $n \geqslant 2$. Then by Lemma 3.13, b is ϕ_n-primary (n-almost primary) to p for all $n \geqslant 2$. Let $a \in L$ be such that $ab \leqslant p$. If $ab \nleqslant \phi_n(p)$ for some $n \geqslant 2$, then as b is ϕ_n-primary to p, we have $a \leqslant \sqrt{p}$, and we are done. Otherwise, suppose $ab \leqslant \phi_n(p)$ for all $n \geqslant 2$. It follows that $ab \leqslant p^n \leqslant p^2$ as $n \geqslant 2$. Since, $p < b$ and p is non-nilpotent, we have $ap \leqslant p^2 \neq 0$. Then by Lemma 1.11 of [10], we have $a \leqslant p \leqslant \sqrt{p}$. Thus, in either case, $a \leqslant \sqrt{p}$, which proves that b is primary to p. □

Definition 3.22. Let $n \geqslant 2$ and $n \in \mathbb{Z}_+$. An element $b \in L$ is said to be n-potent primary to a proper element $p \in L$ if for all $x \in L$, $xb \leqslant p^n$ imply $x \leqslant \sqrt{p}$.

Theorem 3.23. *Let $p \in L$ be a proper element and $b \in L$. Then b is ϕ-primary to p for some $\phi \leqslant \phi_n$ where $n \geqslant 2$ and $n \in \mathbb{Z}_+$ if and only if b is primary to p, provided b is k-potent primary to p for some $k \leqslant n$.*

Proof. If b is primary to p, then b is trivially ϕ-primary to p for every reduction function ϕ and hence for some $\phi \leqslant \phi_n$ where $n \geqslant 2$ and $n \in \mathbb{Z}_+$. Conversely, assume that b is ϕ-primary to p for some $\phi \leqslant \phi_n$ where $n \geqslant 2$ and $n \in \mathbb{Z}_+$. Then by Lemma 3.13, b is ϕ_n-primary (n-almost primary) to p. Let $a \in L$ be such that $ab \leqslant p$. If $ab \nleqslant \phi_k(p) = p^k$, then $ab \nleqslant \phi_n(p) = p^n$ as $k \leqslant n$. Since b is ϕ_n-primary to p, we have $a \leqslant \sqrt{p}$, and we are done. If $ab \leqslant \phi_k(p) = p^k$, then since b is k-potent primary to p, we have $a \leqslant \sqrt{p}$. Therefore, b is primary to p. □

Theorem 3.24. *Let an element $b \in L$ be ϕ-primary to a proper element $p \in L$. If $pb \nleqslant \phi(p)$, then b is primary to p.*

Proof. Let $a \in L$ be such that $ab \leqslant p$. If $ab \nleqslant \phi(p)$, then $a \leqslant \sqrt{p}$, because b is ϕ-primary to p. Now, assume that $ab \leqslant \phi(p)$. Then, as $pb \nleqslant \phi(p)$, we have $db \nleqslant \phi(p)$ for some $d \leqslant p$ in L. Notice that $(a \vee d)b = ab \vee db \leqslant p$ and $(a \vee d)b \nleqslant \phi(p)$. Since b is ϕ-primary to p, we have $a \leqslant (a \vee d) \leqslant \sqrt{p}$, which proves b is primary to p. □

Corollary 3.25. *If an element $b \in L$ is weakly primary to a proper element $p \in L$ such that b is not primary to p, then $pb = 0$.*

Proof. The proof is obvious. □

Theorem 3.26. *Let an element $b \in L$ be ϕ-primary to a proper element $p \in L$. If b is primary to $\phi(p)$, then b is primary to p.*

Proof. The proof is obvious. □

We conclude this paper with the following two results.

Theorem 3.27. *Let $(L,\ m)$ be a quasi-local Noether lattice. If a proper element $p \in L$ is such that $p^2 = m^2 \leqslant p \leqslant m$ and $b \in L$, then b is either ϕ_2-primary to p or $b \leqslant p$.*

Proof. Let $a \in L$ be such that $ab \leqslant p$ and $ab \nleqslant \phi_2(p)$. If $a \nleqslant m$, then $a = 1$. So $ab \leqslant p$ gives $b \leqslant p$. Now if $a \leqslant m$, then $a^2 \leqslant m^2 = p^2 \leqslant p$ and hence $a \leqslant \sqrt{p}$, which implies b is ϕ_2-primary to p. Thus b is either ϕ_2-primary to p or $b \leqslant p$. □

Theorem 3.28. *Let L be a local Noetherian domain. If an element $0 \neq b$ is n-almost primary to a proper element $p \in L$ $\forall n \geqslant 2$, then b is primary to p.*

Proof. Let $a \in L$ be such that $ab \leqslant p$. If $ab \nleqslant p^n$ for any $n \geqslant 2$, then as b is n-almost primary to p, we have $a \leqslant \sqrt{p}$ and we are done. If $ab \nleqslant p^n$ $\forall n \geqslant 1$, then as L is local Noetherian, by Corollary 3.3 of [5], it follows that $ab \leqslant \bigwedge_{n=1}^{\infty} p^n = 0$ and so $ab = 0$. Since L is a domain and $b \neq 0$, we have $a = 0$, which implies $a \leqslant \sqrt{p}$ and hence b is primary to p. □

4. Concluding remarks

In a multiplicative lattice L, we have achieved unification of the following notions: "an element primary to another element", "an element weakly primary to another element", "an element n-almost primary to another element $(n \geqslant 2)$", "an element ω-primary to another element" under one frame by defining the notion of "an element ϕ-primary to another element", where ϕ is a reduction function on L.

This paper paves the way to several promising directions for future research, in particular, "an element ϕ-δ-primary to another element" in multiplicative lattices.

The relations among generalized notions of "an element primary to another element" in L, and their relations with the corresponding generalized notions of "an element prime to another element" in L, as discussed in this paper, is summarized below.

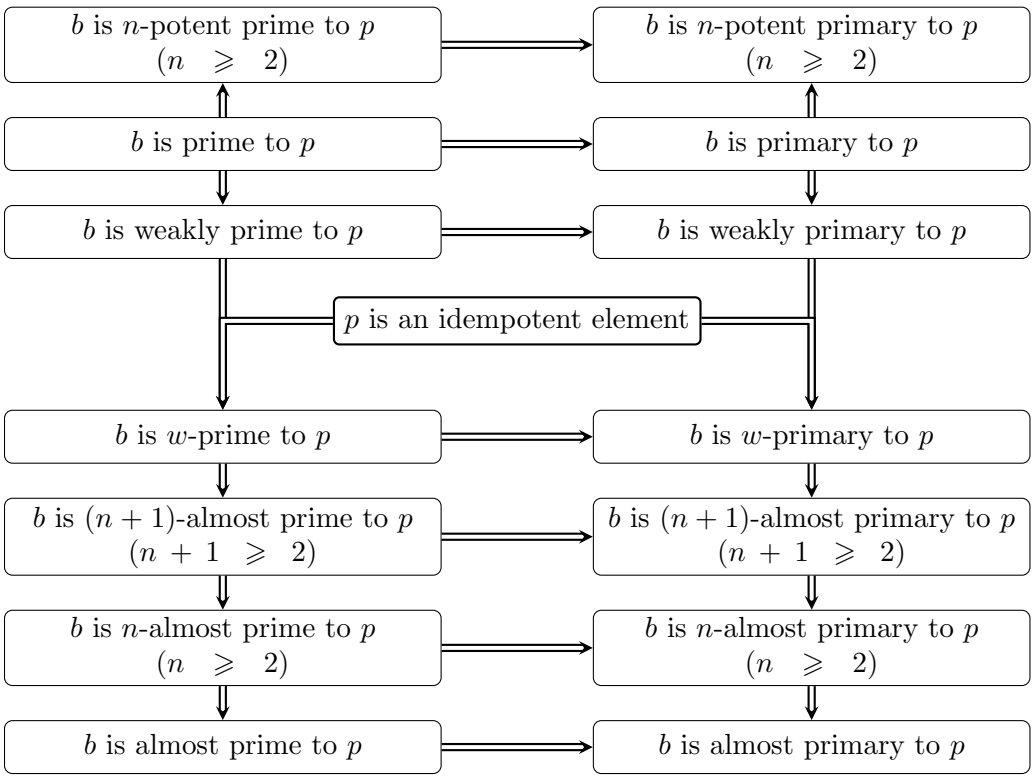


Figure-1
Implications among notions of element"prime to"/"primary to" another element in L

Conflict of interest. The author declares no conflicts of interest.

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Department of Mathematics, St. Xavier's College, Mumbai(An empowered autonomous institute affiliated to the University of Mumbai)
Email address: ashok.bingi@xaviers.edu