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OHAM FOR FUZZY VOLTERRA–FREDHOLM INTEGRAL EQUATIONS

MUATH TALAL MAHMOUD ALZUBI[1,*], FARAH AINI ABDULLAH[1,2], ALI FAREED JAMEEL[3], ADILA AIDA AZAHAR[4], SHAHIDA SHAHNAWAZ[1] and LUBNA ALI KHALED ALOMARI[5]

Abstract. This study formulates the original Optimal Homotopy Asymptotic Method for second-kind linear fuzzy Volterra–Fredholm integral equations. The parametric representation yields lower and upper crisp equations, while a composite operator combines their Volterra and Fredholm terms. The OHAM deformation recursion generates parameter-dependent correction terms, and the convergence-control parameters are determined by nonlinear least-squares minimization of the governing-equation residual. Two benchmark problems demonstrate accurate approximations, preservation of fuzzy ordering, and coincidence of the lower and upper endpoints at the crisp level. Comparisons with published HPM and BWM results illustrate the numerical performance of the formulation.

Keywords. Fuzzy integral equations; optimal homotopy asymptotic method; parametric representation; nonlinear least squares; residual minimization.
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1. Introduction

Fuzzy Volterra-Fredholm integral equations of the second kind (FVFIEs) are a significant category of mathematical frameworks for characterizing systems that exhibit memory effects and global interactions amidst uncertainty. The Volterra integral component signifies the cumulative effect of prior states via its variable upper limit of integration, while the Fredholm integral component delineates global interactions throughout a constant period. Equations of this nature emerge in mathematical frameworks associated with heat transfer, viscoelasticity, population dynamics, biological systems, control theory, and several physical and engineering applications that include uncertain data or parameters.

The mathematical underpinnings of fuzzy analysis were formulated through various key contributions. Dubois and Prade [6] established significant arithmetic functions for fuzzy numbers, while Goetschel and Voxman [11] presented essential principles of fuzzy calculus. Kaleva [12] advanced the theoretical foundation of fuzzy differential equations, while Friedman et al. [8] introduced an initial numerical methodology to address fuzzy differential and integral equations. These

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*Corresponding author.

efforts established the foundation for the later advancement of numerical and semi-analytical techniques for fuzzy integral equations.

A range of numerical methods has been suggested to address fuzzy integral equations. Babolian et al. [5] utilized the Adomian decomposition technique for second-kind linear fuzzy Fredholm integral equations, whereas Abbasbandy et al. [1] formulated a numerical approach for the same class of problems. Homotopy-based methodologies have garnered significant interest due to their ability to yield analytical or semi-analytical approximations without traditional spatial discretization. Ghanbari [10] utilized the Homotopy Analysis Method (HAM) to address second-kind fuzzy Volterra integral equations, while Attari and Yazdani [4] formulated a computational approach grounded in the Homotopy Perturbation Method (HPM) for fuzzy Volterra–Fredholm integral equations. Subsequent advances have concentrated on enhancing convergence and regulating the approximation procedure. Georgieva and Naydenova [9] utilized HAM for two-dimensional linear fuzzy Volterra integral equations, and related studies examined the selection of suitable iterations and the numerical efficiency of homotopy-based approaches for fuzzy integral equations [7, 18]. Ramadan and Ali [19] utilized the Bernoulli wavelet method (BWM) to derive numerical solutions for fuzzy linear Volterra–Fredholm integral equations. Related work published in the *Gulf Journal of Mathematics* includes the homotopy-based treatment of fractional Fredholm–Volterra integro-differential equations by Nasiri and Aminataei [17] and a numerical study of nonlinear quadratic Fredholm integral equations by Mehellou and Khirani [16]. These studies further demonstrate the journal's continuing interest in analytical and numerical methods for integral and integro-differential equations.

The Optimal Homotopy Asymptotic Method (OHAM), developed by Marinca and Herisanu [14, 15], integrates supplementary convergence-control parameters within the framework of homotopy construction. These parameters can be established by minimizing a residual function derived from the governing equation. This characteristic distinguishes OHAM from homotopy techniques whose convergence behavior is predominantly dictated by a constant perturbation series. OHAM has since been utilized for several categories of integral and differential equations. In the context of fuzziness, Kashefi and Ghorbani [13] utilized OHAM for fuzzy Fredholm integral equations. Recently, AlZubi et al. [2, 3] formulated OHAM-based and multistage-OHAM-based approaches for fuzzy Volterra and Fredholm integral equations.

Despite these developments, direct applications of the original OHAM deformation recursion to second-kind linear fuzzy Volterra–Fredholm integral equations remain limited. The mixed equation requires the same parameter-dependent corrections to account simultaneously for its variable- and fixed-limit integral terms while maintaining valid lower and upper r-cut solutions.

Driven by these factors, the current research formulates FVF-OHAM for second-kind linear fuzzy Volterra–Fredholm integral equations. The parametric r-cut representation converts the fuzzy equation into lower and upper crisp equations, and a composite operator combines the Volterra and Fredholm terms.

The correction functions are generated directly from the original OHAM deformation equations. The convergence-control parameters are then determined within the OHAM formulation by nonlinear least-squares minimization of the governing-equation residual.

The proposed approach is examined using two linear fuzzy Volterra–Fredholm benchmark problems with known exact solutions. The first example is compared with the HPM result reported by Attari and Yazdani [4], and the second with the BWM results reported by Ramadan and Ali [19]. The evaluation considers absolute errors, residual norms, preservation of fuzzy ordering, and coincidence of the lower and upper components at the crisp level $r = 1$. Each comparison is stated under the computational settings reported for the corresponding method.

The principal contributions of this study are summarized as follows:

- A unified FVF-OHAM formulation is developed for second-kind linear fuzzy Volterra–Fredholm integral equations using the parametric r-cut representation.
- A compact operator notation is used to treat the Volterra and Fredholm contributions consistently within the OHAM deformation recursion.
- The original OHAM recursion is retained. Its convergence-control parameters are obtained by nonlinear least-squares residual minimization.
- The lower and upper components of the fuzzy solution are reconstructed and assessed using absolute errors, residual norms, preservation of fuzzy ordering, and coincidence of the bounds at $r = 1$.
- The numerical performance of FVF-OHAM is evaluated using two benchmark problems and compared with previously reported HPM and BWM results.

The remainder of this paper is organized as follows. Section 2 introduces the fundamental concepts of fuzzy numbers, parametric r-cut representation, and second-kind linear fuzzy Volterra-Fredholm integral equations. Section 3 presents the proposed FVF-OHAM formulation, including the composite operator, the original deformation recursion, and residual minimization. Section 4 presents two numerical examples and comparisons with previously reported HPM and BWM results. Finally, Section 5 summarizes the main findings and outlines possible directions for future research.

2. Mathematical Preliminaries

We use the standard parametric representation of fuzzy numbers [6, 8, 11, 12]. A fuzzy number $\widetilde{z}$ is represented by its r-cut

$$[\widetilde{z}]^r = [\underline{z}(r), \overline{z}(r)], \qquad 0 \le r \le 1, \tag{2.1}$$

where $\underline{z}$ is nondecreasing, $\overline{z}$ is nonincreasing, and $\underline{z}(r) \le \overline{z}(r)$. The usual componentwise addition and scalar multiplication of fuzzy numbers are assumed. In particular, under the nonnegativity assumptions imposed below, the integral operators act separately on the lower and upper endpoints.

The problem considered is the second-kind linear fuzzy Volterra–Fredholm integral equation

$$\widetilde{u}(x) = \widetilde{f}(x) + \lambda_1 \int_a^x K_1(x,t)\widetilde{u}(t)\,dt + \lambda_2 \int_a^b K_2(x,t)\widetilde{u}(t)\,dt, \qquad x \in [a,b], \tag{2.2}$$

where $\lambda_1, \lambda_2 \geq 0$ and the continuous kernels K_1, K_2 are nonnegative on their domains. Writing $\widetilde{u}(x,r) = [\underline{u}(x,r), \overline{u}(x,r)]$ and $\widetilde{f}(x,r) = [\underline{f}(x,r), \overline{f}(x,r)]$ gives, for $s \in \{_, \bar{}\}$,

$$u^s(x,r) = f^s(x,r) + \lambda_1 \int_a^x K_1(x,t)u^s(t,r)\,dt + \lambda_2 \int_a^b K_2(x,t)u^s(t,r)\,dt. \tag{2.3}$$

Thus, each fixed r-level produces two crisp equations. After their solution, validity of the reconstructed fuzzy approximation is checked by $\underline{U}_m(x,r) \leq \overline{U}_m(x,r)$, with coincidence of the two endpoints at any crisp level.

3. FVF-OHAM Formulation

For either endpoint in Eq. (2.3), define

$$\mathcal{T}[v](x,r) = \lambda_1 \int_a^x K_1(x,t)v(t,r)\,dt + \lambda_2 \int_a^b K_2(x,t)v(t,r)\,dt, \tag{3.1}$$

and the governing operator

$$\mathcal{A}_r[v](x) = v(x,r) - f(x,r) - \mathcal{T}[v](x,r). \tag{3.2}$$

Hence the endpoint equation is $\mathcal{A}_r[u] = 0$. The notation $\mathcal{T}$ is used only to shorten the simultaneous Volterra and Fredholm terms; no independent novelty is claimed for their algebraic sum.

Following the original OHAM framework [14, 15], introduce $q \in [0,1]$ and construct

$$(1-q)[\varphi(x;q,r) - f(x,r)] = H(q,r)\,[\varphi(x;q,r) - f(x,r) - \mathcal{T}[\varphi](x,r)], \tag{3.3}$$

with

$$H(q,r) = \sum_{j=1}^{m} C_j(r)q^j, \qquad \varphi(x;q,r) = u_0(x,r) + \sum_{k=1}^{\infty} u_k(x,r)q^k, \tag{3.4}$$

where $u_0 = f$. The conditions $\varphi(x;0,r) = f(x,r)$ and $\varphi(x;1,r) = u(x,r)$ connect the initial approximation to the required solution.

3.1. Order-by-order OHAM deformation equations.

Substituting Eq. (3.4) into Eq. (3.3) and equating like powers of q produces the original OHAM hierarchy. The coefficient of q^0 gives the zero-order problem

$$u_0(x,r) - f(x,r) = 0, \qquad u_0(x,r) = f(x,r). \tag{3.5}$$

For later use, define the coefficients of the governing-equation expression by

$$\rho_0(x,r) = -\mathcal{T}[u_0](x,r), \tag{3.6}$$

$$\rho_j(x,r) = u_j(x,r) - \mathcal{T}[u_j](x,r), \qquad j \geq 1. \tag{3.7}$$

The coefficient of q^1 yields the first-order deformation problem

$$\begin{aligned} u_1(x,r) &= C_1(r)\rho_0(x,r) \\ &= -C_1(r)\left[\lambda_1 \int_a^x K_1(x,t)u_0(t,r)\,dt + \lambda_2 \int_a^b K_2(x,t)u_0(t,r)\,dt\right]. \end{aligned} \tag{3.8}$$

The coefficient of q^2 gives the second-order deformation problem

$$\begin{aligned} u_2(x,r) - u_1(x,r) &= C_1(r)\rho_1(x,r) + C_2(r)\rho_0(x,r), \\ u_2(x,r) &= u_1(x,r) + C_1(r)\left[u_1(x,r) - \lambda_1 \int_a^x K_1(x,t)u_1(t,r)\,dt \right. \\ &\qquad \left. -\lambda_2 \int_a^b K_2(x,t)u_1(t,r)\,dt\right] \\ &\quad - C_2(r)\left[\lambda_1 \int_a^x K_1(x,t)u_0(t,r)\,dt \right. \\ &\qquad \left. +\lambda_2 \int_a^b K_2(x,t)u_0(t,r)\,dt\right]. \end{aligned} \tag{3.9}$$

In general, equating the coefficient of q^n, $n \geq 2$, gives

$$\begin{aligned} u_n(x,r) - u_{n-1}(x,r) &= \sum_{i=1}^{n} C_i(r)\rho_{n-i}(x,r), \\ u_n(x,r) &= u_{n-1}(x,r) + \sum_{i=1}^{n} C_i(r)\rho_{n-i}(x,r), \qquad n \geq 2. \end{aligned} \tag{3.10}$$

Equivalently, after separating the term containing ρ_0,

$$\begin{aligned} u_n(x,r) &= u_{n-1}(x,r) - C_n(r)\mathcal{T}[u_0](x,r) \\ &\quad + \sum_{i=1}^{n-1} C_i(r)\left(u_{n-i}(x,r) - \mathcal{T}[u_{n-i}](x,r)\right), \qquad n \geq 2. \end{aligned} \tag{3.11}$$

Equations (3.5)– (3.11) constitute the complete order-by-order OHAM formulation for each lower or upper endpoint of the fuzzy Volterra–Fredholm problem. The m-th order approximation is

$$U_m(x,r;\mathbf{C}) = \sum_{k=0}^{m} u_k(x,r;\mathbf{C}), \qquad \mathbf{C} = (C_1,\ldots,C_m)^T. \tag{3.12}$$

Because the corrections contain products such as C_1^2, both U_m and its residual generally depend nonlinearly on $\mathbf{C}$.

The residual is obtained from the governing equation itself:

$$\mathcal{R}_m(x,r;\mathbf{C}) = \mathcal{A}_r[U_m](x) = U_m(x,r;\mathbf{C}) - f(x,r) - \mathcal{T}[U_m](x,r). \tag{3.13}$$

The proper convergence-control parameters are determined within OHAM by minimizing the least-squares functional

$$J(\mathbf{C};r) = \int_a^b \mathcal{R}_m^2(x,r;\mathbf{C})\,dx, \qquad \frac{\partial J}{\partial C_j} = 0, \qquad j = 1,\ldots,m. \tag{3.14}$$

These conditions form a nonlinear algebraic system. Thus, least squares is only the optimization criterion used to determine the OHAM parameters; it is not an independent solution method. The lower and upper parameter vectors are optimized separately, and the fuzzy approximation is

$$\widetilde{U}_m(x,r) = [\underline{U}_m(x,r), \overline{U}_m(x,r)]. \tag{3.15}$$

When an exact solution is available, it is excluded from the optimization and used only afterward to report

$$E_m^s(x,r) = |u_{\text{exact}}^s(x,r) - U_m^s(x,r)|, \qquad s \in \{_, \bar{}\}. \tag{3.16}$$

The computational procedure is summarized as follows: (i) apply the r-cut representation and form the two endpoint equations; (ii) set $u_0 = f$ and generate $u_1, \ldots, u_m$ from Eqs. (3.6)– (3.10); (iii) form U_m and $\mathcal{R}_m$; (iv) minimize J to determine $\mathbf{C}^*$; and (v) reconstruct the fuzzy solution and evaluate ordering, endpoint coincidence, residuals, and, when available, a posteriori errors. No Picard or independently generated operator basis replaces the OHAM corrections.

4. Numerical Examples and Comparisons

This section evaluates the proposed FVF-OHAM using two second-kind linear fuzzy Volterra-Fredholm integral equations with known exact solutions. The first example is compared with the homotopy perturbation method (HPM) results reported by Attari and Yazdani [4], while the second example is compared with the Bernoulli wavelet method (BWM) results reported by Ramadan and Ali [19].

4.1. Example 1: A Linear Fuzzy Volterra–Fredholm Integral Equation.

Consider the second-kind linear fuzzy Volterra–Fredholm integral equation [4]

$$\widetilde{u}(x) = \widetilde{f}(x) + \int_0^x K_1(x,t)\widetilde{u}(t)\,dt + \int_0^1 K_2(x,t)\widetilde{u}(t)\,dt, \qquad 0 \le x \le 1, \tag{4.1}$$

where

$$K_1(x,t) = \frac{\sin(\pi t)}{2}, \qquad K_2(x,t) = \frac{\cos^2(\pi x)}{2}. \tag{4.2}$$

For $0 \le r \le 1$, the lower and upper forcing functions are

$$\underline{f}(x,r) = -\frac{r\cos^2(\pi x)}{\pi} + r\sin(\pi x) + \frac{r[-2\pi x + \sin(2\pi x)]}{8\pi}, \tag{4.3}$$

$$\overline{f}(x,r) = -\frac{(2-r)\cos^2(\pi x)}{\pi} + (2-r)\sin(\pi x) + \frac{(2-r)[-2\pi x + \sin(2\pi x)]}{8\pi}. \tag{4.4}$$

The exact parametric solution, used only to assess the final approximation, is

$$\widetilde{u}_{\text{exact}}(x,r) = [r\sin(\pi x), (2-r)\sin(\pi x)]\,. \tag{4.5}$$

Original FVF-OHAM implementation. For each component, $u_0 = f$, and the correction terms $u_1, \ldots, u_8$ are generated directly from Eqs. (3.6)– (3.10). Thus,

$$U_8(x,r) = \sum_{k=0}^{8} u_k(x,r), \qquad \mathcal{R}_8(x,r) = U_8(x,r) - f(x,r) - \mathcal{T}[U_8](x,r). \tag{4.6}$$

The convergence-control parameters are obtained by minimizing

$$J(C_1, \ldots, C_8; r) = \int_0^1 \mathcal{R}_8^2(x,r)\, dx, \tag{4.7}$$

through the nonlinear least-squares conditions $\partial J/\partial C_j = 0$, $j = 1, \ldots, 8$. The integral operators were evaluated on a uniform grid of 801 points using composite Simpson quadrature (with a local Simpson 3/8 closure when required). The nonlinear least-squares iteration was initialized with $C_1 = -1$ and $C_2 = \cdots = C_8 = 0$. The exact solution in Eq. (4.5) was not used in this optimization.

Both forcing components are scalar multiples of the same unit-scale function. Since Eq. (4.1) is linear, the lower and upper residual functionals differ from the unit-scale functional only by the positive factors r^2 and $(2-r)^2$, respectively. Consequently, they have the same minimizing convergence-control parameters for every nonzero component. The optimized parameters are listed in Table 1.

TABLE 1. Common eighth-order FVF-OHAM convergence-control parameters for Example 1.

C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
-1.004096458371234	$-1.205554994200 \times 10^{-3}$	$-1.168066729299 \times 10^{-2}$	$4.099451604950 \times 10^{-4}$	$1.366572627156 \times 10^{-5}$	$-2.801962870741 \times 10^{-4}$	$1.404940673455 \times 10^{-6}$	$-3.618086200384 \times 10^{-8}$

The optimized unit-scale squared residual was $J = 1.384111498545151 \times 10^{-23}$. The required fuzzy ordering $\underline{U}_8(x,r) \le \overline{U}_8(x,r)$ was preserved over the full computational domain, and the two bounds coincided at $r = 1$ to machine precision. The global maximum lower and upper absolute errors were 1.612×10^{-9} and 3.223×10^{-9}, respectively; both occurred on the first integration panel near $x = 0$. At the comparison points used in the earlier study, the errors were substantially smaller. Table 2 summarizes the mean absolute error over the lower and upper components. Because the two components are scaled versions of the same unit solution, the reported mean error is independent of r, up to floating-point rounding.

TABLE 2. Mean absolute errors of the eighth-order original FVF-OHAM for Example 1 at the selected spatial points.

x	Mean absolute error
0.25	6.807×10^{-13}
0.50	5.103×10^{-12}
0.75	2.021×10^{-12}
1.00	7.466×10^{-12}

At $x = 0.25$, Attari and Yazdani [4] reported an HPM absolute error of 1.3455×10^{-4}, whereas the corresponding mean absolute error of the present eighth-order

FVF-OHAM is 6.807×10^{-13}. Under the stated computational settings, this represents an improvement factor of approximately 1.98×10^{8}. This comparison supports the accuracy of the residual-optimized convergence-control parameters while retaining the original OHAM deformation recursion.

4.2. **Example 2: A Second-Kind Linear Fuzzy Volterra–Fredholm Integral Equation.** Consider the benchmark problem reported by Ramadan and Ali [19]:

$$\widetilde{u}(x) = \widetilde{f}(x) + \int_0^1 (x+t)\widetilde{u}(t)\,dt + \int_0^x \widetilde{u}(t)\,dt, \qquad 0 \le x \le 1. \tag{4.8}$$

Let

$$\underline{a}(r) = 0.9 + 0.1r, \qquad \overline{a}(r) = 1.25 - 0.25r, \qquad 0 \le r \le 1, \tag{4.9}$$

and define, for either endpoint $a(r)$,

$$f(x;r) = x^2 - \frac{x^3}{3} - \frac{x}{3} - \frac{1}{4} + a(r) - 4a(r)x. \tag{4.10}$$

The exact parametric solution, used only for the a posteriori error assessment, is

$$\widetilde{u}_{\text{exact}}(x;r) = \left[x^2 + 1.8 + 0.2r,\ x^2 + 2.5 - 0.5r\right]. \tag{4.11}$$

For each endpoint, introduce the composite operator

$$\mathcal{T}[v](x) = \int_0^1 (x+t)v(t)\,dt + \int_0^x v(t)\,dt, \tag{4.12}$$

so that $u = f + \mathcal{T}[u]$. The original OHAM deformation equation is applied without replacing its correction functions by an independently generated operator basis. With $H(p) = \sum_{j=1}^{8} C_j p^j$, $u_0 = f$, $\rho_0 = -\mathcal{T}[u_0]$, and $\rho_j = u_j - \mathcal{T}[u_j]$ for $j \ge 1$, equating equal powers of p gives

$$u_1 = C_1\rho_0, \tag{4.13}$$

$$u_n = u_{n-1} + \sum_{i=1}^{n} C_i\rho_{n-i}, \qquad n = 2, \dots, 8, \tag{4.14}$$

$$U_8 = \sum_{n=0}^{8} u_n. \tag{4.15}$$

Accordingly, least squares is used only as the optimization criterion for finding the proper convergence-control parameters; it is not introduced as an independent solution method. After the optimal parameter vectors have been substituted into the truncated OHAM series, any available exact solution is used only for the a posteriori error calculation. Consequently, U_8 depends nonlinearly on the convergence-control parameters. They are determined solely by minimizing the governing-equation residual

$$J(C_1, \dots, C_8; r) = \int_0^1 \left[U_8(x;r) - f(x;r) - \mathcal{T}[U_8](x;r)\right]^2 dx. \tag{4.16}$$

All polynomial operator evaluations are performed analytically. A 32-point Gauss–Legendre rule, which is exact for the residual polynomial in exact arithmetic, is used in the nonlinear least-squares optimization. A reproducible multistart strategy is employed to reduce sensitivity to local minima. The common crisp problem is optimized first at $r = 1$, and its solution is then used for continuation toward the remaining fuzzy levels.

Table 3 lists representative optimized parameters. At $r = 1$, the lower and upper problems are identical and therefore use the same optimized parameter vector.

TABLE 3. Representative eighth-order original FVF-OHAM parameters for Example 2.

Component	r	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
Lower	0	-0.922552112	0.292122862	0.946528568	0.507165680	0.407337845	-1.009531906	-1.188883591	-1.559785156
Upper	0	-0.918479941	0.298351902	0.952610853	0.508640840	0.400221512	-1.026791530	-1.212068658	-1.578974861
Common	1	-0.953671400	0.240437562	0.912791242	0.497706457	0.498633922	-0.895411996	-1.010724484	-1.522365886

The results at $r = 0$ are reported in Table 4. The largest error among these reported points is 2.569×10^{-7}, whereas the maximum errors over the complete verification grid and all fuzzy levels are 3.752×10^{-7} and 4.306×10^{-7} for the lower and upper components, respectively.

TABLE 4. Exact and eighth-order original FVF-OHAM solutions for Example 2 at $r = 0$.

x	$\underline{u}_{\text{exact}}$	$\underline{U}_8$	$\underline{E}_8$	$\overline{u}_{\text{exact}}$	$\overline{U}_8$	$\overline{E}_8$
0.0	1.800000000	1.800000019	1.859×10^{-8}	2.500000000	2.500000024	2.405×10^{-8}
0.2	1.840000000	1.839999833	1.669×10^{-7}	2.540000000	2.539999751	2.487×10^{-7}
0.4	1.960000000	1.960000124	1.239×10^{-7}	2.660000000	2.660000185	1.850×10^{-7}
0.6	2.160000000	2.159999827	1.730×10^{-7}	2.860000000	2.859999743	2.569×10^{-7}
0.8	2.440000000	2.440000129	1.292×10^{-7}	3.140000000	3.140000192	1.920×10^{-7}
1.0	2.800000000	2.799999987	1.274×10^{-8}	3.500000000	3.499999978	2.237×10^{-8}

For additional validation, the largest squared residual over all fuzzy levels is 3.510×10^{-14}. The computed endpoints preserve $\underline{U}_8(x, r) \le \overline{U}_8(x, r)$ throughout the verification grid, with no detected ordering violation. Moreover, $\underline{U}_8(x, 1) = \overline{U}_8(x, 1)$ to machine precision.

Table 5 compares the new original-OHAM results at $r = 0$ with the BWM errors reported by Ramadan and Ali [19]. The comparison is mixed at the interior points: BWM reports smaller errors at several of them, including exact zeros at $x = 0$ and $x = 0.2$. Near the right endpoint, however, the present eighth-order FVF-OHAM gives substantially smaller errors. At $x = 1$, the lower and upper errors are reduced by factors of approximately 353 and 228, respectively. Over the full OHAM verification grid, its maximum error 4.306×10^{-7} is 91.6% smaller than the maximum reported BWM error 5.1×10^{-6}.

5. Conclusion

This paper has developed FVF-OHAM, a formulation of the Optimal Homotopy Asymptotic Method for solving second-kind linear fuzzy Volterra-Fredholm

TABLE 5. Absolute-error comparison between BWM and the eighth-order original FVF-OHAM for Example 2 at $r = 0$.

x	BWM lower	FVF-OHAM lower	BWM upper	FVF-OHAM upper
0.0	0	1.859×10^{-8}	0	2.405×10^{-8}
0.2	0	1.669×10^{-7}	0	2.487×10^{-7}
0.4	0	1.239×10^{-7}	9.6×10^{-9}	1.850×10^{-7}
0.6	1.0×10^{-9}	1.730×10^{-7}	7.4×10^{-9}	2.569×10^{-7}
0.8	3.2×10^{-8}	1.292×10^{-7}	5.3×10^{-8}	1.920×10^{-7}
1.0	4.5×10^{-6}	1.274×10^{-8}	5.1×10^{-6}	2.237×10^{-8}

integral equations. The parametric cut r representation was used to transform the fuzzy equation into two crisp Volterra-Fredholm integral equations corresponding to the lower and upper solution limits. A composite integral operator was introduced to incorporate the Volterra and Fredholm terms within a unified formulation. The correction functions were generated directly from the original OHAM deformation recursion, and the convergence-control parameters were determined within that formulation through nonlinear least-squares minimization of the governing-equation residual.

The method was evaluated using two benchmark problems with known exact solutions. In Example 1, the eighth-order FVF-OHAM produced substantially smaller errors than the published HPM result at the reported comparison point. In Example 2, comparison with BWM was mixed at the interior points, whereas FVF-OHAM was markedly more accurate near $x = 1$ and had a smaller maximum error under the reported computational settings. In both examples, the approximations preserved the required fuzzy ordering and the lower and upper components coincided at the crisp level $r = 1$.

The present study is restricted to second-kind linear fuzzy Volterra–Fredholm integral equations, although the OHAM corrections and residual generally depend nonlinearly on the convergence-control parameters. Future work may extend the formulation to nonlinear fuzzy Volterra–Fredholm equations and establish additional existence, uniqueness, stability, and convergence results under suitable operator conditions.

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[1]School of Mathematical Sciences, Universiti Sains Malaysia, 11800 USM, Pulau Pinang, Malaysia.

[2]Nonlinear Dynamics Research Center, Ajman University, Ajman, United Arab Emirates.

[3]Faculty of Education and Arts, Sohar University, Sultanate of Oman.

[4]Department of Mathematics, Universiti Sains Malaysia, Malaysia.

[5]Ministry of Education, Hashemite Kingdom of Jordan.
Email address: moath.alzoubi@yahoo.com