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AN M-ITERATION PROCESS FOR KHAN MAPPINGS

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Abstract. We study Khan mappings, governed by a rational inequality, from the viewpoint of iterative fixed point approximation. A convex combination estimate shows that such a mapping is dominated by the larger of its two displacements. From this we deduce strong convergence of the M-iteration process to the unique fixed point in a uniformly convex Banach space, together with stability and a data dependence estimate. Every approximate fixed point sequence converges strongly, even without continuity. A two-parameter condition is introduced which contains the rational, the nonexpansive and the Suzuki classes, and exhibits a sharp threshold for an intrinsic displacement bound.

Keywords. Khan mapping, M-iteration, fixed point, uniformly convex Banach space, stability
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1. Introduction and preliminaries

Throughout, X is a real Banach space, $C \subset X$ a nonempty subset, and $\mathrm{Fix}(T) = \{x \in C : Tx = x\}$ the fixed-point set of a self-mapping $T\colon C \to C$. The development of metric fixed-point theory for nonexpansive and generalized nonexpansive mappings stems from the works of Browder, Göhde and Kirk, and the role played by the geometry of X (uniform convexity, normal structure, quasi-normal structure) is by now classical [15, 5].

In 1976, M. Khan [18] introduced a class of mappings governed by a rational expression; a corrected version of his theorem was given by Fisher [13]. Generalized contractive conditions of this kind remain an active topic; see, for instance, [4]. We recall the nonexpansive-type version studied in [1].

Definition 1.1. Let (X, d) be a metric space. A self-mapping $T\colon X \to X$ is a *Khan mapping* if, for all $x, y \in X$ with $x \neq y$,

$$d(Tx, Ty) \;\leq\; \frac{d(x, Tx)\, d(x, Ty) + d(y, Ty)\, d(y, Tx)}{d(x, Ty) + d(y, Tx)}, \tag{1.1}$$

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together with $d(x,Ty)+d(y,Tx)\neq 0$ for $x\neq y$. If there is $k\in[0,1)$ such that $d(Tx,Ty)\le k\cdot$(right-hand side of (1.1)) for $x\neq y$ (and $d(Tx,Ty)=0$ when the denominator vanishes), then T is a *Khan contraction*.

Throughout, "Khan mapping" refers exclusively to the rational class of Definition 1.1, and not to the unrelated Picard–Mann hybrid iteration of S. H. Khan [19], to which part of the literature also attaches that name. We work in a normed space, so $d(x,y)=\|x-y\|$. The following two estimates, due to Ray–Singh [23] (see also [1, Remark 3.1]), will be used repeatedly.

Lemma 1.2. *Let C be a nonempty subset of a Banach space X and let $T\colon C\to C$ be a Khan mapping. Then, for all $x_0\in C$ and all integers $n,m\ge 1$,* (i) $\|T^{n+1}x_0-T^nx_0\|\le\|T^nx_0-T^{n-1}x_0\|$; (ii) $\|T^mx_0-T^nx_0\|\le\|x_0-Tx_0\|$.

We also recall the geometric core established in [1].

Theorem 1.3. *Let C be a weakly compact convex subset of a Banach space X possessing quasi-normal structure, and let $T\colon C\to C$ be a continuous Khan mapping. Then T has a unique fixed point $z_0\in C$, and T is quasi-nonexpansive, i.e. $\|Tz-z_0\|\le\|z-z_0\|$ for all $z\in C$.*

Recall that a Banach space X is *uniformly convex* (UC) if for every $\varepsilon\in(0,2]$ there is $\delta(\varepsilon)>0$ such that $\|x\|\le 1$, $\|y\|\le 1$, $\|x-y\|\ge\varepsilon$ imply $\|(x+y)/2\|\le 1-\delta(\varepsilon)$. Every UC space is reflexive and strictly convex; closed bounded convex subsets of UC spaces are weakly compact and have normal structure, hence quasi-normal structure [15].

We shall use the iteration introduced by Ullah and Arshad [29].

Definition 1.4. Let C be a convex subset of X, $T\colon C\to C$, and $\{\alpha_n\}\subset[a,b]\subset(0,1)$. For $x_1\in C$, the *M-iteration* $\{x_n\}$ is

$$z_n=(1-\alpha_n)x_n+\alpha_nTx_n,\qquad y_n=Tz_n,\qquad x_{n+1}=Ty_n,\qquad n\ge 1. \tag{1.2}$$

We write Φ_n for the operator $p\mapsto T\big(T((1-\alpha_n)p+\alpha_nTp)\big)$, so that $x_{n+1}=\Phi_n(x_n)$.

Finally we recall the Senter–Dotson condition [25], the convexity lemma of Schu [24], and a standard recurrence estimate [6, Lemma 1.6].

Definition 1.5. $T\colon C\to C$ with $\operatorname{Fix}(T)\neq\varnothing$ satisfies *Condition (I)* if there is a nondecreasing $f\colon[0,\infty)\to[0,\infty)$ with $f(0)=0$ and $f(r)>0$ for $r>0$ such that $\|x-Tx\|\ge f(\operatorname{dist}(x,\operatorname{Fix}(T)))$ for all $x\in C$.

Lemma 1.6. *Let X be UC and $\{\alpha_n\}\subset[a,b]\subset(0,1)$. If $\{u_n\},\{v_n\}\subset X$ satisfy $\limsup_n\|u_n\|\le c$, $\limsup_n\|v_n\|\le c$ and $\lim_n\|(1-\alpha_n)u_n+\alpha_nv_n\|=c$ for some $c\ge 0$, then $\lim_n\|u_n-v_n\|=0$.*

Lemma 1.7. *Let $\{c_n\}\subset[0,\infty)$ and $\{\varepsilon_n\}\subset[0,\infty)$ with $\varepsilon_n\to 0$, and let $\theta\in[0,1)$. If $c_{n+1}\le\theta c_n+\varepsilon_n$ for all n, then $c_n\to 0$.*

2. A convex-combination estimate

The following elementary observation is the lever for the rest of the paper.

Lemma 2.1. *Let $T\colon C \to C$ be a Khan mapping and $x, y \in C$ with $x \neq y$ and $\|x - Ty\| + \|y - Tx\| \neq 0$. Then*

$$\|Tx - Ty\| \leq \max\{\|x - Tx\|, \|y - Ty\|\}. \tag{2.1}$$

If T is a Khan contraction with constant k, then $\|Tx - Ty\| \leq k \max\{\|x - Tx\|, \|y - Ty\|\}$, and consequently

$$\|Tx - Ty\| \leq \frac{k}{1-k}\|x - y\| + \frac{2k}{1-k}\|x - Tx\|. \tag{2.2}$$

Proof. Put $A = \|x - Tx\|$, $B = \|y - Ty\|$, $p = \|x - Ty\|$, $q = \|y - Tx\|$, with $p + q > 0$. By (1.1),

$$\|Tx - Ty\| \leq \frac{Ap + Bq}{p+q} = \frac{p}{p+q}A + \frac{q}{p+q}B,$$

a convex combination of A and B; hence it is at most $\max\{A, B\} \leq A + B$, proving (2.1). The contraction case multiplies the bound by k. For (2.2), from $\|Tx - Ty\| \leq k(A + B)$ and $B = \|y - Ty\| \leq \|y - x\| + \|x - Tx\| + \|Tx - Ty\| = \|x - y\| + A + \|Tx - Ty\|$ we get $\|Tx - Ty\| \leq kA + k(\|x - y\| + A + \|Tx - Ty\|)$, i.e. $(1 - k)\|Tx - Ty\| \leq k\|x - y\| + 2kA$, which is (2.2). □

Remark 2.2. Estimate (2.1) re-proves uniqueness at once: if $Tx = x$, $Ty = y$ and $x \neq y$, then the denominator equals $2\|x - y\| > 0$ and $\|x - y\| = \|Tx - Ty\| \leq \max\{0, 0\} = 0$, a contradiction. Inequality (2.2) shows that a Khan contraction with $k < \frac{1}{2}$ is an almost contraction in the sense of Berinde [6], with constant $\delta = \frac{k}{1-k} < 1$ and $L = \frac{2k}{1-k}$. This is used for data dependence in Section 4.

3. Strong convergence of the M-iteration

Theorem 3.1. *Let X be a uniformly convex Banach space, $C \subset X$ a nonempty closed bounded convex set, and $T\colon C \to C$ a continuous Khan mapping. Then the M-iteration $\{x_n\}$ defined by (1.2), with $\{\alpha_n\} \subset [a, b] \subset (0, 1)$, converges strongly to the unique fixed point z_0 of T.*

Proof. Since X is UC it is reflexive and strictly convex, so C is weakly compact and has normal, hence quasi-normal, structure. By Theorem 1.3, T has a unique fixed point z_0 and is quasi-nonexpansive. Moreover, applying (1.1) with $y = z_0 = Tz_0$ gives, for $x \neq z_0$,

$$\|Tx - z_0\| = \|Tx - Tz_0\| \leq \frac{\|x - Tx\|\,\|x - z_0\|}{\|x - z_0\| + \|Tx - z_0\|} \leq \|x - Tx\|,$$

the last step because $\|x - z_0\| \leq \|x - z_0\| + \|Tx - z_0\|$. Combining with $\|Tx - z_0\| \geq \|x - z_0\| - \|x - Tx\|$ yields $\|x - z_0\| \leq 2\|x - Tx\|$, that is

$$\|x - Tx\| \;\geq\; \tfrac{1}{2}\|x - z_0\| \;=\; \tfrac{1}{2}\operatorname{dist}(x, \operatorname{Fix}(T)), \tag{3.1}$$

so T satisfies Condition (I) with $f(r) = r/2$.

We next establish Fejér monotonicity. Using quasi-nonexpansiveness along (1.2),

$$\|z_n - z_0\| \leq (1 - \alpha_n)\|x_n - z_0\| + \alpha_n\|Tx_n - z_0\| \leq \|x_n - z_0\|,$$

$\|y_n - z_0\| = \|Tz_n - z_0\| \le \|z_n - z_0\|$, and $\|x_{n+1} - z_0\| = \|Ty_n - z_0\| \le \|y_n - z_0\|$. Hence $\|x_{n+1} - z_0\| \le \|y_n - z_0\| \le \|z_n - z_0\| \le \|x_n - z_0\|$, so $\{\|x_n - z_0\|\}$ is nonincreasing and converges to some $r \ge 0$. The squeeze gives $\|z_n - z_0\| \to r$ and $\|y_n - z_0\| \to r$.

For asymptotic regularity, if $r = 0$ then $x_n \to z_0$ and $\|x_n - Tx_n\| \le \|x_n - z_0\| + \|z_0 - Tx_n\| \le 2\|x_n - z_0\| \to 0$. If $r > 0$, write $z_n - z_0 = (1 - \alpha_n)(x_n - z_0) + \alpha_n(Tx_n - z_0)$ with $\|x_n - z_0\| \to r$, $\|Tx_n - z_0\| \le \|x_n - z_0\| \to r$, and $\|z_n - z_0\| \to r$. Lemma 1.6 (with $u_n = x_n - z_0$, $v_n = Tx_n - z_0$, $c = r$) gives $\|x_n - Tx_n\| = \|u_n - v_n\| \to 0$.

Finally, by (3.1) we have $\frac{1}{2}\operatorname{dist}(x_n, \operatorname{Fix}(T)) \le \|x_n - Tx_n\| \to 0$, hence $\operatorname{dist}(x_n, \operatorname{Fix}(T)) = \|x_n - z_0\| \to 0$, i.e. $x_n \to z_0$ strongly. □

Remark 3.2. Two points are worth stressing. (a) No compactness or demicompactness assumption is needed: Khan mappings satisfy Condition (I) automatically via (3.1). (b) In the example of Section 5 the M-iteration is observed to converge faster than the one-step Krasnoselskii–Mann process of [1, Prop. 4.4], although no general comparative-rate theorem is claimed here.

4. Stability and data dependence for Khan contractions

For a Khan contraction the geometric machinery of Section 3 is not needed: existence, uniqueness and a sharp contraction estimate toward the fixed point come directly from (1.1).

Lemma 4.1. *Let $T\colon C \to C$ be a Khan contraction with constant $k \in [0, 1)$ and (unique) fixed point z_0. Then*

$$\|Tx - z_0\| \le k\|x - z_0\| \qquad \text{for all } x \in C. \tag{4.1}$$

Proof. For $x = z_0$ this is trivial. For $x \ne z_0$, apply Definition 1.1 with $y = z_0 = Tz_0$: $\|Tx - z_0\| = \|Tx - Tz_0\| \le k\,\|x - Tx\|\,\|x - z_0\|/(\|x - z_0\| + \|Tx - z_0\|)$. Writing $a = \|x - z_0\|$, $b = \|Tx - z_0\|$ and using $\|x - Tx\| \le a + b$ gives $b \le k(a + b)\,a/(a + b) = ka$. □

Lemma 4.1 already yields convergence of the M-iteration for Khan contractions, without uniform convexity or continuity, with a computable geometric rate.

Lemma 4.2. *Under the hypotheses of Lemma 4.1, with $\{\alpha_n\} \subset [a, b] \subset (0, 1)$,*

$$\|\Phi_n(p) - z_0\| \le \theta\|p - z_0\| \quad (p \in C), \qquad \theta := k^2\big(1 - a(1 - k)\big) < 1.$$

In particular the M-iterates satisfy $\|x_n - z_0\| \le \theta^{n-1}\|x_1 - z_0\| \to 0$.

Proof. Let $z = (1 - \alpha_n)p + \alpha_n Tp$, $w = Tz$, $\Phi_n(p) = Tw$. By (4.1), $\|z - z_0\| \le (1 - \alpha_n)\|p - z_0\| + \alpha_n\|Tp - z_0\| \le (1 - \alpha_n(1 - k))\|p - z_0\|$, and applying (4.1) twice more, $\|\Phi_n(p) - z_0\| = \|Tw - z_0\| \le k\|w - z_0\| \le k^2\|z - z_0\| \le k^2(1 - \alpha_n(1 - k))\|p - z_0\|$. Since $\alpha_n \ge a$ and $1 - k > 0$, $1 - \alpha_n(1 - k) \le 1 - a(1 - k)$, and $\theta = k^2(1 - a(1 - k)) < k^2 < 1$. □

Stability.

Definition 4.3. Let $\{x_n\}$, $x_{n+1} = \Phi_n(x_n)$, be the M-iteration. It is *T-stable* if for every $\{p_n\} \subset C$, setting $\varepsilon_n := \|p_{n+1} - \Phi_n(p_n)\|$, one has $\lim_n \varepsilon_n = 0 \iff \lim_n p_n = z_0$.

Theorem 4.4. *Let X be a Banach space, $C \subset X$ closed convex, and $T\colon C \to C$ a Khan contraction with constant $k \in [0,1)$. Then the M-iteration is T-stable.*

Proof. Let z_0 be the unique fixed point and $\theta = k^2(1 - a(1-k)) < 1$ as in Lemma 4.2.

($\Rightarrow$) Suppose $\varepsilon_n \to 0$. By the triangle inequality and Lemma 4.2, $\|p_{n+1} - z_0\| \le \|p_{n+1} - \Phi_n(p_n)\| + \|\Phi_n(p_n) - z_0\| \le \varepsilon_n + \theta\|p_n - z_0\|$. By Lemma 1.7, $\|p_n - z_0\| \to 0$, i.e. $p_n \to z_0$.

($\Leftarrow$) Suppose $p_n \to z_0$. Then $\varepsilon_n = \|p_{n+1} - \Phi_n(p_n)\| \le \|p_{n+1} - z_0\| + \|z_0 - \Phi_n(p_n)\| \le \|p_{n+1} - z_0\| + \theta\|p_n - z_0\| \to 0$. □

Data dependence.

Theorem 4.5. *Let X be a Banach space, $C \subset X$ closed convex, and let $T\colon C \to C$ be a Khan contraction with $k < \frac{1}{2}$, so that (Remark 2.2) T is an almost contraction with $\delta = \frac{k}{1-k} < 1$, $L = \frac{2k}{1-k}$; let z_0 be its fixed point. Let $S\colon C \to C$ admit a fixed point w_0 whose M-iteration $\{u_n\}$ (with the same $\{\alpha_n\} \subset [a,b] \subset (0,1)$) converges to w_0, and suppose $\sup_{x\in C}\|Tx - Sx\| \le \eta$. Then, with $\rho := \delta^2(1 - a(1-\delta)) \in [0,1)$,*

$$\|z_0 - w_0\| \le \frac{\delta^2 + \delta + 1}{1 - \rho}\,\eta.$$

Proof. Let $\{x_n\}$ be the M-iteration of T, which converges to z_0 by Lemma 4.2; write its steps z_n, y_n and those of S as $p_n = (1-\alpha_n)u_n + \alpha_n S u_n$, $q_n = Sp_n$, $u_{n+1} = Sq_n$. Using $\|Tx - Ty\| \le \delta\|x - y\| + L\|x - Tx\|$ and $\|Tv - Sv\| \le \eta$,

$$\begin{aligned}
\|z_n - p_n\| &\le (1-\alpha_n)\|x_n - u_n\| + \alpha_n\big(\delta\|x_n - u_n\| + L\|x_n - Tx_n\| + \eta\big)\\
&= (1 - \alpha_n(1-\delta))\|x_n - u_n\| + \alpha_n L\|x_n - Tx_n\| + \alpha_n\eta,\\
\|y_n - q_n\| &= \|Tz_n - Sp_n\| \le \delta\|z_n - p_n\| + L\|z_n - Tz_n\| + \eta,\\
\|x_{n+1} - u_{n+1}\| &= \|Ty_n - Sq_n\| \le \delta\|y_n - q_n\| + L\|y_n - Ty_n\| + \eta.
\end{aligned}$$

Substituting the first two lines into the third and putting $a_n = \|x_n - u_n\|$,

$$a_{n+1} \le \delta^2(1 - \alpha_n(1-\delta))a_n + (\delta^2\alpha_n + \delta + 1)\eta + \sigma_n,$$

where $\sigma_n = L(\delta^2\alpha_n\|x_n - Tx_n\| + \delta\|z_n - Tz_n\| + \|y_n - Ty_n\|)$. Since $x_n \to z_0$, also $z_n \to z_0$ and $y_n \to z_0$ (Lemma 4.2 and (4.1)), so each displacement, hence σ_n, tends to 0. Using $\alpha_n \ge a$ and $\alpha_n \le 1$, $a_{n+1} \le \rho\, a_n + (\delta^2 + \delta + 1)\eta + \sigma_n$, $\rho = \delta^2(1 - a(1-\delta)) < 1$. As $a_n \to \|z_0 - w_0\|$, passing to the limit gives $\|z_0 - w_0\| \le \rho\|z_0 - w_0\| + (\delta^2 + \delta + 1)\eta$, i.e. the claim. □

5. Numerical experiment

We illustrate Theorem 3.1 on $T\colon [-1,1]\to[-1,1]$, $T(x)=-\frac{1}{2}x$, in the (uniformly convex) space $\mathbb{R}$. Writing $a=\frac{3}{2}|x|$, $b=\frac{3}{2}|y|$, $p=|x+y/2|$, $q=|y+x/2|$, the inequality $\|Tx-Ty\|\le\frac{2}{3}(ap+bq)/(p+q)$ reduces to $2(|x|p+|y|q)-|x-y|(p+q)\ge 0$, which a short case analysis on the signs of x and y confirms, with equality precisely when $y=-x\neq 0$. Hence T is a continuous Khan contraction with sharp constant $k=\frac{2}{3}$ and unique fixed point $z_0=0$. By contrast, the isometry $x\mapsto 1-x$ is not a Khan mapping, since the pairs $(x,1-x)$ make the denominator in (1.1) vanish—a concrete witness of the strictness of Definition 1.1.

Table 1 reports the error $\|x_n-z_0\|$ from $x_1=1$ with $\alpha_n\equiv\frac{1}{2}$ for the Picard, Krasnoselskii–Mann and M-iterations. Since this T is linear with $\|T'\|=\frac{1}{2}$, the observed per-step factors are exactly $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{16}$ respectively, while the guaranteed class bound of Lemma 4.2, $\theta=k^2(1-\frac{1}{2}(1-k))=\frac{20}{54}\approx 0.37$, is looser, as expected. To reach tolerance 10^{-8} the M-iteration needs 7 steps and Krasnoselskii–Mann 14, while Picard has not reached it after 14.

n	Picard	Krasnoselskii–Mann	M-iteration
0	1.000	1.000	1.000
1	5.000×10^{-1}	2.500×10^{-1}	6.250×10^{-2}
2	2.500×10^{-1}	6.250×10^{-2}	3.906×10^{-3}
3	1.250×10^{-1}	1.562×10^{-2}	2.441×10^{-4}
4	6.250×10^{-2}	3.906×10^{-3}	1.526×10^{-5}
5	3.125×10^{-2}	9.766×10^{-4}	9.537×10^{-7}
6	1.562×10^{-2}	2.441×10^{-4}	5.960×10^{-8}
7	7.812×10^{-3}	6.104×10^{-5}	3.725×10^{-9}
8	3.906×10^{-3}	1.526×10^{-5}	2.328×10^{-10}

Table 1. Error $\|x_n-z_0\|$ for $T(x)=-\frac{1}{2}x$, $x_1=1$, $\alpha_n\equiv\frac{1}{2}$.

6. A unified class and an application to integral equations

The rational inequality (1.1) couples the displacements $\|x-Tx\|$, $\|y-Ty\|$ rather than the distance $\|x-y\|$, so it is not directly matched by the solution operator of an integral equation, which is naturally Lipschitz. We therefore introduce a slightly larger class that contains both Khan contractions and Banach contractions, for which the iteration theory of Sections 3–4 still holds and through which integral equations enter transparently.

Definition 6.1. Let (X,d) be a metric space. A self-mapping $T\colon X\to X$ is a *generalized Khan-type contraction* if there is $k\in[0,1)$ such that, for all $x,y\in X$ with $x\neq y$,

$$d(Tx,Ty)\le k\,N(x,y),\qquad N(x,y)=\max\{d(x,y),R(x,y)\},\tag{6.1}$$

where $R(x,y) = \dfrac{d(x,Tx)\,d(x,Ty) + d(y,Ty)\,d(y,Tx)}{d(x,Ty) + d(y,Tx)}$, with the convention $R(x,y) = 0$ when the denominator vanishes.

Proposition 6.2. (a) *Every Khan contraction with constant k is a generalized Khan-type contraction with the same k.* (b) *Every Banach contraction ($d(Tx,Ty) \le c\,d(x,y)$, $c<1$) is a generalized Khan-type contraction with $k=c$.*

Proof. (a) $d(Tx,Ty) \le k\,R(x,y) \le k\,N(x,y)$. (b) $d(Tx,Ty) \le c\,d(x,y) \le c\,N(x,y)$. □

Theorem 6.3. *Let (X,d) be complete and $T\colon X \to X$ a generalized Khan-type contraction with constant $k<1$. Then T has a unique fixed point z_0, and the Picard sequence $\{T^n x_0\}$ converges to z_0 for every $x_0 \in X$.*

Proof. Write $x_n = T^n x_0$ and assume $x_n \neq x_{n+1}$ for all n (otherwise some x_n is a fixed point and the orbit is eventually constant). With $Tx_{n-1} = x_n$ and $Tx_n = x_{n+1}$, and since $d(x_n, Tx_{n-1}) = 0$, $R(x_{n-1},x_n) = d(x_{n-1},x_n)d(x_{n-1},x_{n+1})/d(x_{n-1},x_{n+1}) = d(x_{n-1},x_n)$ when $d(x_{n-1},x_{n+1}) > 0$, and $R(x_{n-1},x_n) = 0$ otherwise; in either case $R(x_{n-1},x_n) \le d(x_{n-1},x_n)$, so $N(x_{n-1},x_n) = d(x_{n-1},x_n)$. Hence $d(x_n,x_{n+1}) = d(Tx_{n-1},Tx_n) \le k\,d(x_{n-1},x_n) \le \cdots \le k^n d(x_0,x_1)$, so $\{x_n\}$ is Cauchy and converges to some z_0. Moreover $R(x_n,z_0) \to 0$ as $n \to \infty$ (numerator $\to 0$, denominator $\to d(z_0,Tz_0)$; if $d(z_0,Tz_0) = 0$ we are done). Thus $N(x_n,z_0) \to 0$ and $d(x_{n+1},Tz_0) = d(Tx_n,Tz_0) \le k\,N(x_n,z_0) \to 0$, while $x_{n+1} \to z_0$; hence $Tz_0 = z_0$. Uniqueness: if $Tz_0 = z_0$, $Tw_0 = w_0$, $z_0 \neq w_0$, then $R(z_0,w_0) = 0$, $N(z_0,w_0) = d(z_0,w_0)$ and $d(z_0,w_0) \le k\,d(z_0,w_0)$ forces $z_0 = w_0$. □

Proposition 6.4. *Let C be a closed convex subset of a Banach space X and $T\colon C \to C$ a generalized Khan-type contraction with constant $k<1$ and fixed point z_0. Then $\|Tx - z_0\| \le k\|x - z_0\|$ for all $x \in C$; consequently the M-iteration converges to z_0 with $\|x_n - z_0\| \le \theta^{n-1}\|x_1 - z_0\|$, $\theta = k^2(1-a(1-k))$ (Lemma 4.2), the process is T-stable (Theorem 4.4), and for $k < \frac{1}{2}$ it is data-dependent with $\delta = \frac{k}{1-k}$ (Theorem 4.5).*

Proof. Taking $y = z_0$ in (6.1), $R(x,z_0) = \|x - Tx\|\,\|x - z_0\|/(\|x - z_0\| + \|Tx - z_0\|) \le \|x - z_0\|$ (since $\|x - Tx\| \le \|x - z_0\| + \|Tx - z_0\|$), so $N(x,z_0) = \|x - z_0\|$ and $\|Tx - z_0\| \le k\|x - z_0\|$. This is exactly (4.1); Lemma 4.2 and Theorems 4.4, 4.5 used only (4.1) (and, for data dependence, the almost-contraction estimate, which follows from (6.1) since $d(Tx,Ty) \le k\max\{d(x,y), d(x,Tx) + d(y,Ty)\}$ yields (2.2) for $k < \frac{1}{2}$). Hence all conclusions carry over. □

We now apply this to integral equations. Let $C[a,b]$ carry the sup-norm $\|x\|_\infty = \sup_{t\in[a,b]}|x(t)|$, under which it is a Banach space.

Theorem 6.5. *Consider*

$$x(t) = g(t) + \lambda \int_a^b K(t,s)\,f\big(s,x(s)\big)\,ds, \qquad t \in [a,b], \tag{6.2}$$

where $g \in C[a,b]$, $K \in C([a,b]^2)$ with $\kappa := \sup_t \int_a^b |K(t,s)|\,ds$, and $f\colon [a,b]\times\mathbb{R} \to \mathbb{R}$ is continuous with $|f(s,u) - f(s,v)| \le L_f|u - v|$. If

$$c := |\lambda|\,\kappa\,L_f < 1, \tag{6.3}$$

then (6.2) *has a unique solution* $x^* \in C[a,b]$*, and for any* $x_1 \in C[a,b]$ *the* M*-iteration converges uniformly to* x^* *with* $\|x_n - x^*\|_\infty \le \theta^{n-1}\|x_1 - x^*\|_\infty$*,* $\theta = c^2(1 - a_0(1-c))$*,* $a_0 = \inf_n \alpha_n$*.*

Proof. The operator $(Tx)(t) = g(t) + \lambda \int_a^b K(t,s) f(s,x(s))\,ds$ maps $C[a,b]$ into itself. For $x, y \in C[a,b]$,

$$|(Tx)(t)-(Ty)(t)| \le |\lambda| \int_a^b |K(t,s)|\,|f(s,x(s))-f(s,y(s))|\,ds \le |\lambda| L_f \Big(\int_a^b |K(t,s)|\,ds \Big) \|x-y\|_\infty,$$

so $\|Tx-Ty\|_\infty \le c\|x-y\|_\infty$ with $c < 1$. By Proposition 6.2(b), T is a generalized Khan-type contraction with $k = c$; Theorem 6.3 gives the unique solution x^*, and Proposition 6.4 the M-iteration estimate. □

Corollary 6.6. *Under the hypotheses of Theorem 6.5 with* $\int_a^b$ *replaced by* $\int_a^t$ *in* (6.2)*, and* $\kappa_V := \sup_t \int_a^t |K(t,s)|\,ds$*, the same conclusions hold provided* $|\lambda| \kappa_V L_f < 1$*.*

Example 6.7. For (6.2) with $[a,b] = [0,1]$, $g(t) = t$, $\lambda = \frac{1}{2}$, $K(t,s) = ts$, $f(s,u) = u$, one has $c = \frac{1}{4} < 1$ and the exact solution $x^*(t) = \frac{6}{5}t$. Since the kernel is degenerate and f, g are linear, T maps $x(t) = ct$ to $(Tx)(t) = t(1 + c/6)$, so from $x_1 \equiv 0$ every iterate stays linear and the computation reduces to the scalar recursion $c \mapsto 1 + c/6$ in exact arithmetic, with no quadrature error. Table 2 reports the resulting exact values of $\|x_n - x^*\|_\infty = |c_n - \frac{6}{5}|$ with $\alpha_n \equiv \frac{1}{2}$: the M-iteration decays geometrically with observed ratio ≈ 0.016, in agreement with θ, while Picard is markedly slower and Krasnoselskii–Mann slower still, the kernel operator being positive.

n	Picard	Krasnoselskii–Mann	M-iteration
0	1.200	1.200	1.200
1	2.000×10^{-1}	7.000×10^{-1}	1.944×10^{-2}
2	3.333×10^{-2}	4.083×10^{-1}	3.151×10^{-4}
3	5.556×10^{-3}	2.382×10^{-1}	5.105×10^{-6}
4	9.259×10^{-4}	1.389×10^{-1}	8.273×10^{-8}
5	1.543×10^{-4}	8.105×10^{-2}	1.340×10^{-9}

TABLE 2. Exact error $\|x_n - x^*\|_\infty$ for the Fredholm equation of Example 6.7.

Remark 6.8. Proposition 6.2 makes the role of each ingredient transparent: the integral operator enters (6.1) through the metric term $d(x,y)$ of N, while the rational term R accommodates the Khan mappings of Sections 2–4.

7. Demiclosedness and strong convergence of approximate fixed point sequences

The argument of Theorem 3.1 yields strong convergence without recourse to Opial's property. The reason is structural: a Khan mapping satisfies an intrinsic

displacement lower bound that forces every approximate fixed point sequence to converge strongly, so the weak-topology machinery (demiclosedness via Opial) is not needed.

Proposition 7.1. *Let X be a Banach space, $C \subset X$ a weakly compact convex set, and $T\colon C \to C$ a continuous Khan mapping with (unique) fixed point z_0. Then:*

(i) $\|x - Tx\| \geq \frac{1}{2}\|x - z_0\|$ *for all* $x \in C$ *(Condition (I) with* $f(r) = r/2$*);*
(ii) *every sequence* $\{x_n\} \subset C$ *with* $\|x_n - Tx_n\| \to 0$ *converges strongly to* z_0*;*
(iii) $\mathrm{I} - T$ *is demiclosed at* 0*: if* $x_n \rightharpoonup z$ *and* $\|x_n - Tx_n\| \to 0$*, then* $z = z_0$ *and* $(\mathrm{I} - T)z = 0$*;*
(iv) *consequently every iteration whose iterates form an approximate fixed point sequence—in particular the M- and Krasnoselskii–Mann processes when X is uniformly convex, and the Picard process when T is a Khan contraction—converges strongly, hence weakly, to* z_0*.*

Proof. (i) Taking $y = z_0 = Tz_0$ in (1.1), for $x \neq z_0$ one has $\|Tx - z_0\| \leq \|x - Tx\|\,\|x - z_0\|/(\|x - z_0\| + \|Tx - z_0\|)$. With $a = \|x - z_0\|$, $b = \|Tx - z_0\|$ this reads $b(a+b) \leq \|x - Tx\|\,a$, whence $\|x - Tx\| \geq b(a+b)/a \geq b$; combined with the reverse triangle inequality $\|x - Tx\| \geq a - b$, we get $\|x - Tx\| \geq \max\{a - b, b\} \geq a/2$. (ii) By (i), $\|x_n - z_0\| \leq 2\|x_n - Tx_n\| \to 0$. (iii) Strong convergence to z_0 implies weak convergence to z_0; as weak limits are unique, $z = z_0 \in \mathrm{Fix}(T)$. (iv) Immediate from (ii), since in each case $\|x_n - Tx_n\| \to 0$ (Theorem 3.1, Lemma 4.2, Proposition 6.4). □

Remark 7.2. A nonexpansive mapping need not satisfy Condition (I), and its approximate fixed point sequences need not converge strongly, which is why the demiclosedness principle is invoked there through Opial's property. For Khan mappings the rational inequality forces the lower bound (i), so the weak-topology machinery is unnecessary; Section 8 makes this dichotomy precise.

8. The intrinsic lower bound in context: a comparison with λ-partially nonexpansive mappings

Proposition 7.1 isolates the feature responsible for the strong convergence obtained above without any compactness or Opial assumption: a Khan mapping satisfies (3.1), so Condition (I) holds *intrinsically* with $f(r) = r/2$. Is such a bound generic among generalized nonexpansive classes, or specific to the rational inequality (1.1)? We answer by comparing with a neighbouring class whose convergence theory must instead borrow the weak-topology machinery.

Recall Suzuki's condition (E) [28, 14]: a mapping $T\colon C \to C$ satisfies *condition* (E_μ) (for some $\mu \geq 1$) if $\|x - Ty\| \leq \mu\|x - Tx\| + \|x - y\|$ for all $x, y \in C$. In 2024, Llorens-Fuster [20] introduced the *λ-partially nonexpansive* (λ-PNE) mappings: for $\lambda \in (0, 1)$ and $T_\lambda = (1 - \lambda)\mathrm{I} + \lambda T$, the mapping T is λ-PNE if $\|Tx - T(T_\lambda x)\| \leq \lambda\|x - Tx\|$ for all $x \in C$. This class properly contains the directionally nonexpansive mappings of Kirk, and the averaged map T_λ is asymptotically regular [20, Thm. 5.6]; hence the Krasnoselskii orbit $x_{n+1} = T_\lambda(x_n)$ is

an approximate fixed point sequence in every closed convex invariant subset, i.e. every λ-PNE mapping has property (A).

Definition 8.1. A mapping $T\colon C \to C$ is an (L)-*type mapping* if (L.a) it has property (A) (an approximate fixed point sequence exists in every nonempty closed convex T-invariant subset), and (L.b) for every approximate fixed point sequence $\{x_n\}$ for T in C and every $x \in C$, $\limsup_n \|x_n - Tx\| \le \limsup_n \|x_n - x\|$.

Proposition 8.2. *Every λ-PNE mapping satisfying condition (E) is an (L)-type mapping.*

Proof. Property (A) is [20, Theorem 5.6]. Let $\{x_n\}$ be any approximate fixed point sequence for T in C. By (E_μ) with first variable x_n and second variable x, $\|x_n - Tx\| \le \mu\|x_n - Tx_n\| + \|x_n - x\|$; taking upper limits and using $\|x_n - Tx_n\| \to 0$ yields $\limsup_n \|x_n - Tx\| \le \limsup_n \|x_n - x\|$, which is (L.b). □

By the fixed-point theory of the (L)-class, a λ-PNE mapping satisfying condition (E) therefore has a fixed point whenever the space has normal structure, and under Opial's property every approximate fixed point sequence converging weakly does so to a fixed point [21, Thms. 4.4, 4.6]. For the strictly smaller class of mappings satisfying condition (L-1), Shukla, Panicker and Vijayasenan [26, Thm. 5] obtain the demiclosedness principle in uniformly convex spaces without invoking Opial's property. Crucially, however, none of these classes possesses the lower bound (3.1).

Proposition 8.3. *Condition (E) does not imply Condition (I): there is no constant $c > 0$ such that every mapping satisfying condition (E), on every domain, obeys $\|x - Tx\| \ge c\,\operatorname{dist}(x, \operatorname{Fix}(T))$.*

Proof. Let C be the closed unit ball of ℓ^2 and $S\colon C \to C$ the right shift, $S(\xi_1, \xi_2, \dots) = (0, \xi_1, \xi_2, \dots)$. As an isometry, S is nonexpansive, hence λ-PNE for every $\lambda \in (0,1)$ and satisfies (E_1), with $\operatorname{Fix}(S) \cap C = \{0\}$. For $N \ge 1$ set $x^{(N)} = N^{-1/2}(\underbrace{1, \dots, 1}_{N}, 0, \dots) \in C$. Then $\|x^{(N)}\| = 1$, so $\operatorname{dist}(x^{(N)}, \operatorname{Fix}(S)) = 1$, while $x^{(N)} - Sx^{(N)} = N^{-1/2}(1, 0, \dots, 0, -1, 0, \dots)$ gives $\|x^{(N)} - Sx^{(N)}\| = \sqrt{2/N} \to 0$. Hence no such $c > 0$ exists. □

In other words, evaluating (E_μ) at a fixed point yields only the *upper* quasi-nonexpansive estimate $\|p - Tx\| \le \|p - x\|$ and no displacement lower bound. Consequently the strong and weak convergence of the Krasnoselskii orbit for λ-PNE mappings satisfying condition (E) genuinely requires one of three external ingredients—compactness of C, Opial's property, or Condition (I) imposed as an additional hypothesis—none of which is needed for the Khan mappings of Sections 3–6.

Remark 8.4. This sharpens Remark 7.2. For a Khan mapping the rational inequality (1.1) forces the lower bound (3.1), so Condition (I) is automatic and the weak-topology machinery is bypassed; condition (E), by contrast, may fail for Khan mappings, and where it holds it supplies only the upper estimate. The two classes are thus complementary: the $(E)/(L)$ class possesses the upper estimate

and the demiclosedness principle but not the intrinsic lower bound, whereas the rational class possesses the intrinsic lower bound—the stronger property for iterative approximation, since by Proposition 7.1(ii) it forces strong convergence of *every* approximate fixed point sequence directly. The lower bound (3.1) is therefore a feature of the rational structure of (1.1), not of generalized nonexpansiveness in itself.

These observations raise three natural questions: can the continuity hypothesis be removed from Theorem 1.3; does the intrinsic lower bound survive when the constant k of Section 6 is replaced by a comparison function; and is there a single condition housing both the rational class and the $(E)/(L)$ class? The remaining sections answer the second and third questions in full and reduce the first to its combinatorial core.

9. Removing continuity: approximate fixed point sequences suffice

In Theorem 1.3—hence in Theorem 3.1 and Proposition 7.1—continuity enters only through the *existence* of the fixed point. We now show that it can be dispensed with entirely once an approximate fixed point sequence is available: completeness alone identifies the limit. The key is that Lemma 2.1 bounds the distance between *images* by the larger of the two *displacements*, so that T maps approximate fixed point sequences to Cauchy sequences.

Theorem 9.1. *Let C be a nonempty closed subset of a Banach space X and let $T\colon C \to C$ be a Khan mapping. The following are equivalent:*

(i) $\mathrm{Fix}(T) \neq \varnothing$;
(ii) $\inf_{x\in C}\|x - Tx\| = 0$, *i.e. T admits an approximate fixed point sequence in C.*

In this case the fixed point z_0 is unique, every approximate fixed point sequence converges strongly to z_0, and $\|x-z_0\| \le 2\|x-Tx\|$ for every $x \in C$. No continuity, convexity, boundedness or reflexivity is assumed.

Proof. (i)⇒(ii) is trivial (take $x_n \equiv z_0$). For (ii)⇒(i), let $\{x_n\} \subset C$ with $\varepsilon_n := \|x_n - Tx_n\| \to 0$.

We first show that $\{x_n\}$ is Cauchy. For $x_n \neq x_m$, Lemma 2.1 gives $\|Tx_n - Tx_m\| \le \max\{\varepsilon_n, \varepsilon_m\}$, and the same bound holds trivially when $x_n = x_m$. Hence

$$\|x_n - x_m\| \le \varepsilon_n + \|Tx_n - Tx_m\| + \varepsilon_m \le 2(\varepsilon_n + \varepsilon_m) \longrightarrow 0 \qquad (n, m \to \infty),$$

so $\{x_n\}$ is Cauchy and converges to some $z \in C$ (C being closed in a Banach space), with $Tx_n \to z$ as well.

It remains to show that $Tz = z$. Suppose $\rho := \|z - Tz\| > 0$. If $x_n = z$ for infinitely many n, then $\varepsilon_n = \rho$ along that subsequence, contradicting $\varepsilon_n \to 0$; so $x_n \neq z$ for all large n, and (1.1) applies to the pair (x_n, z):

$$\|Tx_n - Tz\| \;\le\; \frac{\varepsilon_n\,\|x_n - Tz\| + \rho\,\|z - Tx_n\|}{\|x_n - Tz\| + \|z - Tx_n\|}.$$

As $n \to \infty$ we have $\|x_n - Tz\| \to \|z - Tz\| = \rho > 0$ and $\|z - Tx_n\| \to 0$, so the numerator tends to 0 while the denominator tends to ρ: the right-hand side tends

to 0. But the left-hand side tends to $\|z - Tz\| = \rho$. Hence $\rho \le 0$, a contradiction, and $Tz = z$.

Uniqueness is Remark 2.2. The estimate $\|x - z_0\| \le 2\|x - Tx\|$ is (3.1), whose derivation used only the existence of z_0, never continuity; it makes the convergence of every approximate fixed point sequence immediate: $\|x_n - z_0\| \le 2\varepsilon_n \to 0$. □

Corollary 9.2. *In Theorem 1.3, continuity is needed only to* produce *an approximate fixed point sequence, never to identify its limit: a Khan self-mapping of a weakly compact convex set with quasi-normal structure has a (unique) fixed point as soon as it admits an approximate fixed point sequence. Consequently the open continuity problem of* [1] *reduces to the following:* does every Khan self-mapping of such a set admit an approximate fixed point sequence?

Remark 9.3. Theorem 9.1 strengthens Proposition 7.1 in two directions at once: the identification of the limit is *strong* rather than weak, and it requires neither the continuity used there (through Theorem 1.3) nor any geometry of X. In the language of Section 8, Khan mappings satisfy condition (L.b) along approximate fixed point sequences in the strong quantitative form $\|Tx_n - Ty\| \le \max\{\varepsilon_n, \|y - Ty\|\}$, and it is exactly this displacement-dominated estimate that permits a limit identification with no weak topology at all.

The same mechanism extends to the generalized class of Section 6 when the constant k is replaced by a comparison function, and it settles the second question in the affirmative for the Boyd–Wong class.

Theorem 9.4. *Let $\psi\colon [0,\infty) \to [0,\infty)$ be upper semicontinuous with $\psi(t) < t$ for $t > 0$ (a* Boyd–Wong comparison function [9]*), let C be a nonempty closed bounded subset of a Banach space with $D := \operatorname{diam} C$, and let $T\colon C \to C$ satisfy*

$$\|Tx - Ty\| \;\le\; \psi\big(N(x,y)\big) \qquad (x, y \in C,\ x \ne y),$$

with $N(x,y) = \max\{\|x - y\|, R(x,y)\}$ as in Definition 6.1. Then:

(i) $\operatorname{Fix}(T) \ne \varnothing$ *if and only if* $\inf_{x\in C}\|x - Tx\| = 0$*; in that case the fixed point z_0 is unique and every approximate fixed point sequence converges strongly to z_0;*

(ii) $\|Tx - z_0\| \le \psi(\|x - z_0\|)$ *for all $x \in C$, and consequently*

$$\|x - Tx\| \;\ge\; \|x - z_0\| - \psi\big(\|x - z_0\|\big) \qquad (x \in C);$$

(iii) *T satisfies Condition* (I) *with the nondecreasing modulus*

$$f(0) = 0, \qquad f(r) = \min_{r \le s \le D}\big(s - \psi(s)\big) > 0 \quad (0 < r \le D),$$

an exact analogue of (3.1).

Proof. Throughout, recall from the proof of Lemma 2.1 that $R(x,y)$ is a weighted mean of the two displacements, so $R(x,y) \le \max\{\|x - Tx\|, \|y - Ty\|\}$, and, as in the proof of Proposition 6.4, $R(x, z_0) \le \|x - z_0\|$ whenever $Tz_0 = z_0$.

(ii) If $Tz_0 = z_0$ and $x \ne z_0$, then $N(x, z_0) = \max\{\|x - z_0\|, R(x, z_0)\} = \|x - z_0\|$, so $\|Tx - z_0\| \le \psi(\|x - z_0\|)$; the displacement bound follows from $\|x - z_0\| \le \|x - Tx\| + \|Tx - z_0\|$.

(iii) The function $h(s) = s - \psi(s)$ is lower semicontinuous and strictly positive on $(0, D]$; on the compact interval $[r, D]$ it attains a strictly positive minimum, so $f(r) > 0$, and f is nondecreasing because the minimum is taken over a shrinking interval. Since $\mathrm{Fix}(T) = \{z_0\}$ (uniqueness below), $\mathrm{dist}(x, \mathrm{Fix}(T)) = \|x - z_0\| \le D$, and by (ii), $\|x - Tx\| \ge h(\|x - z_0\|) \ge f(\|x - z_0\|) = f\big(\mathrm{dist}(x, \mathrm{Fix}(T))\big)$.

(i) Necessity is trivial. For sufficiency, let $\varepsilon_n = \|x_n - Tx_n\| \to 0$ and put $L := \limsup_{n,m\to\infty} \|x_n - x_m\| \le D < \infty$. Suppose $L > 0$ and choose $n_k, m_k \to \infty$ with $d_k := \|x_{n_k} - x_{m_k}\| \to L$. Since $R(x_{n_k}, x_{m_k}) \le \max\{\varepsilon_{n_k}, \varepsilon_{m_k}\} \to 0$ and $d_k \to L > 0$, eventually $N(x_{n_k}, x_{m_k}) = d_k$, whence $d_k \le \varepsilon_{n_k} + \|Tx_{n_k} - Tx_{m_k}\| + \varepsilon_{m_k} \le \varepsilon_{n_k} + \varepsilon_{m_k} + \psi(d_k)$. Taking upper limits and using the upper semicontinuity of ψ at L gives $L \le \psi(L) < L$, a contradiction; hence $L = 0$, $\{x_n\}$ is Cauchy, and $x_n \to z \in C$, $Tx_n \to z$. If $\rho := \|z - Tz\| > 0$, then (passing to a tail with $x_n \ne z$, as in Theorem 9.1)

$$R(x_n, z) = \frac{\varepsilon_n \|x_n - Tz\| + \rho \|z - Tx_n\|}{\|x_n - Tz\| + \|z - Tx_n\|} \longrightarrow \frac{0 \cdot \rho + \rho \cdot 0}{\rho} = 0,$$

so $N(x_n, z) = \max\{\|x_n - z\|, R(x_n, z)\} \to 0$ with $N(x_n, z) > 0$, and $\|Tx_n - Tz\| \le \psi(N(x_n, z)) < N(x_n, z) \to 0$, while $\|Tx_n - Tz\| \to \rho$: again $\rho \le 0$, so $Tz = z$. Uniqueness: if $Tz = z$, $Tw = w$ and $z \ne w$, then $R(z, w) = 0$ and $\|z - w\| \le \psi(\|z - w\|) < \|z - w\|$, which is absurd. Finally, by (iii), any approximate fixed point sequence satisfies $f\big(\mathrm{dist}(x_n, \mathrm{Fix}(T))\big) \le \varepsilon_n \to 0$, hence $\|x_n - z_0\| \to 0$ (if $\mathrm{dist}(x_n, \mathrm{Fix}(T)) \ge \rho' > 0$ along a subsequence, then $\varepsilon_n \ge f(\rho') > 0$ there, which is absurd). □

Remark 9.5. Upper semicontinuity is used only in the Cauchy step of (i) and in the positivity of the modulus in (iii); for a merely nondecreasing (Matkowski [22]) comparison function, ψ may jump at the critical value and both arguments break down. Whether the conclusions persist in that case is posed as question (b′) in Section 11.

10. A unifying condition and the phase transition at $\nu = 1$

We finally exhibit a single two-parameter condition containing the rational class, the nonexpansive class and the mappings satisfying condition (E), in which the dichotomy of Section 8 becomes a sharp threshold phenomenon.

Definition 10.1. Let $\mu \ge 0$ and $\nu \in [0, 1]$. A mapping $T\colon C \to C$ satisfies *condition* $(R_{\mu,\nu})$ if

$$\|Tx - Ty\| \;\le\; \mu \max\big\{\|x - Tx\|,\, \|y - Ty\|\big\} \;+\; \nu \|x - y\| \qquad (x, y \in C). \tag{10.1}$$

We call the regime $\nu < 1$ *subcritical* and $\nu = 1$ *critical*.

Proposition 10.2. (a) *Every Khan mapping satisfies* $(R_{1,0})$, *and every Khan contraction with constant* k *satisfies* $(R_{k,0})$ (Lemma 2.1). (b) *Every nonexpansive mapping satisfies* $(R_{0,1})$, *and every Banach* c*-contraction satisfies* $(R_{0,c})$. (c) *Every generalized Khan-type contraction of Definition 6.1 satisfies* $(R_{k,k})$. (d) *Every mapping satisfying condition* (E_μ) *satisfies* $(R_{1+\mu,1})$.

Proof. (a) and (b) are immediate. (c) $\|Tx-Ty\| \le k\max\{\|x-y\|, R(x,y)\} \le k\|x-y\| + k\,R(x,y) \le k\|x-y\| + k\max\{\|x-Tx\|, \|y-Ty\|\}$, using again $R \le$ max of the two displacements. (d) $\|Tx-Ty\| \le \|Tx-x\| + \|x-Ty\| \le \|x-Tx\| + \mu\|x-Tx\| + \|x-y\|$. □

Theorem 10.3. *Let C be a nonempty closed subset of a Banach space and let $T\colon C \to C$ satisfy $(R_{\mu,\nu})$ with $\nu < 1$. Assume moreover that $I - T$ is* strongly demiclosed at 0*: whenever $x_n \to z$ strongly and $\|x_n - Tx_n\| \to 0$, then $Tz = z$. Then:*

(i) *every approximate fixed point sequence of T is Cauchy and converges strongly to a fixed point; consequently* $\mathrm{Fix}(T) \neq \varnothing$ *if and only if* $\inf_{x\in C}\|x-Tx\| = 0$*, and* $\mathrm{Fix}(T)$ *is then a singleton* $\{z_0\}$*;*

(ii) *Condition* (I) *holds intrinsically, with the* linear *modulus*

$$\|x - Tx\| \;\ge\; \frac{1-\nu}{1+\mu}\,\mathrm{dist}\big(x, \mathrm{Fix}(T)\big) \qquad (x \in C).$$

Strong demiclosedness at 0 *is automatic when T is continuous, when T satisfies condition (E), and when T is a Khan mapping.*

Proof. (i) For an approximate fixed point sequence, (10.1) gives $\|x_n - x_m\| \le \varepsilon_n + \|Tx_n - Tx_m\| + \varepsilon_m \le (\varepsilon_n + \varepsilon_m) + \mu\max\{\varepsilon_n, \varepsilon_m\} + \nu\|x_n - x_m\|$, whence

$$(1-\nu)\,\|x_n - x_m\| \;\le\; (1+\mu)\,(\varepsilon_n + \varepsilon_m) \longrightarrow 0.$$

So $\{x_n\}$ is Cauchy, $x_n \to z \in C$, and strong demiclosedness gives $Tz = z$. If $p, q \in \mathrm{Fix}(T)$, then (10.1) yields $\|p - q\| \le \nu\|p - q\|$, so $p = q$.

(ii) With $z_0 \in \mathrm{Fix}(T)$: $\|Tx - z_0\| = \|Tx - Tz_0\| \le \mu\|x - Tx\| + \nu\|x - z_0\|$, hence $\|x - z_0\| \le \|x - Tx\| + \|Tx - z_0\| \le (1+\mu)\|x - Tx\| + \nu\|x - z_0\|$, which rearranges to the claim.

For the last assertion: continuity is clear; for condition (E), $\|x_n - Tz\| \le \mu\varepsilon_n + \|x_n - z\| \to 0$ forces $Tz = \lim x_n = z$; for Khan mappings this is Step 2 of the proof of Theorem 9.1. □

Example 10.4. At $(\mu,\nu) = (1,0)$ the modulus is $\frac{1-0}{1+1} = \frac{1}{2}$: Theorem 10.3(ii) recovers the intrinsic bound (3.1) for Khan mappings *exactly*. At $(k,0)$ it gives $\frac{1}{1+k}$, which for Khan contractions *sharpens* the bound $(1-k)$ obtainable from Lemma 4.1, since $\frac{1}{1+k} - (1-k) = \frac{k^2}{1+k} > 0$. At $(0,c)$ it predicts $\|x - Tx\| \ge (1-c)\,\mathrm{dist}(x, \mathrm{Fix}(T))$ for a Banach c-contraction, giving the modulus $\frac{3}{4}$ for the Fredholm operator of Example 6.7.

Proposition 10.5. *At $\nu = 1$ the hypothesis $(R_{\mu,\nu})$ alone no longer yields conclusion* (ii) *of Theorem 10.3: the right shift on the closed unit ball of ℓ^2 satisfies $(R_{0,1})$ yet admits no displacement lower bound* (Proposition 8.3)*. Since $(R_{\mu,\nu})$ implies $(R_{\mu,\nu'})$ for $\nu \le \nu'$, the representation of a map by a pair (μ,ν) is not unique, so this is a statement about what the hypothesis guarantees, not a classification of mappings. For $\nu < 1$, Condition* (I) *follows from $(R_{\mu,\nu})$ with no geometric assumption on the space; at $\nu = 1$ it does not, and an intrinsic bound must be established by other means.*

Remark 10.6. Condition (10.1) is a max-form relative of the Hardy–Rogers and Bogin conditions $\|Tx-Ty\| \le a\|x-y\|+b\,(\|x-Tx\|+\|y-Ty\|)+c\,(\|x-Ty\|+\|y-Tx\|)$ [17, 7], and the Cauchy mechanism in the subcritical regime is in the spirit of Górnicki's asymptotic-regularity theorems for Kannan-type mappings [16]; for recent Kannan-type developments see also [10]. To the best of our knowledge the subcritical/critical split expressed through Condition (I) has not been isolated in this form before.

11. Concluding remarks and remaining problems

The three questions raised at the end of Section 8 are settled as follows: the comparison-function question affirmatively for every Boyd–Wong comparison function (Theorem 9.4); the common-framework question by Section 10; and the continuity question, reduced by Theorem 9.1 and Corollary 9.2 to the existence of approximate fixed point sequences. Three problems remain open: whether every Khan self-mapping of a weakly compact convex set with quasi-normal structure admits an approximate fixed point sequence, which by Theorem 9.1 would remove continuity from Theorem 1.3 entirely; whether Theorem 9.4 persists for merely nondecreasing (Matkowski [22]) comparison functions, for which upper semicontinuity may fail at jump points; and which structural refinements of the μ-term restore an intrinsic displacement bound on the critical surface $\nu = 1$, the rational weights of (1.1) being one such mechanism.

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