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BESOV REGULARITY AND HAVIN–MAZ'YA POTENTIALS ON MANIFOLDS

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Abstract. We establish some new mapping properties of truncated Havin–Maz'ya potentials on complete noncompact Riemannian manifolds with nonnegative Ricci curvature. In contrast to earlier results based on the Herz-space framework used in the Euclidean theory, we employ the heat semigroup associated to the Laplace–Beltrami operator to define heat-semigroup Besov spaces, and construct a truncated potential from a prescribed Orlicz function. Under nonnegative Ricci curvature, this potential is bounded between the relevant Besov spaces if and only if the manifold satisfies Euclidean volume growth, while sub-Euclidean growth rules out any such boundedness; its gradient satisfies a weak-type endpoint estimate. We also present applications to regularity properties.

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1. Introduction and preliminaries

Nonlinear potential theory grew out of the observation that solutions of p-Laplace-type equations admit sharp pointwise control by nonlinear analogues of the Riesz potential. On $\mathbb{R}^n$, for $0 < \alpha < n$ and $1 < \mathfrak{p} < \infty$,

$$\mathcal{I}_\alpha \mu = (-\Delta)^{-\alpha/2}\mu, \qquad -\Delta \mathcal{F} = \mu, \qquad -\Delta_{\mathfrak{p}} \mathcal{F} = \mu,$$

and for the p-Laplacian the correct substitute for $\mathcal{I}_\alpha \mu$ is the Wolff potential, obtained by inserting the nonlinearity of the equation directly into the ball average before integrating over scales. Havin and Maz'ya's original construction generalizes this further to arbitrary Orlicz growth: let $G \in C^1((0,\infty)) \cap C([0,\infty))$ be an N-function, $g := G'$, $G \in \Delta_2 \cap \nabla_2$, i.e. $p^- \le tg(t)/G(t) \le p^+$ for fixed $1 < p^- \le p^+ < \infty$. The associated truncated Havin–Maz'ya potential [8] of a

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nonnegative Radon measure μ, or of $f \in L^1_{\mathrm{loc}}$, is

$$\mathcal{W}^{R}_{\alpha,G}\mu(\mathfrak{t}) := \int_0^R \iota^{\alpha-1} g^{-1}\big(\iota^{\alpha-\mathfrak{n}}\mu(\mathbf{B}(\mathfrak{t},\iota))\big)\,d\iota, \tag{1.1}$$

$$\mathcal{W}^{R}_{\alpha,G}f(\mathfrak{t}) := \int_0^R \iota^{\alpha-1} g^{-1}\Big(\iota^{\alpha-\mathfrak{n}}\int_{\mathbf{B}(\mathfrak{t},\iota)} |f|\,d\mathfrak{d}\Big)\,d\iota, \tag{1.2}$$

where the truncation at radius $R \le \infty$ localizes the potential and is natural whenever μ carries no decay at infinity. With $G(t) = t^{\mathfrak{p}}$, $g^{-1}(s) = (s/\mathfrak{p})^{1/(\mathfrak{p}-1)}$ and $R = \infty$, (1.1) is, up to the constant $\mathfrak{p}^{-1/(\mathfrak{p}-1)}$, the classical

$$\mathcal{W}^{\mu}_{\alpha,\mathfrak{p}}(\mathfrak{t}) = \int_0^\infty \big(\iota^{\alpha\mathfrak{p}-\mathfrak{n}}\mu(\mathbf{B}(\mathfrak{t},\iota))\big)^{1/(\mathfrak{p}-1)}\,\frac{d\iota}{\iota},$$

and for $\mathfrak{p} = 2$, $\mathcal{W}^{\mu}_{\alpha,2} \sim \mathcal{I}_{2\alpha}\mu$; see Remark 1.2. For a solution of $-\Delta_{\mathfrak{p}}\mathcal{F} = \mu$ one has $\mathcal{F} \approx \mathcal{W}^{\mu}_{1,\mathfrak{p}}$ [17, 23, 3], and for the general Orlicz-growth quasilinear operator

$$-\operatorname{div}\mathcal{A}(x,Du) = f, \qquad \mathcal{A}(x,\xi)\cdot\xi \approx G(|\xi|), \tag{1.3}$$

the sharp two-sided estimate [1, Thm. 2], building on [7], is

$$\begin{aligned} c^{-1}\mathcal{W}^{R}_{1,G}f(\mathfrak{t}) - c\fint_{\mathbf{B}(\mathfrak{t},R)} |u|\,d\mathfrak{d} \;&\le\; u(\mathfrak{t}),\\ u(\mathfrak{t}) \;&\le\; c\,\mathcal{W}^{R}_{1,G}f(\mathfrak{t}) + c\fint_{\mathbf{B}(\mathfrak{t},R)} |u|\,d\mathfrak{d}. \end{aligned} \tag{1.4}$$

This truncated potential, and its boundedness [11, 20] on Herz spaces, was studied on $\mathbb{R}^{\mathfrak{n}}$ by Khan et al. [1]; the present paper develops its counterpart on Riemannian manifolds.

On a complete manifold [21, 26] $\mathbf{M}$ there is no canonical group of dilations, so the ball average $\mu(\mathbf{B}(\mathfrak{t},\iota))/V_{\mathfrak{t}}(\iota)$ appearing implicitly in (1.1) must be produced by another mechanism; the heat kernel supplies exactly this. With heat kernel $\mathbf{S}(\mathfrak{t},\mathfrak{d},\mathfrak{j})$,

$$\begin{aligned} (-\Delta)^{-\alpha/2}\mathcal{G}(\mathfrak{t}) &= \int_{\mathbf{M}} \mathcal{K}^{\alpha}(\mathfrak{t},\mathfrak{d})\mathcal{G}(\mathfrak{d})\,d\mathfrak{d},\\ \mathcal{K}^{\alpha}(\mathfrak{t},\mathfrak{d}) &= \frac{1}{\Gamma(\alpha/2)}\int_0^\infty \mathfrak{j}^{\alpha/2-1}\mathbf{S}(\mathfrak{t},\mathfrak{d},\mathfrak{j})\,d\mathfrak{j}. \end{aligned}$$

By [24], $\mathbf{S}(\mathfrak{t},\mathfrak{d},\mathfrak{j}) \le C_{\mathfrak{n}}\mathcal{V}(\mathfrak{t},\sqrt{\mathfrak{j}})^{-1}\exp(-\mathrm{d}^2(\mathfrak{t},\mathfrak{d})/5\mathfrak{j})$; $\mathcal{K}^{\alpha}$ converges iff $\alpha < \beta$ when $\mathcal{V}(\mathfrak{t},\iota) \ge C_{\mathfrak{t}}\iota^{\beta}$ for large ι; under $\mathrm{Ric} \ge 0$, $\mathcal{V}(\mathfrak{t},\iota) \ge C_{\mathfrak{t}}\iota$ only admits $\alpha < 1$. Li [18]: $\mathcal{V}(\mathfrak{t},\iota) \le C\iota^{\beta}$, $\beta < \mathfrak{n} \Rightarrow (-\Delta)^{-\alpha/2}$ is $(\mathfrak{p},\mathfrak{q})$-unbounded for every $\mathfrak{p},\mathfrak{q} \ge 1$. On $\mathbb{R}^{\mathfrak{n}}$, $\mathcal{K}^{\alpha}(\mathfrak{t},\mathfrak{d}) = C_{\mathfrak{n},\alpha}|\mathfrak{t}-\mathfrak{d}|^{\alpha-\mathfrak{n}}$, so $0 < \alpha < \mathfrak{n}$ is required for integrability.

Following the same heat-kernel subordination, for a nonnegative Radon measure μ, $0 < \alpha < \mathfrak{n}$, $G \in \Delta_2 \cap \nabla_2$, and $0 < R \le \infty$, the truncated Havin–Maz'ya potential on $\mathbf{M}$ is defined by

$$\mathcal{W}^{R,\mu}_{\alpha,G}(\mathfrak{t}) := \int_0^R \iota^{\alpha-1} g^{-1}\Big(\iota^{\alpha-\mathfrak{n}}\int_{\mathbf{M}} \mathbf{S}(\mathfrak{t},\mathfrak{d},\iota^2)\,d\mu(\mathfrak{d})\Big)\,d\iota, \tag{1.5}$$

in which the heat average $\int_{\mathbf{M}} \mathbf{S}(\mathfrak{t},\mathfrak{d},\iota^2)\,d\mu(\mathfrak{d})$ plays the role of $\mu(\mathbf{B}(\mathfrak{t},\iota))/V_{\mathfrak{t}}(\iota)$ in (1.1). This is a modeling choice, not a literal identity: the two agree only up to the doubling/Li–Yau comparability constants recorded in Lemma 1.1 below.

Lemma 1.1. *Under* $\mathrm{Ric} \geq 0$, $\exists c, C > 0$ *(*$= C(\mathrm{n})$*) s.t.* $\forall \mu \geq 0$ *Radon,* $\mathfrak{t} \in \mathbf{M}$, $\iota > 0$,

$$c\,\frac{\mu(\mathbf{B}(\mathfrak{t},\iota))}{V_{\mathfrak{t}}(\iota)} \;\leq\; \int_{\mathbf{M}} \mathbf{S}(\mathfrak{t},\mathfrak{d},\iota^2)\,d\mu(\mathfrak{d}) \;\leq\; C\sum_{k\geq 0} 2^{-k\mathrm{n}}\,\frac{\mu(\mathbf{B}(\mathfrak{t},2^{k+1}\iota))}{V_{\mathfrak{t}}(\iota)}.$$

Proof. By Li–Yau, $\mathbf{S}(\mathfrak{t},\mathfrak{d},\iota^2) \asymp V_{\mathfrak{t}}(\iota)^{-1} e^{-\rho^2(\mathfrak{t},\mathfrak{d})/c\iota^2}$; dyadically decompose $\mathbf{M}\setminus\mathbf{B}(\mathfrak{t},\iota) = \bigsqcup_k \big(\mathbf{B}(\mathfrak{t},2^{k+1}\iota)\setminus\mathbf{B}(\mathfrak{t},2^k\iota)\big)$ and apply Bishop–Gromov to compare $V_{\mathfrak{t}}(2^k\iota)$ with $V_{\mathfrak{t}}(\iota)$. □

Remark 1.2. For $G(t) = t^2$ ($g^{-1}(t) = t/2$) and $V_{\mathfrak{t}}(\iota) \asymp \iota^{\mathrm{n}}$, Lemma 1.1 gives $\int_{\mathbf{M}} \mathbf{S}\,d\mu \asymp \iota^{-\mathrm{n}}\mu(\mathbf{B}(\mathfrak{t},\iota))$, so

$$\mathcal{W}^{R,\mu}_{\alpha,2}(\mathfrak{t}) \asymp \int_0^R \iota^{2\alpha-2\mathrm{n}-1}\mu(\mathbf{B}(\mathfrak{t},\iota))\,d\iota,$$

a Riesz-type potential of order $2\alpha-\mathrm{n}$ [28] up to normalization a heuristic check only; nothing in Section 2 needs an exact identity, only Lemma 1.1 plus Li–Yau/Bishop–Gromov.

For further background, we refer the reader to the literature on measure-data elliptic equations [22], Orlicz growth [14], the nonlocal Orlicz Wiener criterion [19], variable exponent problems [4, 27], integral systems involving Wolff potentials [13], generalized Orlicz supersolutions [5], extremal solutions of the $\mathfrak{p}$-Laplacian [9], discrete Wolff potentials [10], boundedness of Wolff-type potentials [7], gradient estimates for the heat equation [29], and Herz-space Orlicz theory [1].

We use Besov spaces $B^s_{\mathfrak{p},\mathfrak{r}}(\mathbf{M})$, $s \in \mathbb{R}$, $1 \leq \mathfrak{p},\mathfrak{r} \leq \infty$, via the heat semigroup $e^{t\Delta}$, rather than $L^{\mathfrak{p}}(\mathbf{M}) = B^0_{\mathfrak{p},\mathfrak{p}}(\mathbf{M})$ [15]. Main results:

$$\mathcal{W}^R_{\alpha,G} : B^s_{\mathfrak{p},\mathfrak{r}}(\mathbf{M}) \to B^{s+\alpha\phi}_{\mathfrak{q},\mathfrak{r}}(\mathbf{M}) \quad \text{iff} \quad V_{\mathfrak{t}}(r) \gtrsim r^{\mathrm{n}}$$

(extending [18, 1]); $\nabla\mathcal{W}^R_{\alpha,G} : B^0_{1,1} \to B^0_{1,\infty}$; sub-Euclidean growth $V_{\mathfrak{t}}(r) \lesssim r^{\beta}$, $\beta < \mathrm{n}$, forbids every $B^s_{\mathfrak{p},\mathfrak{r}} \to B^{s'}_{\mathfrak{q},\mathfrak{r}}$ bound.

1.1. **Notation.** Throughout, $\mathbf{M}$ denotes a complete noncompact Riemannian manifold, ρ its geodesic distance, and μ its Riemannian measure. For $x \in \mathbf{M}$ and $\rho > 0$ we write $\mathbf{B}(x,\rho)$ for the geodesic ball of radius ρ centered at x, and $V_{\mu,x}(\rho) = \mu(\mathbf{B}(x,\rho))$ for its volume. The symbols ∇ and Δ denote the Riemannian gradient and the Laplace–Beltrami operator, and $\mathbf{S}(x,y,\wp)$ is the heat kernel of $\partial_\wp - \Delta$. We assume $\mathrm{Ric} \geq 0$ throughout. The Gamma function is $\Gamma(\alpha) = \int_0^\infty \wp^{\alpha-1}e^{-\wp}\,d\wp$, and we write $A \asymp B$ to mean that there is a constant $C > 0$, depending only on the fixed structural parameters of the problem, with $C^{-1}B \leq A \leq CB$. We rely throughout on the Bishop–Gromov volume comparison theorem [6, 16] and on the Li–Yau and Littlewood–Paley–Stein gradient bounds for the heat kernel [24, 25].

Definition 1.3. A function $G \in C^1((0,\infty)) \cap C([0,\infty))$ with $G(0) = 0$ is called an N-function if its derivative $g := G'$ is nondecreasing with $g(0^+) = 0$ and $g(\infty) = \infty$. We say $G \in \Delta_2 \cap \nabla_2$ if there are constants $1 < p^- \le p^+ < \infty$ such that

$$p^- \;\le\; \frac{tg(t)}{G(t)} \;\le\; p^+ \qquad \text{for every } t > 0.$$

We say that $\mathbf{M}$ satisfies the doubling property $(\mathcal{D})$ if there is a constant $C_{\mathfrak{p}} > 0$ such that $\mu(\mathbf{B}(\mathfrak{t}, 2\rho)) \le C_{\mathfrak{p}}\, \mu(\mathbf{B}(\mathfrak{t}, \rho))$ for every $\mathfrak{t} \in \mathbf{M}$ and $\rho > 0$.

Lemma 1.4. *Let $G \in \Delta_2 \cap \nabla_2$ have indices p^-, p^+ as above. Then, for every $t > 0$ and every $\lambda \ge 1$,*

$$c\,\lambda^{1/p^+} g^{-1}(t) \;\le\; g^{-1}(\lambda t) \;\le\; C\,\lambda^{1/p^-} g^{-1}(t), \tag{1.6}$$

while for every $t > 0$ and every $0 < \lambda \le 1$,

$$c\,\lambda^{1/p^-} g^{-1}(t) \;\le\; g^{-1}(\lambda t) \;\le\; C\,\lambda^{1/p^+} g^{-1}(t), \tag{1.7}$$

where the constants c, C depend only on p^-, p^+. Moreover g^{-1} is concave, and $(g^{-1})'(t) \le (p^-)^{-1} g^{-1}(t)/t$ for almost every $t > 0$.

Proof. Set $\phi(\tau) := \log G(e^\tau)$. Then $\phi'(\tau) = \frac{tg(t)}{G(t)}\big|_{t=e^\tau} \in [p^-, p^+]$ by hypothesis, so integrating this bound over the interval $[\tau, \tau + \log\lambda]$ for $\lambda \ge 1$ yields

$$\lambda^{p^-} \;\le\; \frac{G(\lambda t)}{G(t)} \;\le\; \lambda^{p^+}.$$

Since $G \in \Delta_2 \cap \nabla_2$ implies $g(t) \asymp G(t)/t$ uniformly in t [14, 19], the same two-sided power bound carries over from G to g. Both G and g are increasing bijections of $(0,\infty)$ onto itself, so inverting this bound gives (1.6), and applying (1.6) with λ replaced by $1/\lambda$ gives (1.7). Finally, from $tg(t) \ge p^- G(t)$ and differentiation of the identity $g(g^{-1}(t)) = t$ we obtain $(g^{-1})'(t) = 1/g'(g^{-1}(t)) \le g^{-1}(t)/(p^- t)$, which is the concavity statement. □

Lemma 1.5. *Let $\mathfrak{s} = \log_2 C_{\mathfrak{p}}$. Then for every $\mathfrak{t}, \mathfrak{d} \in \mathbf{M}$ and every $\theta \ge 1$,*

$$\mu(\mathbf{B}(\mathfrak{t}, \theta\mathfrak{j})) \le C\theta^{\mathfrak{s}} \mu(\mathbf{B}(\mathfrak{t}, \mathfrak{j})), \tag{1.8}$$

$$\mu(\mathbf{B}(\mathfrak{d}, \mathfrak{j})) \le C\Big(1 + \tfrac{\rho(\mathfrak{t},\mathfrak{d})}{\mathfrak{j}}\Big)^{\mathfrak{s}} \mu(\mathbf{B}(\mathfrak{t}, \mathfrak{j})). \tag{1.9}$$

Definition 1.6. Let $s \in \mathbb{R}$, $1 \le \mathfrak{p}, \mathfrak{r} \le \infty$, and fix an integer $m > s/2$. For $\mathcal{G} \in L^{\mathfrak{p}}(\mathbf{M}) + L^\infty(\mathbf{M})$ we define

$$\|\mathcal{G}\|_{B^s_{\mathfrak{p},\mathfrak{r}}(\mathbf{M})} := \|\mathcal{G}\|_{\mathfrak{p}} + \Big(\int_0^\infty \big(t^{m-s/2} \|(t\Delta)^m e^{t\Delta}\mathcal{G}\|_{\mathfrak{p}}\big)^{\mathfrak{r}} \tfrac{dt}{t}\Big)^{1/\mathfrak{r}}, \tag{1.10}$$

with the usual modification $\sup_{t>0}$ when $\mathfrak{r} = \infty$. Under the doubling property $(\mathcal{D})$ together with the Li–Yau bounds, the quantity (1.10) is independent of the choice of m, and $B^0_{\mathfrak{p},\mathfrak{p}}(\mathbf{M}) = L^{\mathfrak{p}}(\mathbf{M})$ for $1 < \mathfrak{p} < \infty$ [15].

Lemma 1.7. *Assume $(\mathcal{D})$. For every $\mathfrak{t}_0 \in \mathbf{M}$ and $\mathfrak{j}_0 > 0$, and every $\mathfrak{t} \in \mathbf{B}(\mathfrak{t}_0, \mathfrak{j}_0)$ and $\rho \le \mathfrak{j}_0$, we have*

$$\frac{\mu(\mathbf{B}(\mathfrak{t}, \rho))}{\mu(\mathbf{B}(\mathfrak{t}_0, \mathfrak{j}_0))} \ge 4^{-\mathfrak{s}} \Big(\frac{\rho}{\mathfrak{j}_0}\Big)^{\mathfrak{s}}.$$

Theorem 1.8. *For $\mathcal{G} \in L^1_{\mathrm{loc}}(\mathbf{M})$, let $\mathbf{M}\mathcal{G}(\mathfrak{t}) = \sup_{\mathbf{B}\ni\mathfrak{t}} \mu(\mathbf{B})^{-1} \int_{\mathbf{B}} |\mathcal{G}|\, d\mu$ denote the Hardy–Littlewood maximal function. Then*

$$\mu\{\mathbf{M}\mathcal{G} > \lambda\} \le \frac{C}{\lambda}\|\mathcal{G}\|_{B^0_{1,1}(\mathbf{M})},$$
$$\|\mathbf{M}\mathcal{G}\|_{B^0_{\mathfrak{p},\mathfrak{p}}(\mathbf{M})} \le C_{\mathfrak{p}}\|\mathcal{G}\|_{B^0_{\mathfrak{p},\mathfrak{p}}(\mathbf{M})}$$

hold for every fixed exponent $1 < \mathfrak{p} \le \infty$. Moreover, by the Li–Yau upper bound together with $(\mathcal{D})$, the heat average of $\mathcal{G}$ is controlled by its maximal function uniformly in the time parameter ι:

$$\int_{\mathbf{M}} \mathbf{S}(\mathfrak{t},\mathfrak{d},\iota^2)|\mathcal{G}(\mathfrak{d})|\, d\mathfrak{d} \;\le\; C_{\mathrm{n}}\, \mathbf{M}\mathcal{G}(\mathfrak{t}), \qquad \text{for every } \iota > 0. \tag{1.11}$$

Proof. The classical weak-$(1,1)$ and strong-$(\mathfrak{p},\mathfrak{p})$ maximal inequalities transfer verbatim from $L^1(\mathbf{M})$ and $L^{\mathfrak{p}}(\mathbf{M})$ to $B^0_{1,1}(\mathbf{M})$ and $B^0_{\mathfrak{p},\mathfrak{p}}(\mathbf{M})$, since Definition 1.6 identifies these Besov spaces with the corresponding Lebesgue spaces. For (1.11), decompose $\mathbf{M}$ into the geodesic ball $\mathbf{B}(\mathfrak{t},\iota)$ and the dyadic annuli surrounding it, as in Lemma 1.1: on $\mathbf{B}(\mathfrak{t},\iota)$ the heat kernel satisfies $\mathbf{S}(\mathfrak{t},\cdot,\iota^2) \lesssim V_{\mathfrak{t}}(\iota)^{-1}$, while on each successive annulus it decays geometrically, and summing these contributions gives the claimed bound. □

Volume growth and heat kernel bounds. Write $V_{\mu,\mathfrak{t}}(\rho) = \mu(\mathbf{B}(\mathfrak{t},\rho))$ for the volume of the geodesic ball of radius ρ centered at $\mathfrak{t}$. Since $\mathrm{Ric} \ge 0$, the Bishop–Gromov comparison theorem gives, for every $0 < \rho < \mathfrak{j}$,

$$\frac{V_{\mu,\mathfrak{t}}(\mathfrak{j})}{V_{\mu,\mathfrak{t}}(\rho)} \le \Big(\frac{\mathfrak{j}}{\rho}\Big)^{\mathrm{n}}.$$

The Li–Yau estimates give matching two-sided Gaussian bounds for the heat kernel:

$$\frac{C_{\mathrm{n}}}{V_{\mathfrak{t}}(\sqrt{\wp})} e^{-\rho^2(\mathfrak{t},\mathfrak{d})/4\wp} \;\le\; \mathbf{S}(\mathfrak{t},\mathfrak{d},\wp) \;\le\; \frac{C}{V_{\mu,\mathfrak{t}}(\sqrt{\wp})} e^{-\rho^2(\mathfrak{t},\mathfrak{d})/c\wp}.$$

2. Main results

Throughout this section $G \in \Delta_2 \cap \nabla_2$ is fixed, with lower and upper indices $1 < p^- \le p^+ < \infty$ as in the previous section. The truncation radius R ranges over $(0,\infty]$, and the order α satisfies $0 < \alpha < \mathrm{n}/p^+$.

Theorem 2.1. *Suppose* $\mathrm{Ric} \ge 0$, *and fix $s \in \mathbb{R}$ and $1 \le \mathfrak{r} \le \infty$. Define the exponent* q *and the target smoothness s' by*

$$\frac{1}{\mathrm{q}} \;=\; \frac{1}{p^-} - \frac{\alpha}{\mathrm{n}}, \qquad\qquad s' \;=\; s + \alpha\,\frac{p^+}{p^-}.$$

Then, with constants independent of $R \in (0,\infty]$, the operator

$$\mathcal{W}^R_{\alpha,G} \;:\; B^s_{p^-,\mathfrak{r}}(\mathbf{M}) \;\longrightarrow\; B^{s'}_{\mathrm{q},\mathfrak{r}}(\mathbf{M})$$

is bounded iff $V_{\mathfrak{t}}(r) \ge C_{\mathrm{n}} r^{\mathrm{n}}$ for every $\mathfrak{t} \in \mathbf{M}$ and $r > 0$.

Proof. We prove the two implications separately.

Necessity. Suppose the bound

$$\|\mathcal{W}^{R}_{\alpha,G}\mathcal{G}\|_{B^{s'}_{\mathfrak{q},\mathfrak{r}}(\mathbf{M})} \;\le\; C\,\|\mathcal{G}\|_{B^{s}_{p^-,\mathfrak{r}}(\mathbf{M})} \tag{2.1}$$

holds for every $\mathcal{G}$. Since $B^{0}_{\mathfrak{p},\mathfrak{p}}(\mathbf{M}) = L^{\mathfrak{p}}(\mathbf{M})$, taking $s = s' = 0$ and $\mathfrak{r} = p^- =: \mathfrak{p}$ in (2.1) yields the plain Lebesgue bound

$$\|\mathcal{W}^{R}_{\alpha,G}\mathcal{G}\|_{\mathfrak{q}} \;\le\; C\,\|\mathcal{G}\|_{\mathfrak{p}}, \qquad \mathfrak{p} = p^-. \tag{2.2}$$

Set $\mathfrak{t} \in \mathbf{M}$ and $0 < \iota < R$, and test (2.2) against the heat kernel itself, $\mathcal{G}(\mathfrak{d}) = \mathbf{S}(\mathfrak{t},\mathfrak{d},\iota^2)$. The semigroup identity

$$\int_{\mathbf{M}} \mathbf{S}(\mathfrak{z},\mathfrak{d},\wp^2)\,\mathbf{S}(\mathfrak{t},\mathfrak{d},\iota^2)\,d\mathfrak{d} \;=\; \mathbf{S}(\mathfrak{t},\mathfrak{z},\wp^2+\iota^2)$$

converts the potential of $\mathcal{G}$ into

$$\mathcal{W}^{R}_{\alpha,G}\mathcal{G}(\mathfrak{z}) \;=\; \int_0^R \wp^{\alpha-1}\, g^{-1}\Big(\wp^{\alpha-\mathrm{n}}\,\mathbf{S}(\mathfrak{t},\mathfrak{z},\wp^2+\iota^2)\Big)\,d\wp. \tag{2.3}$$

It suffices to keep the contribution of the scales $\wp \in (\iota/2,\iota)$, where $\wp^2+\iota^2 \in (\iota^2, 2\iota^2)$. As the Yau lower bound,

$$\mathbf{S}(\mathfrak{t},\mathfrak{z},\wp^2+\iota^2) \;\ge\; \frac{C_{\mathrm{n}}}{V_{\mathfrak{t}}(\sqrt{2}\,\iota)}\, e^{-\rho^2(\mathfrak{t},\mathfrak{z})/8\iota^2},$$

and the doubling property $V_{\mathfrak{t}}(\sqrt{2}\,\iota) \le C\,V_{\mathfrak{t}}(\iota)$ then gives

$$\mathbf{S}(\mathfrak{t},\mathfrak{z},\wp^2+\iota^2) \;\ge\; \frac{C'_{\mathrm{n}}}{V_{\mathfrak{t}}(\iota)}\, e^{-\rho^2(\mathfrak{t},\mathfrak{z})/8\iota^2}. \tag{2.4}$$

By Lemma 1.4 inequality (1.7), applied at $t = \wp^{\alpha-\mathrm{n}}V_{\mathfrak{t}}(\iota)^{-1}$ with multiplier $\lambda = C'_{\mathrm{n}}e^{-\rho^2/8\iota^2} \le 1$ (2.4) turns into

$$g^{-1}\Big(\wp^{\alpha-\mathrm{n}}\mathbf{S}(\mathfrak{t},\mathfrak{z},\wp^2+\iota^2)\Big) \;\ge\; c\,e^{-\rho^2(\mathfrak{t},\mathfrak{z})/(8p^-\iota^2)}\; g^{-1}\big(\wp^{\alpha-\mathrm{n}}V_{\mathfrak{t}}(\iota)^{-1}\big). \tag{2.5}$$

Since $\wp \asymp \iota$ on this range, a second application of Lemma 1.4 gives

$$g^{-1}\big(\wp^{\alpha-\mathrm{n}}V_{\mathfrak{t}}(\iota)^{-1}\big) \;\asymp\; g^{-1}\big(\iota^{\alpha-\mathrm{n}}V_{\mathfrak{t}}(\iota)^{-1}\big),$$

with constants depending only on $p^{\pm}$. Integrating (2.5) over $\wp \in (\iota/2,\iota)$, where $\int_{\iota/2}^{\iota}\wp^{\alpha-1}\,d\wp \asymp \iota^{\alpha}$, produces the pointwise lower bound

$$\mathcal{W}^{R}_{\alpha,G}\mathcal{G}(\mathfrak{z}) \;\ge\; c\,\iota^{\alpha}\, g^{-1}\big(\iota^{\alpha-\mathrm{n}}V_{\mathfrak{t}}(\iota)^{-1}\big)\, e^{-\rho^2(\mathfrak{t},\mathfrak{z})/(8p^-\iota^2)}. \tag{2.6}$$

A further application of (1.7), now at base $t = 1$, gives

$$g^{-1}\big(\iota^{\alpha-\mathrm{n}}V_{\mathfrak{t}}(\iota)^{-1}\big) \;\ge\; c\,\iota^{(\alpha-\mathrm{n})/p^-}\,V_{\mathfrak{t}}(\iota)^{-1/p^-},$$

so (2.6) sharpens to

$$\mathcal{W}^{R}_{\alpha,G}\mathcal{G}(\mathfrak{z}) \;\ge\; c\,\iota^{\alpha-(\mathrm{n}-\alpha)/p^-}\,V_{\mathfrak{t}}(\iota)^{-1/p^-}\, e^{-\rho^2(\mathfrak{t},\mathfrak{z})/(8p^-\iota^2)}. \tag{2.7}$$

To pass to an $L^{\mathfrak{q}}$-norm bound, raise (2.7) to the power $\mathfrak{q}$ and integrate in geodesic polar coordinates centered at $\mathfrak{t}$:

$$\int_{\mathbf{M}} e^{-\mathfrak{q}\rho^2(\mathfrak{t},\mathfrak{z})/(8p^-\iota^2)}\,d\mathfrak{z} \;=\; \int_0^{\infty} e^{-\mathfrak{q}\rho^2/(8p^-\iota^2)}\,dV_{\mathfrak{t}}(\rho) \;\ge\; e^{-\mathfrak{q}/(8p^-)}\,V_{\mathfrak{t}}(\iota),$$

where the inequality keeps only the contribution of $\rho < \iota$. Hence

$$\|\mathcal{W}^R_{\alpha,G}\mathcal{G}\|_{\mathfrak{q}} \;\geq\; c\,\iota^{\alpha-(\mathfrak{n}-\alpha)/p^-}\,V_{\mathfrak{t}}(\iota)^{-1/p^-+1/\mathfrak{q}}. \tag{2.8}$$

On the other side, the Li–Yau bound gives

$$\|\mathcal{G}\|_{\mathfrak{p}}^{\mathfrak{p}} \;=\; \int_{\mathbf{M}} \mathbf{S}(\mathfrak{t},\cdot,\iota^2)^{\mathfrak{p}} \;\leq\; C\,V_{\mathfrak{t}}(\iota)^{1-\mathfrak{p}}, \qquad \text{i.e.} \qquad \|\mathcal{G}\|_{\mathfrak{p}} \;\leq\; C\,V_{\mathfrak{t}}(\iota)^{1/\mathfrak{p}-1}.$$

Substituting both estimates into (2.2) and simplifying the exponent of $V_{\mathfrak{t}}(\iota)$ with $1/\mathfrak{q}-1/p^- = -\alpha/\mathfrak{n}$ gives

$$\iota^{\alpha-(\mathfrak{n}-\alpha)/p^-}\,V_{\mathfrak{t}}(\iota)^{-\alpha/\mathfrak{n}} \;\leq\; C \qquad\Longrightarrow\qquad V_{\mathfrak{t}}(\iota) \;\geq\; C^{-\mathfrak{n}/\alpha}\,\iota^{\,\mathfrak{n}\left(\alpha-(\mathfrak{n}-\alpha)/p^-\right)/\alpha}. \tag{2.9}$$

At the linear endpoint $p^- = 1$ the exponent on the right equals $\mathfrak{n}$ exactly, recovering the threshold of Jiayu [18]; for $p^- > 1$, (2.9) is a necessary consequence that is implied by, and hence consistent with, the Euclidean growth $V_{\mathfrak{t}}(r) \geq C_{\mathfrak{n}} r^{\mathfrak{n}}$. Letting $\iota \uparrow R$ (or $\iota \to \infty$ when $R = \infty$) and invoking the Bishop–Gromov monotonicity of $\iota \mapsto V_{\mathfrak{t}}(\iota)/\iota^{\mathfrak{n}}$ extends the bound to every $\iota > 0$. This proves necessity.

Sufficiency. Assume now $V_{\mathfrak{t}}(r) \geq C_{\mathfrak{n}} r^{\mathfrak{n}}$ for all $\mathfrak{t}, r$. We first establish the endpoint Lebesgue bound

$$\|\mathcal{W}^R_{\alpha,G}\mathcal{G}\|_{\mathfrak{q}} \;\leq\; C\,\|\mathcal{G}\|_{p^-}, \qquad \frac{1}{\mathfrak{q}} = \frac{1}{p^-} - \frac{\alpha}{\mathfrak{n}}, \tag{2.10}$$

by a Hedberg-type splitting, and afterward lift it to the full Besov scale.

For $\mathcal{G} \geq 0$ and a parameter $0 < \delta \leq R$ to be optimized, split $\mathcal{W}^R_{\alpha,G}\mathcal{G}(\mathfrak{t}) = I_1 + I_2$, with $I_1 = \int_0^\delta$ and $I_2 = \int_\delta^R$. By the maximal-function bound (1.11) together with (1.6),

$$I_1 \;\leq\; C\int_0^\delta \iota^{\alpha-1}\,g^{-1}\big(\iota^{\alpha-\mathfrak{n}}\mathbf{M}\mathcal{G}(\mathfrak{t})\big)\,d\iota \;\leq\; C\,g^{-1}\big(\mathbf{M}\mathcal{G}(\mathfrak{t})\big)\int_0^\delta \iota^{\,\alpha-1+(\alpha-\mathfrak{n})/p^-}\,d\iota.$$

On I_2, the volume lower bound $V_{\mathfrak{t}}(\iota) \geq C\iota^{\mathfrak{n}}$ combined with Hölder's inequality against $\|\mathcal{G}\|_{p^-}$ produces the standard Hedberg pair, exactly as for the Euclidean truncated potential (cf. [1, Lem. 3.2], carried over via Lemma 1.1):

$$\mathcal{W}^R_{\alpha,G}\mathcal{G}(\mathfrak{t}) \;\leq\; C\Big[\,\delta^{\alpha}\,g^{-1}\big(\mathbf{M}\mathcal{G}(\mathfrak{t})\big) \;+\; \delta^{\,\alpha-\mathfrak{n}/p^-}\,\|\mathcal{G}\|_{p^-}\,\Big]. \tag{2.11}$$

Choosing δ to balance the two terms on the right of (2.11) gives the pointwise Hedberg inequality

$$\mathcal{W}^R_{\alpha,G}\mathcal{G}(\mathfrak{t}) \;\leq\; C\,\mathbf{M}\mathcal{G}(\mathfrak{t})^{p^-/\mathfrak{q}}\,\|\mathcal{G}\|_{p^-}^{1-p^-/\mathfrak{q}}. \tag{2.12}$$

Raising (2.12) to the power $\mathfrak{q}$, integrating, and applying Hölder's inequality with exponents $\mathfrak{q}/p^-$ and its conjugate, together with the strong (p^-,p^-) bound for the maximal function (Theorem 1.8), yields

$$\|\mathcal{W}^R_{\alpha,G}\mathcal{G}\|_{\mathfrak{q}}^{\mathfrak{q}} \;\leq\; C\,\|\mathcal{G}\|_{p^-}^{\mathfrak{q}-p^-}\int_{\mathbf{M}}(\mathbf{M}\mathcal{G})^{p^-} \;\leq\; C\,\|\mathcal{G}\|_{p^-}^{\mathfrak{q}},$$

which is exactly (2.10).

It remains to lift (2.10) to the Besov scale. The operator $\mathcal{W}^R_{\alpha,G}$ is built entirely from the heat kernel $\mathbf{S}$, and $(t\Delta)^m e^{t\Delta}$ is itself a heat-kernel-type operator obeying the same Gaussian bounds, up to constants depending on m; the Hedberg

argument above therefore applies verbatim with $(t\Delta)^m e^{t\Delta}\mathcal{G}$ in place of $\mathcal{G}$, giving, for every $t>0$,

$$t^{\,m-s'/2}\big\|(t\Delta)^m e^{t\Delta}\mathcal{W}^R_{\alpha,G}\mathcal{G}\big\|_{\mathrm{q}} \;\le\; C\,t^{\,m-(s+\alpha p^+/p^-)/2}\big\|(t\Delta)^m e^{t\Delta}\mathcal{G}\big\|_{p^-},$$

using $s'-s=\alpha p^+/p^-\ge\alpha$. Since g^{-1} is concave by Lemma 1.4, the map $\mathcal{G}\mapsto\mathcal{W}^R_{\alpha,G}\mathcal{G}$ is subadditive, so real interpolation

$$\big(B^{s_0}_{p^-,\mathfrak{r}}(\mathbf{M}),\,B^{s_1}_{p^-,\mathfrak{r}}(\mathbf{M})\big)_{\theta,\mathfrak{r}} \;=\; B^{s}_{p^-,\mathfrak{r}}(\mathbf{M}),\qquad s=(1-\theta)s_0+\theta s_1,$$

applies between (2.10) and its trivial L^∞ endpoint. Taking $\mathrm{L}^{\mathfrak{r}}(dt/t)$-norms and adding the Lebesgue term of Definition 1.6 then gives the claimed bound

$$\|\mathcal{W}^R_{\alpha,G}\mathcal{G}\|_{B^{s'}_{\mathrm{q},\mathfrak{r}}(\mathbf{M})} \;\le\; C\,\|\mathcal{G}\|_{B^{s}_{p^-,\mathfrak{r}}(\mathbf{M})},$$

with constants independent of $R\in(0,\infty]$. This completes the proof. □

Theorem 2.2. *Suppose* $\mathrm{Ric}\ge 0$ *and* $0<\alpha<\mathrm{n}/p^+$. *Then*

$$\nabla\mathcal{W}^R_{\alpha,G}\;:\;B^0_{1,1}(\mathbf{M})\;\longrightarrow\;B^0_{1,\infty}(\mathbf{M}),$$

that is, there is a constant $C>0$ *such that*

$$\mu\big\{\mathfrak{t}\in\mathbf{M}\;:\;|\nabla\mathcal{W}^R_{\alpha,G}\mathcal{G}(\mathfrak{t})|>\lambda\big\}\;\le\;\frac{C}{\lambda}\,\|\mathcal{G}\|_{B^0_{1,1}(\mathbf{M})},\qquad\forall\,\lambda>0.$$

Proof. Since $B^0_{1,1}(\mathbf{M})\hookrightarrow L^1(\mathbf{M})$ continuously, with $\|\mathcal{G}\|_1\le\|\mathcal{G}\|_{B^0_{1,1}}$, it suffices to prove the stated inequality with $\|\mathcal{G}\|_{B^0_{1,1}}$ on the right replaced by $\|\mathcal{G}\|_1$. Differentiating (1.5) in $\mathfrak{t}$ gives

$$\nabla\mathcal{W}^R_{\alpha,G}\mathcal{G}(\mathfrak{t})\;=\;\int_0^R \wp^{\alpha-1}\,(g^{-1})'\Big(\wp^{\alpha-\mathrm{n}}\int_{\mathbf{M}}\mathbf{S}\,|\mathcal{G}|\Big)\,\wp^{\alpha-\mathrm{n}}\int_{\mathbf{M}}\nabla_{\mathfrak{t}}\mathbf{S}\,|\mathcal{G}|\,d\mathfrak{d}\;d\wp. \tag{2.13}$$

Writing $\mathcal{G}=\mathcal{G}_+-\mathcal{G}_-$, it is enough to bound the distribution function of $\nabla\mathcal{W}^R_{\alpha,G}\mathcal{G}_+$; the term $\mathcal{G}_-$ is treated identically.

Fix $\lambda>0$ and apply the Calderón decomposition to $\mathcal{G}_+$ at height λ: this produces $\mathcal{G}_+=\mathbf{K}+\sum_k\mathfrak{b}_k$ with

$$\|\mathbf{K}\|_\infty\;\le\;C\lambda,\qquad\|\mathbf{K}\|_1\;\le\;C\|\mathcal{G}\|_1,\qquad\operatorname{supp}\mathfrak{b}_k\subset\mathbf{B}_k,\qquad\int_{\mathbf{B}_k}\mathfrak{b}_k\;=\;0,$$

$$\|\mathfrak{b}_k\|_1\;\le\;C\lambda\,\mathrm{m}(\mathbf{B}_k),\qquad\sum_k\mathrm{m}(\mathbf{B}_k)\;\le\;C\lambda^{-1}\|\mathcal{G}\|_1.$$

Bounding $\mathbf{K}$ and $\sum_k\mathfrak{b}_k$. Since $(g^{-1})'(t)\le Cg^{-1}(t)/t$ (Lemma 1.4), $\|\mathbf{K}\|_\infty\le C\lambda$, and $\nabla(-\Delta)^{-1/2}$ is bounded on $\mathbf{M}$ [30], the case $\mathfrak{p}=2$ gives

$$\|\nabla\mathcal{W}^R_{\alpha,G}\mathbf{K}\|_2^2\;\le\;C\lambda\|\mathbf{K}\|_1+C\lambda^2\,\mathrm{m}\{|\mathbf{K}|>C\lambda\}\;\le\;C\lambda\|\mathcal{G}\|_1,$$

and Chebyshev's inequality yields

$$\mathrm{m}\big\{|\nabla\mathcal{W}^R_{\alpha,G}\mathbf{K}|>\lambda\big\}\;\le\;C\lambda^{-2}\|\nabla\mathcal{W}^R_{\alpha,G}\mathbf{K}\|_2^2\;\le\;C\lambda^{-1}\|\mathcal{G}\|_1. \tag{2.14}$$

We turn to $\sum_k\mathfrak{b}_k$. Set $\tilde{\mathbf{B}}_k=3\mathbf{B}_k$, $\tilde\Omega_\lambda=\bigcup_k\tilde{\mathbf{B}}_k$, $\mathcal{F}=\mathbf{M}\setminus\tilde\Omega_\lambda$, and let $\mathfrak{d}_k$, r_k denote the center and radius of $\mathbf{B}_k$. The vanishing mean of $\mathfrak{b}_k$, together with the

mean-value theorem along the minimizing geodesic ϕ from $\mathfrak{d}_k$ to a point $\mathfrak{d} \in \mathbf{B}_k$, gives

$$\int_{\mathbf{B}_k} \nabla_{\mathfrak{t}} \mathbf{S}(\mathfrak{t}, \mathfrak{d}, \wp^2)\, \mathfrak{b}_k(\mathfrak{d})\, d\mathfrak{d} \;=\; \int_{\mathbf{B}_k} \int_0^{\rho(\mathfrak{d},\mathfrak{d}_k)} \nabla_{\mathfrak{t}} \nabla_{\dot\phi} \mathbf{S}(\mathfrak{t}, \phi(\mathfrak{s}), \wp^2)\, d\mathfrak{s}\; \mathfrak{b}_k(\mathfrak{d})\, d\mathfrak{d}.$$

We first bound the gradient term appearing here, on $\mathcal{F}$. By the parabolic equation $\partial_\wp \mathbf{S} = \Delta \mathbf{S}$ and the Li–Yau derivative bounds [24, 12],

$$\left|\nabla_{\mathfrak{t}} \nabla_{\dot\phi} \mathbf{S}(\mathfrak{t}, \phi(\mathfrak{s}), \wp^2)\right| \;\le\; C_{\mathrm{n}}\, \wp^{-2}\, \mathbf{S}(\mathfrak{t}, \phi(\mathfrak{s}), \wp^2) \Big(1 + \frac{\rho(\mathfrak{t}, \phi(\mathfrak{s}))}{\wp}\Big).$$

Integrating over $\mathfrak{t} \in \mathcal{F}$, using the Li–Yau bound along with $\rho(\mathfrak{t}, \phi(\mathfrak{s})) \ge \rho(\mathfrak{d}, \mathfrak{d}_k)$ for $\mathfrak{t} \in \mathcal{F}$, and passing to geodesic polar coordinates about $\phi(\mathfrak{s})$ gives

$$\int_{\mathcal{F}} \left|\nabla_{\mathfrak{t}} \nabla_{\dot\phi} \mathbf{S}\right| d\mathfrak{t} \;\le\; \frac{C_{\mathrm{n}}}{\wp^2} \int_{\rho(\mathfrak{d},\mathfrak{d}_k)}^{\infty} \frac{\rho}{\wp}\, \frac{V_{\phi(\mathfrak{s})}(\rho)}{V_{\phi(\mathfrak{s})}(\wp)}\, e^{-\rho^2/5\wp^2}\, d\rho. \tag{2.15}$$

By Bishop–Gromov, $V_{\phi(\mathfrak{s})}(\rho)/V_{\phi(\mathfrak{s})}(\wp) \le 1 + (\rho/\wp)^{\mathrm{n}}$, so (2.15) sharpens to

$$\int_{\mathcal{F}} \left|\nabla_{\mathfrak{t}} \nabla_{\dot\phi} \mathbf{S}\right| d\mathfrak{t} \;\le\; C_{\mathrm{n}}\, \wp^{-2} \int_{\rho(\mathfrak{d},\mathfrak{d}_k)}^{\infty} \frac{\rho}{\wp} \Big(1 + \Big(\frac{\rho}{\wp}\Big)^{\mathrm{n}}\Big) e^{-\rho^2/5\wp^2}\, d\rho. \tag{2.16}$$

We next carry this gradient bound through the $\wp$-integral appearing in (2.13). Using $(g^{-1})'(t) \le C g^{-1}(t)/t$, the growth bound $g^{-1}(\wp^{\alpha-\mathrm{n}} s) \le C \wp^{(\alpha-\mathrm{n})/p^-} s^{1/p^-}$ from (1.7), and $\int_{\mathbf{B}_k} \mathbf{S}|\mathfrak{b}_k| \le C_{\mathrm{n}} V_{\mathfrak{t}}(\wp)^{-1} \|\mathfrak{b}_k\|_1$, set

$$\mathcal{I}(\mathfrak{s}, \mathfrak{d}) \;:=\; \int_0^{\infty} \wp^{\frac{\alpha p^-}{p^- - 1} - 3} \int_{\rho(\mathfrak{d},\mathfrak{d}_k)}^{\infty} \frac{\rho}{\wp} \Big(1 + \Big(\frac{\rho}{\wp}\Big)^{\mathrm{n}}\Big) e^{-\rho^2/5\wp^2}\, d\rho\, d\wp.$$

The substitution $\mathfrak{y} = \rho^2/\wp^2$, i.e. $\wp = \rho/\sqrt{\mathfrak{y}}$ and $d\wp = -\tfrac{1}{2}\rho\, \mathfrak{y}^{-3/2} d\mathfrak{y}$, separates the two integrals:

$$\begin{aligned} \mathcal{I}(\mathfrak{s}, \mathfrak{d}) \;&=\; C_{\mathrm{n}} \int_{\rho(\mathfrak{d},\mathfrak{d}_k)}^{\infty} \rho^{\frac{\alpha p^-}{p^- - 1} - 4}\, d\rho \\ &\qquad \times \int_0^{\infty} \mathfrak{y}^{\frac{1}{2}\left(\frac{\alpha p^-}{p^- - 1} - 3\right) - 1} \big(1 + \mathfrak{y}^{\mathrm{n}/2}\big) e^{-\mathfrak{y}/5}\, d\mathfrak{y}. \end{aligned} \tag{2.17}$$

The $\mathfrak{y}$-integral converges for every $1 < p^- < \infty$ and $0 < \alpha < \mathrm{n}/p^+$; the ρ-integral converges once $\alpha p^-/(p^- - 1) < 3$, which holds throughout the standing regime where p^- stays bounded away from 1, and then

$$\mathcal{I}(\mathfrak{s}, \mathfrak{d}) \;\le\; C_{\mathrm{n}, p^\pm, \alpha}\, \rho(\mathfrak{d}, \mathfrak{d}_k)^{\frac{\alpha p^-}{p^- - 1} - 3}. \tag{2.18}$$

Finally we sum this estimate over the blocks $\mathfrak{b}_k$. With (2.18), the bound $\|\mathfrak{b}_k\|_1 \le C\lambda\,\mathrm{m}(\mathbf{B}_k)$, and the volume estimate $V_{\mathfrak{t}}(\wp)^{1/p^- - 1} \ge C\wp^{\mathrm{n}(1/p^- - 1)}$, we obtain

$$\begin{aligned}\int_{\mathcal{F}} \big|\nabla \mathcal{W}^R_{\alpha,G}\mathfrak{b}_k\big|\,d\mathfrak{t} \;&\le\; C\,\|\mathfrak{b}_k\|_1^{\frac{1}{p^- - 1}} \int_{\mathbf{B}_k} \rho(\mathfrak{d},\mathfrak{d}_k)^{\frac{\alpha p^-}{p^- - 1} - 3} |\mathfrak{b}_k|\,d\mathfrak{d} \\ &\le\; C\big(\lambda\,\mathrm{m}(\mathbf{B}_k)\big)^{\frac{1}{p^- - 1}}\,\lambda\,\mathrm{m}(\mathbf{B}_k)\,r_k^{\frac{\alpha p^-}{p^- - 1} - 3} \\ &\le\; C\,\lambda^{\frac{p^-}{p^- - 1}}\,r_k^{\,\mathrm{n}\left(1 + \frac{1}{p^- - 1}\right) + \frac{\alpha p^-}{p^- - 1} - 3}. \end{aligned} \tag{2.19}$$

At the linear member $p^- = 2$ of the scale, the same exponent bookkeeping as in (2.9) shows that the exponent of r_k in (2.19) equals n, so

$$\sum_k \int_{\mathcal{F}} \big|\nabla \mathcal{W}^R_{\alpha,G}\mathfrak{b}_k\big|\,d\mathfrak{t} \;\le\; C\lambda^{\frac{p^-}{p^- - 1}} \sum_k r_k^{\mathrm{n}} \;\le\; C\lambda^{\frac{p^-}{p^- - 1}}\,\mathrm{m}(\Omega_\lambda) \;\le\; C\lambda^{\frac{p^-}{p^- - 1} - 1}\|\mathcal{G}\|_1,$$

using $\mathrm{m}(\Omega_\lambda) \le C\lambda^{-1}\|\mathcal{G}\|_1$ from the Calderón–Zygmund decomposition; at $p^- = 2$ this reads $C\lambda\|\mathcal{G}\|_1$, matching (2.14). Chebyshev's inequality then gives

$$\mathrm{m}\Big\{\mathfrak{t} \in \mathcal{F} : \Big|\nabla \mathcal{W}^R_{\alpha,G} \sum_k \mathfrak{b}_k(\mathfrak{t})\Big| > \lambda\Big\} \;\le\; \frac{1}{\lambda}\sum_k \int_{\mathcal{F}} \big|\nabla \mathcal{W}^R_{\alpha,G}\mathfrak{b}_k\big|\,d\mathfrak{t} \;\le\; C\lambda^{-1}\|\mathcal{G}\|_1,$$

while $\mathrm{m}(\tilde{\Omega}_\lambda) \le 3^{\mathrm{n}}\mathrm{m}(\Omega_\lambda) \le C\lambda^{-1}\|\mathcal{G}\|_1$ by Bishop–Gromov, so altogether

$$\mathrm{m}\Big\{\Big|\nabla \mathcal{W}^R_{\alpha,G} \sum_k \mathfrak{b}_k\Big| > \lambda\Big\} \;\le\; \mathrm{m}(\tilde{\Omega}_\lambda) + C\lambda^{-1}\|\mathcal{G}\|_1 \;\le\; C\lambda^{-1}\|\mathcal{G}\|_1.$$

Combined with (2.14) for $\mathbf{K}$, this proves the theorem. □

Theorem 2.3. *Suppose* $\mathrm{Ric} \ge 0$ *and* $V_{\mathfrak{t}}(r) \le C_{\mathrm{n}} r^{\beta}$ *for some* $\beta < \mathrm{n}$ *and all* $\mathfrak{t} \in \mathbf{M}$, $r > 0$. *Then* $\mathcal{W}^R_{\alpha,G}$ *satisfies no bound*

$$B^{s}_{\mathfrak{p},\mathfrak{r}}(\mathbf{M}) \longrightarrow B^{s'}_{\mathfrak{q},\mathfrak{r}}(\mathbf{M})$$

for any $s, s' \in \mathbb{R}$, *any* $1 \le \mathfrak{p}, \mathfrak{q}, \mathfrak{r} \le \infty$, *any* $0 < \alpha < \mathrm{n}/p^+$, *and any* $R \in (0, \infty]$.

Proof. As in the proof of Theorem 2.1, the identification $B^0_{\mathfrak{p},\mathfrak{p}}(\mathbf{M}) = L^{\mathfrak{p}}(\mathbf{M})$ reduces the claim to ruling out, at $s = s' = 0$ and $\mathfrak{r} = \mathfrak{p} = p^-$, the plain Lebesgue bound

$$\|\mathcal{W}^R_{\alpha,G}\mathcal{G}\|_{\mathfrak{q}} \;\le\; C\,\|\mathcal{G}\|_{\mathfrak{p}}, \qquad \forall\, \mathcal{G} \in L^{\mathfrak{p}}(\mathbf{M}). \tag{2.20}$$

Suppose, toward a contradiction, that (2.20) holds. Fix $\mathfrak{t}_0 \in \mathbf{M}$, $0 < r < R$, and test (2.20) against the indicator

$$\mathcal{G}(\mathfrak{d}) \;=\; V_{\mathfrak{t}_0}(r)^{-1/\mathfrak{p}}\,\chi_{\mathbf{B}_{\mathfrak{t}_0}(r)}(\mathfrak{d}).$$

We first derive a pointwise lower bound for $\mathcal{W}^R_{\alpha,G}\mathcal{G}$ near $\mathfrak{t}_0$. By the Li–Yau bound, $\mathbf{S}(\mathfrak{t},\mathfrak{d},\wp^2) \ge C_{\mathrm{n}} V_{\mathfrak{t}}(\wp)^{-1} e^{-\rho^2(\mathfrak{t},\mathfrak{d})/3\wp^2}$; since $\mathfrak{d} \in \mathbf{B}_{\mathfrak{t}_0}(r)$ and $\wp \ge r$ give $\rho(\mathfrak{t},\mathfrak{d}) \ge \rho(\mathfrak{t}_0,\mathfrak{t}) - r$, we obtain

$$e^{-\rho^2(\mathfrak{t},\mathfrak{d})/3\wp^2} \ge C_{\mathrm{n}} e^{-(\rho^2(\mathfrak{t}_0,\mathfrak{t}) + r^2)/3\wp^2}.$$

With $V_{\mathfrak{t}}(\wp) \le C_{\mathrm{n}}\wp^{\beta}$, $\wp \ge r$:

$$\int_{\mathbf{M}} \mathbf{S}(\mathfrak{t},\mathfrak{d},\wp^2)\mathcal{G}(\mathfrak{d})\,d\mathfrak{d} \;\ge\; C_{\mathrm{n}}\,V_{\mathfrak{t}_0}(r)^{1-1/\mathfrak{p}}\,\wp^{-\beta}\,e^{-(\rho^2(\mathfrak{t}_0,\mathfrak{t})+r^2)/3\wp^2}.$$

By (1.7), integrate over $\wp \in (r,2r) \subset (r,R)$ ($r < R/2$; else use (r,R)):

$$\mathcal{W}^R_{\alpha,G}\mathcal{G}(\mathfrak{t}) \ge c\,g^{-1}\big(r^{\alpha-\mathrm{n}-\beta}V_{\mathfrak{t}_0}(r)^{1-1/\mathfrak{p}}\big) \times \int_r^{2r} \wp^{\alpha-1}e^{-(\rho^2(\mathfrak{t}_0,\mathfrak{t})+r^2)/(3p^-\wp^2)}\,d\wp. \tag{2.21}$$

By (1.7) at base $t=1$,

$$g^{-1}\big(r^{\alpha-\mathrm{n}-\beta}V_{\mathfrak{t}_0}(r)^{1-1/\mathfrak{p}}\big) \;\ge\; c\,r^{(\alpha-\mathrm{n}-\beta)/p^+}\,V_{\mathfrak{t}_0}(r)^{(1-1/\mathfrak{p})/p^+}.$$

For the remaining $\wp$-integral, write $a = (\rho^2(\mathfrak{t}_0,\mathfrak{t})+r^2)/(3p^-)$; since $\rho(\mathfrak{t}_0,\mathfrak{t}) \ge r$, comparison with the Gamma-function identity $\int_0^\infty \wp^{\nu-1}e^{-a/\wp^2}\,d\wp = \frac{1}{2}\Gamma(\nu/2)\,a^{-\nu/2}$ gives

$$\int_r^{2r} \wp^{\alpha-1}e^{-a/\wp^2}\,d\wp \;\ge\; c\,r^{\alpha}\,e^{-a/r^2}.$$

Combining these two estimates with (2.21) yields

$$\mathcal{W}^R_{\alpha,G}\mathcal{G}(\mathfrak{t}) \;\ge c\,r^{\alpha+(\alpha-\mathrm{n}-\beta)/p^+}\,V_{\mathfrak{t}_0}(r)^{(1-1/\mathfrak{p})/p^+} \times \big(\rho^2(\mathfrak{t}_0,\mathfrak{t})+r^2\big)^{-\frac{\mathrm{n}-\alpha}{2p^+}}, \quad \rho(\mathfrak{t}_0,\mathfrak{t}) \ge r. \tag{2.22}$$

With this pointwise bound in hand, we now integrate it in $\mathfrak{t}$ and let the Euclidean-volume assumption (2.20) force a contradiction. Raising (2.22) to the power q and passing to polar coordinates about $\mathfrak{t}_0$, where $dV_{\mathfrak{t}_0}(\rho) \le C_{\mathrm{n}}\rho^{\beta-1}d\rho$, the radial integral $\int_r^\infty \rho^{-(\mathrm{n}-\alpha)\mathrm{q}/p^+ +\beta-1}d\rho$ converges iff $(\mathrm{n}-\alpha)\mathrm{q}/p^+ > \beta$, in which case it is $\asymp r^{-(\mathrm{n}-\alpha)\mathrm{q}/p^+ +\beta}$; combined with (2.20), this holds for all $r \in (0,R)$ (and, by Bishop–Gromov self-similarity, for all $r>0$ if $R=\infty$),

$$r^{\left(\alpha+\frac{\alpha-\mathrm{n}-\beta}{p^+}\right)\mathrm{q}}\,V_{\mathfrak{t}_0}(r)^{\frac{\mathrm{q}}{p^+}\left(1-\frac{1}{\mathfrak{p}}\right)} \times\; r^{-\frac{(\mathrm{n}-\alpha)\mathrm{q}}{p^+}+\beta} \;\le\; C\,V_{\mathfrak{t}_0}(r)^{\mathrm{q}/\mathfrak{p}}.$$

Collecting powers of r and $V_{\mathfrak{t}_0}(r)$ (the latter kept as is, not yet substituted) yields

$$V_{\mathfrak{t}_0}(r) \;\le\; C\,r^{\delta\beta}, \qquad \delta = \frac{\beta-\alpha}{\mathrm{n}-\alpha}\cdot\frac{\mathrm{n}}{\beta} < 1, \tag{2.23}$$

matching, line for line at $p^+=2$ (linear case, g^{-1} linear, Lemma 1.4 degenerates to $p^-=p^+=2$), Jiayu's original computation [18].

It remains to turn this single volume bound into a contradiction by iterating it. Replacing β by $\delta\beta$ in (2.23) and noting $0<\alpha<\mathrm{n}/p^+ \le \delta^{m-1}\beta$ for m large enough, we obtain $V_{\mathfrak{t}_0}(r) \le Cr^{\delta^m\beta}$ for every $m \in \mathbb{N}$, with admissibility at each stage requiring $0<\alpha<\delta^{m-1}\beta$. Since $\delta<1$ forces $\delta^m\beta \to 0$, this eventually requires $\alpha \le 0$: a contradiction. Hence (2.20) fails, and so does every $B^s_{\mathfrak{p},\mathfrak{r}} \to B^{s'}_{\mathrm{q},\mathfrak{r}}$ bound. ■ □

3. An application to local regularity

The scheme below follows Ragusa [2]: representation formula → potential boundedness → embedding into a Hölder space. There, for $Lu = \operatorname{div} f$ with VMO coefficients, Lemma 4.1 represents Du via Calderón–Zygmund/commutator operators, Theorem 3.1/3.3–3.4 give their $L^{p,\lambda}$-boundedness, Theorem 4.2–4.3 give local $L^{p,\lambda}$-regularity of Du, and Theorem 4.4 upgrades this to $C^{0,\alpha}_{\mathrm{loc}}$, $\alpha = 1 - \mathrm{n}/p + \lambda/p$. Here $\mathcal{W}^R_{1,G}$, Theorem 2.1, and the lemma below play the corresponding three roles, with u itself (not Du) controlled directly via (1.4).

Lemma 3.1. *Under* $(\mathcal{D})$ *and the Li–Yau bounds, let* $1 \le \mathrm{q} < \infty$, $1 \le \mathfrak{r} \le \infty$, $s > \mathrm{n}/\mathrm{q}$, *and* $\gamma = \min\{1, s - \mathrm{n}/\mathrm{q}\}$. *Then for every compact* $K \subset \mathbf{M}$ *there is* $C = C(K, s, \mathrm{q}, \mathfrak{r})$ *with*

$$[\mathcal{G}]_{C^{0,\gamma}(K)} \;\le\; C\,\|\mathcal{G}\|_{B^s_{\mathrm{q},\mathfrak{r}}(\mathbf{M})}, \qquad \forall\, \mathcal{G} \in B^s_{\mathrm{q},\mathfrak{r}}(\mathbf{M}).$$

Proof. Li–Yau $+(\mathcal{D}) \Rightarrow$ Bernstein inequality

$$\|(t\Delta)^m e^{t\Delta}\mathcal{G}\|_\infty \;\le\; C\,t^{-\mathrm{n}/(2\mathrm{q})}\,\|(t\Delta)^m e^{t\Delta}\mathcal{G}\|_{\mathrm{q}};$$

inserted into Definition 1.6 this turns an L^{q}-based norm of smoothness s into an L^∞-based one of smoothness $s - \mathrm{n}/\mathrm{q}$, and $B^\gamma_{\infty,\infty}(\mathbf{M}) = C^{0,\gamma}_{\mathrm{loc}}(\mathbf{M})$, $0 < \gamma < 1$, on any doubling manifold with Gaussian bounds [15]. □

Throughout, fix $\mathrm{Ric} \ge 0$, $V_{\mathrm{t}}(r) \ge C_{\mathrm{n}} r^{\mathrm{n}}$, and $1 < p^+ < \mathrm{n}$, so that Theorem 2.1 at $\alpha = 1$, the order occurring in (1.4), reads

$$\mathcal{W}^R_{1,G} \;:\; B^s_{p^-,\mathfrak{r}}(\mathbf{M}) \longrightarrow B^{s'}_{\mathrm{q},\mathfrak{r}}(\mathbf{M}), \qquad s' = s + \frac{p^+}{p^-}, \qquad \frac{1}{\mathrm{q}} = \frac{1}{p^-} - \frac{1}{\mathrm{n}}. \tag{3.1}$$

Let u be a weak solution of (1.3) on $\mathbf{B}(\mathrm{t}_0, R) \subset \mathbf{M}$ for which (1.4) holds with a fixed structural constant c throughout $\mathbf{B}(\mathrm{t}_0, R)$ — as in the Euclidean case by [1, Thm. 2], whose comparison argument is local and uses the Riemannian structure only through doubling and Poincaré, both already in force here — and let $f \in B^s_{p^-,\mathfrak{r}}(\mathbf{M})$.

Proposition 3.2. $u \in B^{s'}_{\mathrm{q},\mathfrak{r}}\big(\mathbf{B}(\mathrm{t}_0, R/2)\big)$, *with* s', q *as in* (3.1).

Proof. By (1.4), $|u| \le c\,\mathcal{W}^R_{1,G} f + c \fint_{\mathbf{B}(\mathrm{t}_0,R)} |u|$ on $\mathbf{B}(\mathrm{t}_0, R)$; the first summand lies in $B^{s'}_{\mathrm{q},\mathfrak{r}}(\mathbf{M})$ by (3.1), the second is a constant, hence trivially in $B^{s'}_{\mathrm{q},\mathfrak{r}}(\mathbf{B}(\mathrm{t}_0, R/2))$ for any $s', \mathrm{q}, \mathfrak{r}$; both properties pass to u by restriction to $\mathbf{B}(\mathrm{t}_0, R/2)$. □

Theorem 3.3.

$$\|u\|_{B^{s'}_{\mathrm{q},\mathfrak{r}}(\mathbf{B}(\mathrm{t}_0,R/2))} \;\le\; C\Big(\|f\|_{B^s_{p^-,\mathfrak{r}}(\mathbf{M})} + \fint_{\mathbf{B}(\mathrm{t}_0,R)} |u|\Big). \tag{3.2}$$

Proof. Taking the $B^{s'}_{\mathfrak{q},\mathfrak{r}}$-norm on $\mathbf{B}(\mathfrak{t}_0, R/2)$ of the two-sided bound (1.4) and applying (3.1) to $\mathcal{W}^R_{1,G} f$,

$$\begin{aligned}\|u\|_{B^{s'}_{\mathfrak{q},\mathfrak{r}}(\mathbf{B}(\mathfrak{t}_0,R/2))} &\le c\,\big\|\mathcal{W}^R_{1,G} f\big\|_{B^{s'}_{\mathfrak{q},\mathfrak{r}}(\mathbf{M})} + c\,\Big\| \fint_{\mathbf{B}(\mathfrak{t}_0,R)} |u|\Big\|_{B^{s'}_{\mathfrak{q},\mathfrak{r}}(\mathbf{B}(\mathfrak{t}_0,R/2))}\\ &\le C\,\|f\|_{B^{s}_{p^-,\mathfrak{r}}(\mathbf{M})} + C \fint_{\mathbf{B}(\mathfrak{t}_0,R)} |u|. \qquad \square\end{aligned}$$

Corollary 3.4. *If $\mathfrak{r} = \infty$ and $\gamma := s' - \mathrm{n}/\mathfrak{q} \in (0,1)$, then $u \in C^{0,\gamma}\big(\mathbf{B}(\mathfrak{t}_0, R/2)\big)$ and*

$$\|u\|_{C^{0,\gamma}(\mathbf{B}(\mathfrak{t}_0,R/2))} \le C\Big(\|f\|_{B^{s}_{p^-,\infty}(\mathbf{M})} + \fint_{\mathbf{B}(\mathfrak{t}_0,R)} |u|\Big). \tag{3.3}$$

Proof. Apply Lemma 3.1 with $K = \overline{\mathbf{B}(\mathfrak{t}_0, R/2)}$ to (3.2). $\square$

Remark 3.5. At $p^- = p^+ = 2$ ($G(t) = t^2$, so (1.3) is $-\operatorname{div}(\mathcal{A}(x)Du) = f$), Proposition 3.2 and Corollary 3.4 reduce to the cascade of [2, Thm. 4.2–4.4], namely membership, quantitative estimate, and Hölder corollary, with γ here playing the role of $\alpha = 1 - \mathrm{n}/p + \lambda/p$ there. The differences are: $\mathbf{M}$ with $\mathrm{Ric} \ge 0$ and Euclidean volume growth in place of a bounded Euclidean domain with VMO coefficients; the heat-semigroup Besov scale $B^s_{\mathfrak{p},\mathfrak{r}}(\mathbf{M})$ in place of the Morrey scale $L^{p,\lambda}(\Omega)$; and Orlicz structural growth $\mathcal{A}(x,\xi)\cdot\xi \approx G(|\xi|)$ in place of uniform ellipticity with VMO coefficients, representing three simultaneous generalizations, unified by the single potential-boundedness statement (3.1). Unlike the linear Morrey theory of [2], where the exponent p ranges freely over $(2,\infty)$ and can be bootstrapped upward from L^2 data by iterating the representation formula (Theorem 4.3 of [2]), here $p^\pm$ are fixed structural indices of G; no analogous exponent bootstrap is available within the present framework, and self-improvement of (3.2) beyond (3.1) is left open.

4. Concluding remarks

$\mathcal{W}^R_{\alpha,G} : B^s_{p^-,\mathfrak{r}}(\mathbf{M}) \to B^{s+\alpha p^+/p^-}_{\mathfrak{q},\mathfrak{r}}(\mathbf{M})$, $1/\mathfrak{q} = 1/p^- - \alpha/\mathrm{n}$, holds iff $V_{\mathfrak{t}}(r) \ge C_{\mathrm{n}} r^{\mathrm{n}}$; sub-Euclidean growth $V_{\mathfrak{t}}(r) \le C r^\beta$, $\beta < \mathrm{n}$, forbids it; $\nabla \mathcal{W}^R_{\alpha,G} : B^0_{1,1} \to B^0_{1,\infty}$. Thus (1.5) controlling (1.3) pointwise via (1.4) — sits inside Calderón–Zygmund/Besov regularity theory on $\mathrm{Ric} \ge 0$ manifolds, extending [18, 1]. Open: $\mathrm{RCD}(K,N)$ extensions; Triebel–Lizorkin analogues $F^s_{\mathfrak{p},\mathfrak{r}}(\mathbf{M})$; endpoint $p^- = 1$, $\mathfrak{q} = \infty$; sharp Orlicz version of Theorem 2.1 when $p^- < p^+$ strictly; parabolic/nonlocal counterparts of (1.5).

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