Gulf Journal of Mathematics
Vol 24, Issue 1 (2026) 1-8

ISSN: 2309-4966
https://doi.org/10.56947/sxns6x27

REFINED EXPONENTIAL INEQUALITIES FOR ACCEPTABLE RANDOM VARIABLES

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Abstract. This paper establishes refined exponential inequalities for a wide class of acceptable random variables using a flexible exponential bound from the extended negative dependence framework. New moment bounds and concentration inequalities are derived, where a parameter replaces the classical variance with an absolute moment of a certain order. The corresponding convergence rate for the strong law of large numbers is obtained. The results unify and extend existing bounds for various dependent structures, including widely, extended negatively, negatively, and independent sequences. Applications to autoregressive models and nonparametric regression illustrate the practical utility of the theoretical findings.

Keywords. Wide acceptable random variables, Exponential inequalities, Complete convergence, Dependent sequences, Nonparametric regression.
2020 Mathematics Subject Classification. Primary 60E15, 60F15; Secondary 62J05, 62G08.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on a fixed probability space $(\Omega, \mathcal{F}, P)$. The concept of acceptable random variables was introduced by Giuliano Antonini et al. [2] and later extended by Sung et al. [9] and Shen et al. [7] to include dominating coefficients. We adopt the following wide definition.

Definition 1.1. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be *wide acceptable* if there exists a finite sequence of positive real numbers $\{g(n), n \geq 1\}$ such that for any $\lambda > 0$ and all $n \geq 1$,

$$E \exp\left\{\lambda \sum_{i=1}^{n} X_i\right\} \leq g(n) \prod_{i=1}^{n} E e^{\lambda X_i}. \tag{1.1}$$

The constants $\{g(n), n \geq 1\}$ are called the dominating coefficients.

Remark 1.2. Definition 1.1 encompasses a large class of dependence structures: independent, negatively associated, and negatively orthant dependent sequences

Date: Received: Jul 23, 2026; Revised: Aug 20, 2026; Accepted: Aug 27, 2026.
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correspond to $g(n) = 1$; extended negatively dependent sequences correspond to $g(n) = M$ for some constant $M > 0$; and widely orthant dependent sequences satisfy the condition with $g(n) = \max\{g_u(n), g_l(n)\}$. Therefore, any result established for wide acceptable sequences automatically holds for all these cases.

The exponential inequality for the partial sum $\sum_{i=1}^{n}(X_i - EX_i)$ is a fundamental tool in probability theory. In the classical independent case, the Hoeffding and Bernstein inequalities are well known. For dependent structures, Shen et al. [7] established exponential inequalities for acceptable random variables, and Kim and Kim [4], Wang et al. [10], and Sung [8] obtained various bounds for related dependent sequences.

Recently, Moussaoui and Benaissa [5] introduced the concept of extended linear negative quadrant dependent random variables and established new concentration inequalities. A key technical tool in their work is the following elementary but powerful bound.

Lemma 1.3. *For any $x \in \mathbb{R}$ and $0 < \alpha \leq 1$, we have*

$$e^x \leq 1 + x + |x|^{1+\alpha} e^{2|x|}. \tag{1.2}$$

This inequality offers a flexible parameter α that allows one to interpolate between different moment conditions. In this paper, we systematically replace the classical bound $e^x \leq 1 + x + \frac{1}{2}x^2 e^{|x|}$ used in the literature for acceptable sequences with (1.2), yielding refined exponential inequalities. The results are particularly useful when higher moments are controlled, or when one wishes to exploit the trade-off encoded by the parameter α.

2. New Exponential Inequalities for Bounded Acceptable Sequences

In this section, we establish our main inequality for bounded wide acceptable random variables. The proof relies on Lemma 1.3 instead of the usual second-order Taylor expansion.

Lemma 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of wide acceptable random variables with $EX_n = 0$ for each $n \geq 1$. Assume there exists a sequence of positive numbers $\{c_n, n \geq 1\}$ such that $|X_i| \leq c_i$ for each $i \geq 1$. Then for any $\lambda > 0$ and $0 < \alpha \leq 1$,*

$$E \exp\left\{\lambda \sum_{i=1}^{n} X_i\right\} \leq g(n) \exp\left(\lambda^{1+\alpha} e^{2\lambda M_n} \sum_{i=1}^{n} E|X_i|^{1+\alpha}\right), \tag{2.1}$$

where $M_n = \max_{1\leq j\leq n} c_j$.

Proof. For each i, applying Lemma 1.3 with $x = \lambda X_i$ and taking expectations, we obtain

$$\begin{aligned} Ee^{\lambda X_i} &\leq E\left(1 + \lambda X_i + |\lambda X_i|^{1+\alpha} e^{2|\lambda X_i|}\right) \\ &= 1 + \lambda EX_i + \lambda^{1+\alpha} E\left(|X_i|^{1+\alpha} e^{2\lambda |X_i|}\right) \\ &= 1 + \lambda^{1+\alpha} E\left(|X_i|^{1+\alpha} e^{2\lambda |X_i|}\right) \quad (\text{since } EX_i = 0). \end{aligned}$$

Using the boundedness assumption $|X_i| \le c_i \le M_n$, we have $e^{2\lambda|X_i|} \le e^{2\lambda M_n}$. Thus,

$$Ee^{\lambda X_i} \le 1 + \lambda^{1+\alpha} e^{2\lambda M_n} E|X_i|^{1+\alpha}.$$

By the elementary inequality $1 + u \le e^u$ for $u \ge 0$, we get

$$Ee^{\lambda X_i} \le \exp\left(\lambda^{1+\alpha} e^{2\lambda M_n} E|X_i|^{1+\alpha}\right).$$

Now, by the definition of wide acceptability (1.1), we conclude that

$$E\exp\left\{\lambda \sum_{i=1}^n X_i\right\} \le g(n) \prod_{i=1}^n Ee^{\lambda X_i} \le g(n) \exp\left(\lambda^{1+\alpha} e^{2\lambda M_n} \sum_{i=1}^n E|X_i|^{1+\alpha}\right).$$

This completes the proof. □

Using this moment bound, we immediately obtain a concentration inequality.

Theorem 2.2. *Under the assumptions of Lemma 2.1, for any $\epsilon > 0$, $\lambda > 0$, and $0 < \alpha \le 1$,*

$$P\left(\left|\sum_{i=1}^n X_i\right| \ge \epsilon\right) \le 2g(n) \exp\left(-\lambda\epsilon + \lambda^{1+\alpha} e^{2\lambda M_n} \sum_{i=1}^n E|X_i|^{1+\alpha}\right). \tag{2.2}$$

Proof. By Markov's inequality and Lemma 2.1,

$$P\left(\sum_{i=1}^n X_i \ge \epsilon\right) = P\left(e^{\lambda \sum X_i} \ge e^{\lambda\epsilon}\right) \le e^{-\lambda\epsilon} E e^{\lambda \sum X_i} \le g(n) \exp\left(-\lambda\epsilon + \lambda^{1+\alpha} e^{2\lambda M_n} \sum_{i=1}^n E|X_i|^{1+\alpha}\right).$$

Since $\{-X_n\}$ is also a wide acceptable sequence with the same dominating coefficients $g(n)$, we apply the same argument to obtain the lower tail. Summing the two bounds yields (2.2). □

Corollary 2.3. *Let $\{X_n, n \ge 1\}$ be a sequence of identically distributed wide acceptable random variables with $EX_1 = 0$ and $E|X_1|^{1+\alpha} < \infty$ for some $0 < \alpha \le 1$. Assume there exists a constant $c > 0$ such that $|X_i| \le c$ almost surely for all $i \ge 1$. Then for any $\epsilon > 0$,*

$$P\left(\left|\frac{1}{n}\sum_{i=1}^n X_i\right| \ge \epsilon\right) \le 2g(n) \inf_{\lambda > 0} \exp\left(-n\left[\lambda\epsilon - \lambda^{1+\alpha} e^{2\lambda c} E|X_1|^{1+\alpha}\right]\right).$$

Proof. This follows directly from Theorem 2.2 by noting that $M_n = c$, $\sum_{i=1}^n E|X_i|^{1+\alpha} = nE|X_1|^{1+\alpha}$, and replacing ϵ by $n\epsilon$. □

3. Optimal Choice of λ and Convergence Rate

To obtain an explicit rate, we optimize the exponent in Corollary 2.3. Define

$$H(\lambda) = \lambda\epsilon - \lambda^{1+\alpha} e^{2\lambda c} E|X_1|^{1+\alpha}.$$

Taking the derivative and setting it to zero yields a transcendental equation. However, for large n, we can obtain the asymptotic behavior.

Theorem 3.1. *Under the conditions of Corollary 2.3, suppose that ϵ is fixed and $n \to \infty$. Then there exists a constant $C = C(\alpha, c, \epsilon, E|X_1|^{1+\alpha}) > 0$ such that for all sufficiently large n,*

$$P\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i\right| \geq \epsilon\right) \leq 2g(n)\exp\left(-Cn^{\frac{\alpha}{1+\alpha}}\right).$$

Proof. Choose λ of the form $\lambda = \lambda_0 n^{-\frac{1}{1+\alpha}}$ for some constant $\lambda_0 > 0$ to be determined. Then

$$\lambda\epsilon n = \lambda_0 \epsilon n^{\frac{\alpha}{1+\alpha}},$$

$$\lambda^{1+\alpha} n e^{2\lambda c} = \lambda_0^{1+\alpha} e^{2c\lambda_0 n^{-\frac{1}{1+\alpha}}} n^{\frac{\alpha}{1+\alpha}} \leq \lambda_0^{1+\alpha} e^{2c\lambda_0} n^{\frac{\alpha}{1+\alpha}}.$$

Substituting into the exponent of Corollary 2.3, we get

$$-n\left[\lambda\epsilon - \lambda^{1+\alpha} e^{2\lambda c} E|X_1|^{1+\alpha}\right] \leq -\left(\lambda_0\epsilon - \lambda_0^{1+\alpha} e^{2c\lambda_0} E|X_1|^{1+\alpha}\right) n^{\frac{\alpha}{1+\alpha}}.$$

Choosing λ_0 to maximize the coefficient in parentheses, we can ensure the coefficient $C > 0$. The result follows. □

Remark 3.2. When $\alpha = 1$, the rate becomes $O(n^{-1/2})$, which matches the classical Hoeffding bound for independent variables. For $0 < \alpha < 1$, the exponent $\alpha/(1+\alpha)$ is smaller than $1/2$, reflecting a slower convergence due to heavier tails or weaker moment assumptions.

Remark 3.3. The result of Theorem 3.1 improves upon the corresponding bounds in Shen et al. [7] by replacing the variance term EX_1^2 with the more flexible absolute moment $E|X_1|^{1+\alpha}$.

4. Truncation Technique and Unbounded Case

Following the truncation method, we define for $1 \leq i \leq n$, $n \geq 1$,

$$\begin{aligned}
X_{1,i,n} &= -c_n I(X_i < -c_n) + X_i I(|X_i| \leq c_n) + c_n I(X_i > c_n),\\
X_{2,i,n} &= (X_i - c_n) I(X_i > c_n),\\
X_{3,i,n} &= (X_i + c_n) I(X_i < -c_n).
\end{aligned}$$

Clearly $X_i = X_{1,i,n} + X_{2,i,n} + X_{3,i,n}$. By the properties of wide acceptable sequences (see Proposition 2.1 in Shen et al. [7]), each sequence $\{X_{q,i,n}\}_{i=1}^n$ is still wide acceptable for $q = 1, 2, 3$, with the same dominating coefficients $g(n)$.

Theorem 4.1. *Let $\{X_i, i \geq 1\}$ be a sequence of identically distributed wide acceptable random variables with $EX_1 = 0$ and $E|X_1|^{1+\alpha} < \infty$ for some $0 < \alpha \leq 1$. Let $\{c_n\}$ and $\{\epsilon_n\}$ be sequences of positive numbers such that $\epsilon_n \to 0$. Assume that*

$$\lim_{n\to\infty} \frac{c_n}{\epsilon_n} < \infty, \quad \text{and} \quad \sum_{n=1}^{\infty} g(n)\exp\left(-Cn\epsilon_n^{\frac{1+\alpha}{\alpha}}\right) < \infty.$$

Then $\frac{1}{n}\sum_{i=1}^n X_i \to 0$ completely.

Proof. We decompose the sum as

$$P\left(\left|\sum_{i=1}^{n} X_i\right| \geq 3n\epsilon_n\right) \leq P\left(\left|\sum_{i=1}^{n}(X_{1,i,n} - EX_{1,i,n})\right| \geq n\epsilon_n\right) + P\left(\left|\sum_{i=1}^{n} X_{2,i,n}\right| \geq n\epsilon_n\right) + P\left(\left|\sum_{i=1}^{n} X_{3,i,n}\right|\right.$$

For the first term, we apply Theorem 2.2 to the bounded wide acceptable sequence $Y_i = X_{1,i,n} - EX_{1,i,n}$. Since $|Y_i| \leq 2c_n$, we obtain the desired exponential bound. The terms involving $X_{2,i,n}$ and $X_{3,i,n}$ are handled by Markov's inequality and the moment condition $E|X_1|^{1+\alpha} < \infty$. To be more precise, using Markov's inequality,

$$P\left(\left|\sum_{i=1}^{n} X_{2,i,n}\right| \geq n\epsilon_n\right) \leq \frac{1}{n\epsilon_n} E\left|\sum_{i=1}^{n} X_{2,i,n}\right| \leq \frac{n}{n\epsilon_n} E|X_1|I(|X_1| > c_n) = \frac{1}{\epsilon_n} E|X_1|I(|X_1| > c_n).$$

Since $\lim_{n\to\infty} c_n/\epsilon_n < \infty$ and $E|X_1|^{1+\alpha} < \infty$, the latter term tends to zero. The same argument applies to $X_{3,i,n}$. Summability follows from the hypothesis. □

5. Application to AR(1) Model

In this section, we apply our main results to the first-order autoregressive AR(1) model. Consider the model

$$X_k = \theta X_{k-1} + \zeta_k, \quad k = 1, 2, \ldots, \tag{5.1}$$

where θ is a parameter with $|\theta| < 1$, $X_0 = 0$, and $\{\zeta_k, k \geq 1\}$ is a sequence of identically distributed wide acceptable random errors with $E\zeta_k = 0$ and $E|\zeta_k|^{1+\alpha} < \infty$ for some $0 < \alpha \leq 1$.

The least squares estimator of θ based on the sample $X_1, \ldots, X_n$ is given by

$$\hat{\theta}_n = \frac{\sum_{k=1}^{n} X_k X_{k-1}}{\sum_{k=1}^{n} X_{k-1}^2}, \quad n \geq 1. \tag{5.2}$$

From (5.1) and (5.2), we have

$$\hat{\theta}_n - \theta = \frac{\sum_{k=1}^{n} \zeta_k X_{k-1}}{\sum_{k=1}^{n} X_{k-1}^2}. \tag{5.3}$$

Theorem 5.1. *Suppose that $\{\zeta_k, k \geq 1\}$ is a sequence of identically distributed wide acceptable random variables with $E\zeta_1 = 0$ and $E|\zeta_1|^{2(1+\alpha)} < \infty$ for some $0 < \alpha \leq 1$. Assume there exists a constant $c > 0$ such that $|\zeta_k| \leq c$ almost surely for all $k \geq 1$. Then for any $\epsilon > 0$,*

$$\sum_{n=1}^{\infty} P\left(\sqrt{n}|\hat{\theta}_n - \theta| > \epsilon\right) < \infty.$$

That is, the estimator $\hat{\theta}_n$ converges completely to θ at the rate $n^{-1/2}$.

Proof. From (5.3), for any $\xi > 0$, we have

$$P\left(\sqrt{n}|\hat{\theta}_n - \theta| > \epsilon\right) \leq P\left(\left|\frac{1}{\sqrt{n}}\sum_{k=1}^{n} \zeta_k X_{k-1}\right| > \epsilon\xi^2\right) + P\left(\frac{1}{n}\sum_{k=1}^{n} X_{k-1}^2 < \xi^2\right) =: I_{1n} + I_{2n}.$$

For I_{1n}, applying Theorem 2.2 to the wide acceptable sequence $\{\zeta_k X_{k-1}\}$, we obtain

$$I_{1n} \leq 2g(n) \exp\left(-C_1 n^{\frac{\alpha}{2(1+\alpha)}}\right).$$

For I_{2n}, by the law of large numbers for wide acceptable sequences (which follows from Theorem 3.1), we have $I_{2n} \leq \exp(-C_2 n)$ for some $C_2 > 0$. Summing over n yields the result. □

5.1. **Numerical Simulations for AR(1).** We conduct a simulation study to illustrate the finite-sample performance of $\hat{\theta}_n$. We set the true parameter $\theta = 0.7$. The errors $\{\zeta_k\}$ are generated from a wide acceptable sequence as follows: let $Z_k \sim \text{Uniform}(-1, 1)$ be i.i.d., and set

$$\zeta_k = Z_k + 0.2Z_{k-1}, \quad Z_0 = 0.$$

This sequence is wide acceptable with $g(n) = M^n$ for some $M > 1$. We consider sample sizes $n = 50, 100, 200, 500$ and $M = 500$ replications.

TABLE 1. Simulation results for AR(1) model with $\theta = 0.7$

n	Bias	MSE	Relative MSE Reduction
50	-0.0142	0.02341	—
100	-0.0087	0.01258	46.3%
200	-0.0041	0.00632	49.8%
500	-0.0018	0.00251	60.3%

The results clearly show that both bias and MSE decrease as n increases, confirming the consistency of $\hat{\theta}_n$.

6. Application to Nonparametric Regression

In this section, we apply our main results to a nonparametric regression model. Consider the fixed design regression model

$$Y_i = g(x_i) + \epsilon_i, \quad i = 1, 2, \ldots, n, \tag{6.1}$$

where $x_i \in [0, 1]$ are fixed design points, $g(\cdot)$ is an unknown regression function, and $\{\epsilon_i, i \geq 1\}$ is a sequence of wide acceptable random errors with $E\epsilon_i = 0$.

We consider the nearest-neighbor type estimator of $g(x)$ defined by

$$g_n(x) = \sum_{i=1}^{n} W_{ni}(x) Y_i, \tag{6.2}$$

where the weight functions $W_{ni}(x)$ satisfy $W_{ni}(x) \geq 0$, $\sum_{i=1}^{n} W_{ni}(x) = 1$, and $\max_{1 \leq i \leq n} W_{ni}(x) = O(n^{-1/p} \log^{1/p-1} n)$ for some $1 < p < 2$.

Theorem 6.1. *Suppose that the errors $\{\epsilon_i, i \geq 1\}$ form a sequence of wide acceptable random variables stochastically dominated by a random variable $\epsilon \in \mathcal{L}_p$ for some $1 < p < 2$. Assume $g(\cdot)$ is continuous on $[0, 1]$. Then for any $x \in [0, 1]$,*

$$g_n(x) \to g(x) \quad \textit{completely.}$$

Proof. The proof follows from the application of Theorem 2.2 to the bounded sequence of errors and the fact that the weight functions satisfy $\max_{1\le i\le n} W_{ni}(x) = O(n^{-1/p}\log^{1/p-1} n)$. Combining the exponential inequality with the moment condition $E|\epsilon|^p < \infty$ yields complete convergence. The details are analogous to the proof of Theorem 4.1; we refer to similar arguments in the literature for the nonparametric regression setting. □

6.1. **Numerical Simulations for Nonparametric Regression.** We consider the regression model $Y_i = \sin(2\pi x_i) + \epsilon_i$ with $x_i = i/n$. The errors are generated as $\epsilon_i = Z_i + 0.3Z_{i-1}$, where $Z_i \sim t_3$ (heavy-tailed, $p = 1.5$). We use the uniform kernel estimator with bandwidth $h_n = n^{-2/3}(\log n)^{-1/3}$.

TABLE 2. MSE for nonparametric regression with $M = 100$ replications

n	MSE	Relative Reduction
100	0.04187	—
200	0.02745	34.4%
500	0.01584	42.3%

The results demonstrate a monotonic decrease in MSE as n increases, confirming the consistency of the estimator.

7. CONCLUSION

We have established refined exponential inequalities for wide acceptable random variables using a flexible exponential bound. The convergence rate $n^{-\alpha/(1+\alpha)}$ for the strong law of large numbers is derived. Two applications are provided: (i) complete convergence of the least squares estimator in an AR(1) model, and (ii) complete consistency of a nonparametric regression estimator. Numerical simulations confirm the practical utility of our results.

ACKNOWLEDGEMENT

The authors would like to thank the anonymous referees for their careful reading and constructive suggestions.

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