Gulf Journal of Mathematics ISSN: 2309-4966
Vol 24, Issue 1 (2026) 1-12 https://doi.org/10.56947/2nmzew85

INFINITELY MANY SOLUTIONS FOR CRITICAL p-KIRCHHOFF PROBLEMS

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Abstract. In this paper, we investigate a class of nonlocal Kirchhoff problems involving the fractional p-Laplacian operator with a critical Sobolev exponent in a bounded domain. With the help of the concentration-compactness principle and the dual fountain theorem, we prove that this problem admits an infinite set of solutions with negative energy that tend to 0.

Keywords. Critical exponent, Dual fountain theorem, Fractional p-Laplacian, Kirchhoff equation, Concentration-compactness principle
2020 Mathematics Subject Classification. Primary 35J20; Secondary 35R11, 35B33

1. Introduction

This article is devoted to the study of a critical Kirchhoff problem involving the fractional p-Laplacian operator, formulated as

$$\begin{cases} [m(\|u\|^p)]^{p-1}\left((-\Delta)_p^s u + |u|^{p-2}u\right) = |u|^{p_s^*-2}u + \lambda g(x,u), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain with dimension N, $\lambda > 0$, $0 < s < 1$, $N > ps$, and $p_s^* := \frac{Np}{N-ps}$ is the fractional critical Sobolev exponent for the embedding of $W_0^{s,p}(\Omega)$ into $L^p(\Omega)$. The fractional p-Laplacian operator $(-\Delta)_p^s$ is defined for sufficiently smooth functions by

$$(-\Delta)_p^s u(x) = 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \backslash B_\varepsilon(x)} \frac{|u(y)-u(x)|^{p-2}(u(y)-u(x))}{|x-y|^{N+ps}}\, \mathrm{d}y, \quad x \in \mathbb{R}^N.$$

For further details on this nonlocal operator, we refer to [10, 23] and the references therein.

The function $m : [0,\infty) \to \mathbb{R}^+$ is continuous and satisfies the following assumptions:

(m_1): There exists $\alpha > 0$ such that $m(\xi) \geq \alpha$ for all $\xi \geq 0$,

Date: Received: Jul 23, 2026; Revised: Aug 21, 2026; Accepted: Aug 28, 2026.
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(m_2): There exists $\sigma > 0$ with $m(\xi) \geq \sigma[M(\xi)]^{p-1}\xi$ for all $\xi \geq 0$, where $M(\xi) = \int_0^\xi [m(t)]^{p-1} dt$,

(m_3): The mapping $\xi \mapsto m(\xi)$ is strictly increasing.

A prototypical example satisfying these conditions is given by $m(\xi) = a + b\xi$ for $p = 2$, widely studied since the foundational work in [2]. The assumptions (m_1)-(m_3) ensure coercivity and subcritical growth of the nonlinearity term.

Regarding the nonlinearity g, we impose the following conditions:

(g_1): $\lim_{\xi\to 0} \frac{g(x,\xi)}{|\xi|^\tau} = 0$ for some $\tau \in (0, p-1)$,

(g_2): $\lim_{\xi\to\infty} \frac{g(x,\xi)}{|\xi|^{p-1}} = 0$,

(g_3): There exists $\theta \in (1, p)$ such that $g(x, \xi)\xi \geq \theta G(x, \xi)$ for all $\xi \geq 0$, where $G(x, \xi) = \int_0^\xi g(x, t)dt$.

The problem (1.1) exhibits a nonlocal nature, preventing it from reducing to a pointwise identity. This nonlocal feature poses distinct mathematical challenges, yet makes its study especially intriguing. We highlight that problem (1.1) arises in numerous physical and biological models where the function u depends on its own mean value. For a detailed mathematical and physical background on Kirchhoff-type problems, see [4, 20].

We now focus on the main advancement of our study. In [8], Chen, Kuo, and Wu consider the following Kirchhoff problem:

$$(a + b\|\nabla u\|_{L^2}^2)\Delta u = \lambda g(x, u) + h(x)|u|^{q-2}u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where $a, b > 0$, and g and h are continuous real-valued functions which may change sign. They establish the existence and multiplicity of solutions by employing the classical Nehari manifold approach. For further results concerning Kirchhoff problems involving the classical Laplacian operator, see for example [1, 6, 12, 13, 21, 22]. Moreover, Kirchhoff equations have also been studied using variational methods when involving the p-Laplacian operator (see [15, 16, 17, 18, 19, 25]).

Furthermore, due to the increasing interest in partial differential equations involving nonlocal operators, these studies have been extended to the fractional Laplacian and fractional p-Laplacian operators. In [14], Fiscella and Valdinoci investigate a fractional Choquard equation with critical exponent:

$$\begin{cases} m\left(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+ps}} dxdy\right)(-\Delta)^s u = |u|^{2_s^*-2}u + \lambda g(x, u), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases}$$

where m and g are continuous functions. Using critical point theory, they prove the existence of a nontrivial solution. For recent developments in this direction, the reader may refer to [3, 18, 27, 28, 32].

More recently, Jiang [18] studied problem (1.1) in the case $s = 1$. The author applied the dual fountain theorem to establish the existence of infinitely many solutions for problem (1.1). An essential intermediate step in this approach is the proof of the local Palais-Smale condition using the Lions concentration-compactness principle.

The objective of this paper is to extend the results from [18] to the case $0 < s < 1$. Following the strategy employed in [18], we focus on establishing the existence of infinitely many solutions to problem (1.1).

Compared to [18], the techniques used in the proof of Lemma 2.3 are different. Our techniques extend variational methods to fractional spaces $W^{s,p}(\Omega)$ with $s \in (0,1)$. The primary analytical difficulty stems from the non-locality of the fractional p-Laplacian operator, which significantly complicates the analysis of the problem compared to the local case. Our main result is as follows:

Theorem 1.1. *Suppose that m satisfies (m_1)–(m_3) and g satisfies (g_1)–(g_3). Then, there exists $\Lambda > 0$ such that if $0 < \lambda < \Lambda$, problem (1.1) has a sequence of solutions (u_n) such that $I(u_n) < 0$ and $I(u_n) \to 0$ as $n \to \infty$.*

For the proof of Theorem 1.1, we first employ Lions concentration-compactness principle to establish the compactness of any Palais-Smale $(PS)_c$ sequence associated with the functional I. Then, the existence of infinitely many solutions is obtained by applying the dual fountain theorem.

This paper is organized as follows. In Section 2, we introduce some notation and preliminary results. In the last section, we apply the dual fountain theorem to prove the existence of infinitely many solutions for (1.1).

2. Notations and preliminary results

We start by introducing the notations that will be consistently used throughout this manuscript. Let $\Omega \subset \mathbb{R}^N$ be a bounded open domain with smooth boundary $\partial\Omega$. Let $s \in (0,1)$ and $p > 1$. Set

$$Q := \mathbb{R}^{2N} \backslash (\mathcal{C}\Omega \times \mathcal{C}\Omega)$$

with $\mathcal{C}\Omega = \mathbb{R}^N \backslash \Omega$. We consider the fractional Sobolev space X defined by

$$X := \left\{ u : \mathbb{R}^N \to \mathbb{R} \;\middle|\; u|_\Omega \in L^p(\Omega) \text{ and } \int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\, dxdy < \infty \right\},$$

endowed with the norm

$$\|u\|_X = \left(\int_\Omega |u(x)|^p dx + \int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\, dxdy \right)^{\frac{1}{p}}.$$

Taking into account the Dirichlet boundary condition $u = 0$ in $\mathbb{R}^N \setminus \Omega$, we work in the closed subspace X_0 defined by

$$X_0 = \left\{ u \in X : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \right\}.$$

We endow the space X_0 with the norm defined by

$$\|u\| := \left(\int_\Omega |u(x)|^p dx + \int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\, dxdy \right)^{\frac{1}{p}}. \tag{2.1}$$

Equipped with this norm, $(X_0, \|\cdot\|)$ is a reflexive and separable Banach space. For any $r \in [1, p_s^*]$, the embedding $X_0 \hookrightarrow L^r(\Omega)$ is continuous. Furthermore, this embedding is compact for all $r \in [1, p_s^*)$. For any $1 < r \leq p_s^*$, let $\mathcal{S}_r$ denote the optimal Sobolev embedding constant defined by

$$\mathcal{S}_r := \inf_{u \in X_0 \setminus \{0\}} \frac{\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy}{\left(\int_\Omega |u|^r dx\right)^{\frac{p}{r}}}. \tag{2.2}$$

When $r = p_s^*$, we simplify the notation by setting $\mathcal{S} = \mathcal{S}_{p_s^*}$.

Weak solutions of (1.1) coincide with the critical points of the $\mathcal{C}^1$-functional $I : X \to \mathbb{R}$, defined by

$$I(u) = \frac{1}{p} M(\|u\|^p) - \lambda \int_\Omega G(x, u) dx - \frac{1}{p_s^*} \int_\Omega |u|^{p_s^*} dx.$$

For convenience, we define

$$\mathcal{A}_{s,p}(u, \varphi) := \int_Q \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+ps}} dxdy.$$

Then $u \in X$ is a weak solution of problem (1.1) if

$$\begin{aligned} I'(u)\varphi : &= [m(\|u\|^p)]^{p-1} \left(\mathcal{A}_{p,s}(u, \varphi) + \int_\Omega |u|^{p-2} u \varphi dx \right) \\ &- \lambda \int_\Omega g(x, u) \varphi dx - \int_\Omega |u|^{p_s^*-2} u \varphi dx = 0, \text{ for any } \varphi \in X. \end{aligned}$$

Additionally, for notational simplicity, we introduce the fractional (s, p)-gradient of a function $u \in X$ defined by

$$|D^s u(x)|^p = \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dy,$$

which is well defined for almost every $x \in \mathbb{R}^N$ and satisfies $|D^s u(x)| \in L^p\left(\mathbb{R}^N\right)$.

The following lemma establishes the concentration-compactness principle adapted to the fractional p-Laplacian operator. This result extends the well-known concentration-compactness principle originally proven by Lions (see [24, 33]) to the framework of the fractional p-Laplacian.

Lemma 2.1 ([5], Theorem 1.1)**.** *Let $(u_n) \subset D^{s,p}(\mathbb{R}^N)$ be a weakly convergent sequence with weak limit u. Then there exist two bounded measures μ and ν, an at most countable set J, $x_i \in \mathbb{R}^N$, and positive real numbers μ_i, ν_i, $i \in J$, such that the following convergences hold weakly in the sense of measures,*

$$|D^s u_n|^p dx \rightharpoonup \mu \geq |D^s u|^p dx + \sum_{i \in J} \mu_i \delta_{x_i},$$

$$|u_n|^{p_s^*} dx \rightharpoonup \nu = |u|^{p_s^*} dx + \sum_{i \in J} \nu_i \delta_{x_i},$$

$$S \nu_i^{p/p_s^*} \leq \mu_i, \quad \textit{for all } i \in J.$$

Before proving that the functional I satisfies the Palais–Smale condition at a given level, it is necessary to establish the following preliminary result.

Lemma 2.2. *If $(u_n) \subset X_0$ is a Palais–Smale sequence at level c for the functional I, then there exists a constant $\Lambda > 0$ such that (u_n) is bounded in X_0 whenever $\lambda \in (0, \Lambda)$.*

Proof. Using (g_1)-(g_2), there exist constants $c_1, c_2 > 0$, independent of ξ, such that

$$|g(x,\xi)| \leq c_1|\xi|^{p-1} + c_2|\xi|^{\tau}.$$

Then from (g_3) we obtain

$$G(x,\xi) - \frac{1}{p_s^*}g(x,\xi)\xi \leq \left(\frac{1}{\theta} - \frac{1}{p_s^*}\right)\left(c_1|\xi|^p + c_2|\xi|^{\tau+1}\right).$$

Now, by (m_1), (m_2) and (2.2) we have

$$\begin{aligned}
&c + 1 + \|u_n\| \geq I(u_n) - \frac{1}{p_s^*}\langle I'(u_n), u_n\rangle \\
&= \frac{1}{p}M\left(\|u_n\|^p\right) - \frac{1}{p_s^*}\left[m\left(\|u_n\|^p\right)\right]^{p-1}\|u_n\|^p - \lambda\int_\Omega\left(G(x,u_n) - \frac{1}{p_s^*}g(x,u_n)u_n\right)dx \\
&\geq \left(\frac{\sigma}{p} - \frac{1}{p_s^*}\right)m_0^{p-1}\|u_n\|^p - \lambda\left(\frac{1}{\theta} - \frac{1}{p_s^*}\right)\left(C_1\mathcal{S}_p^{-1}\|u_n\|^p + C_2\mathcal{S}_{\tau+1}^{-\frac{\tau+1}{p}}\|u_n\|^{\tau+1}\right).
\end{aligned}$$

Therefore, for every $\lambda \in (0, \Lambda)$ with Λ sufficiently small, the sequence (u_n) remains bounded in X_0. $\square$

Lemma 2.3. *Define*

$$\begin{aligned}
c^* := &\left[\left(\frac{\sigma}{p} - \frac{1}{p_s^*}\right)\mathcal{S}\alpha^{p-1} - \lambda\left(\frac{1}{\theta} - \frac{1}{p_s^*}\right)C_1|\Omega|^{\frac{p_s^*-p}{p_s^*}}\right]\left(\mathcal{S}\alpha^{p-1}\right)^{\frac{p}{p_s^*-p}} \\
&- \lambda\left(\frac{1}{\theta} - \frac{1}{p_s^*}\right)C_2|\Omega|^{\frac{p_s^*-\tau-1}{p_s^*}}\left(\mathcal{S}\alpha^{p-1}\right)^{\frac{\tau+1}{p_s^*-p}}.
\end{aligned}$$

For $\lambda > 0$ chosen sufficiently small so that $c^* > 0$, the functional I satisfies the Palais–Smale condition in $(0, c^*)$.

Proof. Let (u_n) be a $(PS)_c$ sequence of I with $c \in (0, c^*)$. By Lemma 2.2, (u_n) is bounded in X_0. Then, there exist $u \in X_0$ and a subsequence, still denoted by (u_n), such that

$$\begin{aligned}
u_n &\rightharpoonup u, \quad \text{weakly in } X_0, \\
u_n &\to u, \quad \text{strongly in } L^q(\Omega), \ \ 1 \leq q < p_s^*, \\
u_n &\to u, \quad \text{a.e. in } \ \Omega.
\end{aligned}$$

By applying the concentration-compactness principle (Lemma 2.1), one obtains the existence of a bounded measure ν on $\mathbb{R}^N$ such that

$$|u_n|^{p_s^*}dx \rightharpoonup \nu,$$

there is also a bounded nonnegative μ measures on $\mathbb{R}^N$, such that

$$|D^s u_n|^p \rightharpoonup \mu.$$

Moreover, there exist $J \subset \mathbb{N}$ and

$$\{x_i\}_{i\in J} \subset \mathbb{R}^N, \quad \{\mu_i\}_{i\in J} \subset (0,\infty), \quad \{\nu_i\}_{i\in J} \subset (0,\infty)$$

with

$$\mathcal{S}\nu_i^{p/p_s^*} \leq \mu_i, \tag{2.3}$$

such that

$$\mu \geq |D^s u|^p + \sum_{i\in J} \mu_i \delta_{x_j}, \quad \nu = |u|^{p_{s}^*1} dx + \sum_{i\in J} \nu_i \delta_{x_i}, \tag{2.4}$$

where δ_{x_i} is the Dirac functions at the points x_i of $\mathbb{R}^N$. Let $\psi \in C_0^\infty(\mathbb{R}^N)$ such that

$$\psi(x) = 1 \quad \text{for } x \in B_1(0), \quad \psi(x) = 0 \quad \text{for } x \in \mathbb{R}^N \backslash B_2(0) \text{ and } |D^s\psi| \leq 2,$$

where $B_r(x_0) := \{x \in \mathbb{R}^N;\, |x - x_0| \leq r\}$. Set $\psi_R(x) = \psi\left(\frac{x-x_i}{R}\right)$ with $R > 0$. According to $I'(u_n) \to 0$ as $n \to +\infty$, taking the test function $u_n\psi_R$, then we obtain $\langle I'(u_n), u_n\psi_R\rangle \to 0$ as $n \to \infty$. Then

$$\begin{aligned} &[M(\|u_n\|^p)]^{p-1}\left(\mathcal{A}_{p,s}(u_n, u_n\psi_R) + \int_\Omega |u_n|^p \psi_R dx\right) \\ &\qquad = \lambda \int_\Omega g(x, u_n) u_n \psi dx + \int_\Omega |u_n|^{p_s^*} \psi_R dx + o(1). \end{aligned} \tag{2.5}$$

First, it is easy to obtain

$$\int_\Omega |u_n|^p \psi_R dx \to 0, \int_\Omega g(x, u_n) u_n \psi_R dx \to 0, \text{ as } n \to \infty \text{ and } R \to 0. \tag{2.6}$$

On the other hand,

$$\begin{aligned} &\mathcal{A}_{p,s}(u_n, u_n\psi_R) \\ =&\int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2}\left(u_n(x) - u_n(y)\right)\left(u_n(x)\psi_R(x) - u_n(y)\psi_R(y)\right)}{|x-y|^{N+sp}} dxdy \\ =&\int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^p \psi_R(x)}{|x-y|^{N+sp}} dxdy \\ +&\int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2}\left(u_n(x) - u_n(y)\right)\left(\psi_R(x) - \psi_R(y)\right) u_n(y)}{|x-y|^{N+sp}} dxdy \\ =&A_1 + A_2. \end{aligned}$$

Clearly, $A_1 = \int_{\mathbb{R}^N} |D^s u_n|^p \psi_R dx$, then by using $u_n \rightharpoonup u$ in $D^{s,p}(\mathbb{R}^N)$ and Lemma 2.1, we have

$$\lim_{n\to\infty} A_1 = \int_{\mathbb{R}^N} \psi_R d\mu.$$

Since $\mu - \sum_{i\in J} \mu_i \delta_{x_i}$ has no atoms and $\psi_R \to 0$ as $R \to 0$ for any $x \neq x_i, i \in J$, we conclude that

$$\lim_{R\to 0} \lim_{n\to\infty} A_1 = \mu_i.$$

Now, we estimate A_2. By using the Hölder inequality, we have

$$
\begin{aligned}
A_2 &\le \int_{\mathbb{R}^N} \left(|D^s u_n|^p\right)^{\frac{p-1}{p}} \left(|D^s \psi_R|^p\right)^{\frac{1}{p}} |u_n| dy \\
&\le \|D^s u_n\|_p^{p-1} \left(\int_{\mathbb{R}^N} |u_n|^p |D^s \psi_R|^p dx\right)^{\frac{1}{p}} \\
&\le C \left(\int_{\mathbb{R}^N} |u_n|^p |D^s \psi_R|^p dx\right)^{\frac{1}{p}}.
\end{aligned}
$$

Then, we have

$$
\limsup_{n\to\infty} A_2 \le C \left(\int_{\mathbb{R}^N} |u|^p |D^s \psi_R|^p dx\right)^{\frac{1}{p}}. \tag{2.7}
$$

Now, we show that $\lim_{R\to 0} \int_{\mathbb{R}^N} |u|^p |D^s \psi_R|^p dx = 0$. Indeed, by [5, Corrolary 2.3] and Hölder inequality, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N} |u|^p |D^s \psi_R|^p dx \\
&\le C \left(R^{-sp} \int_{|x|<R} |u|^p dx + R^N \int_{|x|\ge R} \frac{|u|^p}{|x|^{N+sp}} dx \right) \\
&\le C R^{-sp} \left(\int_{|x|<R} |u|^{p_s^*} dx\right)^{\frac{p}{p_s^*}} |B_R|^{\frac{sp}{N}} + C R^N \sum_{k=0}^{\infty} \int_{2^k R \le |x| \le 2^{k+1} R} \frac{|u|^p}{|x|^{N+sp}} dx \\
&\le C \left(\int_{|x|<R} |u|^{p_s^*} dx\right)^{\frac{p}{p_s^*}} \\
&\quad + C \sum_{k=0}^{\infty} 2^{-k(N+sp)} R^{-sp} \left(\int_{|x|\le 2^{k+1}R} |u|^{p_s^*} dx\right)^{\frac{p}{p_s^*}} |B_{2^{k+1}R}|^{\frac{sp}{N}} \\
&= C \left(\int_{|x|<R} |u|^{p_s^*} dx\right)^{\frac{p}{p_s^*}} + C \sum_{k=0}^{\infty} 2^{-kN} \left(\int_{|x|\le 2^{k+1}R} |u|^{p_s^*} dx\right)^{\frac{p}{p_s^*}}.
\end{aligned}
$$

Since $u \in L^{p_s^*}\left(\mathbb{R}^N\right)$, we have $\lim_{R\to 0} \int_{|x|<R} |u|^{p_s^*} dx = 0$. Let $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that $c_2 \sum_{k=k_0+1}^{\infty} 2^{-kN} < \varepsilon$, then

$$
\begin{aligned}
&C \sum_{k=0}^{\infty} 2^{-kN} \left(\int_{|x|\le 2^{k+1}R} |u|^{p_s^*} dx\right)^{\frac{p}{p_s^*}} \\
&\le C \sum_{k=0}^{k_0} 2^{-kN} \left(\int_{|x|\le 2^{k_0+1}R} |u|^{p_s^*} dx\right)^{\frac{p}{p_s^*}} + \varepsilon \|u\|_{p_s^*}^p
\end{aligned}
$$

Then,

$$
\limsup_{R\to 0} C \sum_{k=0}^{\infty} 2^{-kN} \left(\int_{|x|\le 2^{k+1}R} |u|^{p_s^*} dx\right)^{\frac{p}{p_s^*}} \le \varepsilon \|u\|_{p_s^*}^p.
$$

Hence,

$$\lim_{R\to 0}\int_{\mathbb{R}^N}|u|^p|D^s\psi_R|^p dx=0. \tag{2.8}$$

By using (2.7) and (2.8), we have $\lim_{R\to 0}\lim_{n\to\infty}A_2=0$. Thus, we conclude that

$$\lim_{R\to 0}\lim_{n\to\infty}\mathcal{A}(u_n,u_n\psi_R)=\mu_i. \tag{2.9}$$

Therefore, by using (2.5), (2.6) and (2.9) we have $\nu_i\geq\alpha^{p-1}\mu_i$. Combining this with (2.3), we obtain that $\mathcal{S}(\nu_i)^{\frac{p}{p_s^*}}\leq\alpha^{1-p}\nu_i$. This implies

$$\nu_i=0 \ \text{ or } \ \nu_i\geq(\mathcal{S}\alpha^{p-1})^{\frac{p_s^*}{p_s^*-p}}.$$

If $\nu_i\geq(\mathcal{S}\alpha^{p-1})^{\frac{p_s^*}{p_s^*-p}}$, we have

$$\begin{aligned}
&I\left(u_n\right)-\frac{1}{p_s^*}\left\langle I'\left(u_n\right),u_n\right\rangle\\
&\quad\geq\left(\frac{\sigma}{p}-\frac{1}{p_s^*}\right)\left[M\left(\|u_n\|^p\right)\right]^{p-1}\|u_n\|^p-\lambda\left(\frac{1}{\theta}-\frac{1}{p_s^*}\right)\int_\Omega\left(C_1|u_n|^p+C_2|u_n|^{\tau+1}\right)dx\\
&\quad\geq\left(\frac{\sigma}{p}-\frac{1}{p_s^*}\right)m_0^{p-1}\mathcal{S}\|u_n\|^p_{L^{p_s^*}(\Omega)}-\lambda\left(\frac{1}{\theta}-\frac{1}{p_s^*}\right)\\
&\qquad\times\left[C_1\|u_n\|^p_{L^{p^*}(\Omega)}|\Omega|^{\frac{p_s^*-p}{p_s^*}}+C_2\|u_n\|^{\tau+1}_{L^{p^*}(\Omega)}|\Omega|^{\frac{p_s^*-\tau-1}{p_s^*}}\right]\\
&\quad\geq\left[\left(\frac{\sigma}{p}-\frac{1}{p_s^*}\right)\mathcal{S}m_0^{p-1}-\lambda\left(\frac{1}{\theta}-\frac{1}{p_s^*}\right)C_1|\Omega|^{\frac{p_s^*-p}{p_s^*}}\right]\left(\mathcal{S}m_0^{p-1}\right)^{\frac{p}{p_s^*-p}}\\
&\qquad-\lambda\left(\frac{1}{\theta}-\frac{1}{p_s^*}\right)C_2|\Omega|^{\frac{p_s^*-\tau-1}{p_s^*}}\left(\mathcal{S}m_0^{p-1}\right)^{\frac{\tau+1}{p_s^*-p}}.
\end{aligned}$$

We conclude that

$$c=\lim_{n\to\infty}\left(I(u_n)-\frac{1}{p_s^*}\langle I'(u_n),u_n\rangle\right)\geq c^*,$$

which contradicts $c<c^*$. Hence, $\nu_i=0$. So, we have $u_n\to u$ in $L^{p_s^*}(\Omega)$. This completes the proof of Lemma 2.3. □

3. Proof of the main results

In this section, we provide the proof of the main theorem. The key instrument used to achieve this proof is the dual fountain theorem (see [31, Theorem 3.18]), which we recall below. Let X be a Banach space equipped with the norm $\|.\|$, and suppose that

$$X=\overline{\oplus_{j=k}^\infty X_j}$$

with $\dim X_j<\infty$ for each $j\in\mathbb{N}$. We define

$$Y_k=\oplus_{j=1}^k X_j,\ Z_k=\overline{\oplus_{j=k}^\infty X_j}.$$

Definition 3.1. Let $I \in \mathcal{C}^1(X, \mathbb{R})$ and $c \in \mathbb{R}$. The functional I satisfies the $(PS)^*_c$ condition (with respect to (Y_n)) if any sequence $(u_n) \subset X$ such that

$$u_n \in Y_n, \quad I(u_n) \to c, \; I'_{|Y_n}(u_n) \to 0 \text{ in } X^* \quad \text{as } n \to \infty$$

contains a subsequence converging to a critical point of I.

Theorem 3.2 ([31]). *Let $I \in \mathcal{C}^1(X, \mathbb{R})$ be an even functional. If for every $k \geq k_0$, there exists $\rho_k > r_k > 0$ such that*

(a) $a_k := \inf_{\substack{u \in Z_k \\ \|u\| = \rho_k}} I(u) \geq 0,$

(b) $b_k := \sup_{\substack{u \in Y_k \\ \|u\| = r_k}} I(u) < 0,$

(c) $d_k := \inf_{\substack{u \in Z_k \\ \|u\| \leq \rho_k}} I(u) \to 0,$ *as* $k \to \infty,$

(d) *I satisfies the $(PS)^*_c$ condition for every $c \in [d_k, 0)$.*

Then I has a sequence of negative critical values converging to 0.

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. By Lemma 2.3, the functional I satisfies the $(PS)^*_c$ condition with respect to (Y_k). It suffices then to verify the other assumptions of Theorem 3.2.

In order to verify (a), we define

$$\gamma_k := \sup_{\substack{u \in Z_k \\ \|u\| = 1}} \left(\int_\Omega |u|^{\tau+1} dx \right)^{\frac{1}{\tau+1}}, \quad k \geq 1. \tag{3.1}$$

Then, $\gamma_k \to 0$ as $k \to \infty$ For further details, see Lemma 3.8 in [31].

Using assumptions (m_1) and (m_2) we obtain

$$I(u) \geq \frac{\sigma}{p} \alpha^{p-1} \|u\|^p - \lambda \int_\Omega \left(\frac{C_1}{p} |u|^p + \frac{C_2}{\tau+1} |u|^{\tau+1} \right) dx - \frac{1}{p_s^*} \int_\Omega |u|^{p_s^*} dx.$$

Moreover, by (3.1) we have

$$I(u) \geq \frac{\sigma \alpha^{p-1} - \lambda C_1}{p} \|u\|^p - \frac{\lambda C_2}{\tau+1} \gamma_k^{\tau+1} \|u\|^{\tau+1} - \frac{1}{p_s^*} \mathcal{S}^{-\frac{p_s^*}{p}} \|u\|^{p_s^*}.$$

Note that there exists $R > 0$ such that

$$\frac{\sigma \alpha^{p-1} - \lambda C_1}{p} \|u\|^p \geq \frac{2}{p_s^*} \mathcal{S}^{-\frac{p_s^*}{p}} \|u\|^{p_s^*}$$

for all $\|u\| \leq R$, this implies that

$$I(u) \geq \frac{\sigma \alpha^{p-1} - \lambda C_1}{2p} \|u\|^p - \frac{\lambda C_2}{\tau+1} \gamma_k^{\tau+1} \|u\|^{\tau+1}. \tag{3.2}$$

We choose $\rho_k := \left(\frac{2p \lambda C_2 \gamma_k^{\tau+1}}{(\sigma \alpha^{p-1} - \lambda C_1)(\tau+1)} \right)^{\frac{1}{p-\tau-1}}$. Since $\gamma_k \to 0$ as $k \to \infty$, it follows that $\rho_k \to 0$ as $k \to \infty$. There exists $k_0 \in \mathbb{N}^*$ such that $\rho_k \leq R$ when $k \geq k_0$. Then, for $k \geq k_0$, $u \in Z_k$ and $\|u\| = \rho_k$, we obtain $I(u) \geq 0$. This gives (a).

Next we show (b). Using the mean value theorem and (m_3), we obtain for all $\|u\| \leq \rho_k$ that there exists $\tau \in (0,1)$ such that

$$M(\|u\|^p) = [m(\tau\|u\|^p)]^{p-1} \|u\|^p \leq [m(\rho_k^p)]^{p-1} \|u\|^p.$$

By the equivalence of norms on the finite dimensional space Y_k, we can find positive constants C_3, C_4, C_5 such that

$$I(u) \leq \left(\frac{[m(\rho_k^p)]^{p-1}}{p} - C_3 \right) \|u\|^p - C_4\|u\|^{\tau+1} - C_5\|u\|^{p_s^*}.$$

Take r_k sufficiently small, we have $I(u) < 0$ for all $\|u\| = r_k$.

The proof of assertion (c) follows from inequality (3.2). Then, the assumptions stated in Theorem 3.2 are fulfilled. This completes the proof of Theorem 1.1. $\square$

Conflict of interest. The authors have no conflict of interest to declare.

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