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GENERALIZED CARTAN AND CARTAN-HARTOGS DOMAINS

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ABSTRACT. In this paper, we investigate weighted reproducing kernels on Cartan domains (irreducible bounded symmetric domains) and Cartan-Hartogs domains. For a Hilbert space $\mathcal{H}_\beta$ of holomorphic functions equipped with generalized canonical metrics $g(\lambda_j)$, explicit formulas for the weighted reproducing kernels are obtained. These results are further extended to Cartan-Hartogs domains $H_D^{B^{q_0}}(\lambda_j)$.

Keywords. Cartan domains, weighted reproducing kernels, Generalized Cartan-Hartogs domains.
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1. Introduction and preliminaries

Let D in $\mathbb{C}^q$ be a Cartan domain, that is, an irreducible bounded symmetric domain in its Harish-Chandra realization, with its rank r, the characteristic multiplicities a and b, the complex dimension q and the genus γ. Therefore, D is the open unit ball of a Banach space equipped with a JB^*-triple structure.

Therefore

$$\begin{aligned} q &= \frac{r(r-1)}{2}a + rb + r \\ \gamma &= a(r-1) + 2 + b. \end{aligned} \tag{1.1}$$

Cartan-Hartogs domains are defined as a family of Hartogs domains over Cartan domains. We define the generic norm of D by

$$N_D(z,\bar{\eta}) := \frac{1}{(V(D)K_D(z,\bar{\eta}))^{\frac{1}{\gamma}}}, \tag{1.2}$$

where $V(D)$ denotes the volume of D with respect to the Euclidean measure on $\mathbb{C}^q$, and $K_D(z,\bar{\eta})$ is its weighted Bergman kernel. For a Cartan domain D in $\mathbb{C}^q$, positive parameter $\lambda_j, j = 1, 2, \ldots, y$ and a positive integer q_0, we define the generalized Cartan-Hartogs domain by

$$H_D^{B^{q_0}}(\lambda_j) := \left\{ (z,u) \in \mathbb{C}^{q_0} \times D \subset \mathbb{C}^{q_0} \times \mathbb{C}^q \;\middle|\; |u|^2 < N_D^{\lambda_j}(z,\bar{z}) \right\}, \tag{1.3}$$

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where $|\cdot|$ denotes the Euclidean norm on $\mathbb{C}^{q_0}$. The domain D is referred to as the base domain of the Cartan-Hartogs domain $H_D^{B^{q_0}}(\lambda_j)$.

The study of asymptotic expansion of Bergman kernels has attracted considerable attention over the past decades due to its fundamental role in complex geometry, quantization, and Kähler metrics (see [13], [5]).

Motivated in part by the influential contribution of Donaldson [1], substantial progress has been made in understanding the structure and applications of these expansions. We refer the reader to [10] and the references therein for a comprehensive account of the subject.

The investigation of complex manifolds and differential geometry heavily relies on the study of bounded symmetric domains, historically known as Cartan domains. Initial frameworks introduced by Feng [4] established the primary Kähler geometry of Cartan domains, characterizing their explicit Bergman metrics, generic norms, and associated geometric properties.

Expanding significantly upon these single-domain structures, subsequent literature advanced into more intricate product spaces. Specifically, Zedda [12] introduced and studied generalized Cartan-Hartogs domains. While Feng's early constructions provided critical insights into standard Cartan-Hartogs configurations, the transition to Zedda's generalized framework opens up complex analytical challenges regarding function spaces, exact kernel functions, and invariant metrics.

Let D in $\mathbb{C}^n$ be a bounded domain equipped with a strictly plurisubharmonic function ψ. The associated Kähler form $\omega_D = \frac{i}{2\pi}\partial\bar{\partial}\psi$ defines a Kähler metric g_D on D. For each $\beta > 0$, let $\mathcal{H}_\beta$ denote the weighted Hilbert space of holomorphic functions on (D, g_D) that are square-integrable with respect to the weight $e^{(-\psi\beta)}$, i.e.,

$$\mathcal{H}_\beta := \left\{ f \in \mathrm{Hol}(D) \;\middle|\; \int_D |f|^2 e^{(-\psi\beta)} \frac{\omega_D^n}{n!} < \infty \right\}, \tag{1.4}$$

where $\mathrm{Hol}(D)$ is the space of holomorphic functions on D and $\frac{\omega_D^n}{n!} = \det\left(\frac{\partial^2\psi}{\partial z_j \partial \bar{z}_l}\right)\frac{\omega_0^n}{n!}$ denotes the volume form associated with Kähler form ω_D and $\omega_0 = \frac{i}{2\pi}\sum_{j,l=1}^{n} dz_j \wedge d\bar{z}_l$.

Let K_β denotes the weighted Bergman kernel of $\mathcal{H}_\beta$ if $\mathcal{H}_\beta$ is nontrivial. For more on the theory of Hilbert spaces and related frame constructions, (see [6]). In this paper, we investigate the theory of reproducing kernel Hilbert spaces of holomorphic functions endowed with generalized canonical metrics on Cartan domains and Cartan-Hartogs domains.

The paper is organized as follows. In Section 2, we present explicit descriptions of the various classical domains and the associated Cartan-Hartogs domains $H_D^{B^{q_0}}(\lambda_j)$. Finally, in Section 3, we derive an explicit formula for the weighted reproducing kernel corresponding to the Kähler metric $g(\lambda_j)$ on $H_D^{B^{q_0}}(\lambda_j)$.

2. Classical domains

Let $M_{n\times m}$ denote the set of all $n \times m$ complex matrices, let Z^* denote the conjugate transpose of the matrix Z, and I is the identity matrix. A square matrix Z is said to be positive definite if $Z > 0$. For each bounded symmetric domain D (see [2, 3, 7, 11]), we provide the genus $\gamma(D)$, the generic norm $N_D^{\lambda_j}(z,\bar{z})$ of D and the corresponding Cartan-Hartogs domain $H_D^{B^{q_0}}(\lambda_j)$ associated with its type are given as follows:

(1) The Classical domains of type I: If $D = D^{\mathrm{I}}_{n,m} := \{Z \in M_{n\times m}(\mathbb{C}) \mid I - ZZ^* > 0\}$ $(1 \le r \le m)$, then $\gamma(D) = r + m$, $N_D(z,\bar{z}) = \det(I - ZZ^*)$, and

$$H_D^{B^{q_0}}(\lambda_j) = \left\{(z,u) \in \mathbb{C}^{q_0} \times D^{\mathrm{I}}_{n,m} \subset \mathbb{C}^{q_0} \times \mathbb{C}^{n\times m} \;\middle|\; |u|^2 < \det(I - ZZ^*)^{\lambda_j}\right\}.$$

In particular, when $D := \mathbb{B}^n$ denotes the unit ball in $\mathbb{C}^n$, it follows that

$$H_D^{B^{q_0}}(\lambda_j) = \left\{(z,u) \in \mathbb{C}^{q_0} \times D \;\middle|\; |z|^2 < 1 - |u|^{2\lambda_j}\right\}.$$

This domain arises as a natural generalization of the Thullen domains.

(2) The Classical domains of type II: If $D = D^{\mathrm{II}}_{n} := \left\{Z \in M_{n\times n}(\mathbb{C}) \;\middle|\; Z = -Z^T,\ I - ZZ^* > 0\right\}$ $(n \ge 4)$, then $\gamma(D) = 2n-2$ (or $4n + 2\varepsilon - 2$, $\varepsilon \in \{0,1\}$ depending on convention), $N_D(z,\bar{z}) = \sqrt{\det(I - ZZ^*)}$, and

$$H_D^{B^{q_0}}(\lambda_j) = \left\{(z,u) \in \mathbb{C}^{q_0} \times D^{\mathrm{II}}_{n} \subset \mathbb{C}^{q_0} \times \mathbb{C}^{n(n-1)/2} \;\middle|\; |u|^2 < (\det(I - ZZ^*))^{\frac{\lambda_j}{2}}\right\}.$$

(3) The Classical domains of type III: If $D = D^{\mathrm{III}}_{n} := \left\{Z \in M_{n\times n}(\mathbb{C}) \;\middle|\; Z = Z^T,\ I - ZZ^* > 0\right\}$ $(n \ge 2)$, then $\gamma(D) = n+1$, $N_D(z,\bar{z}) = \det(I - ZZ^*)$, and

$$H_D^{B^{q_0}}(\lambda_j) = \left\{(z,u) \in \mathbb{C}^{q_0} \times D^{\mathrm{III}}_{n} \subset \mathbb{C}^{q_0} \times \mathbb{C}^{n(n+1)/2} \;\middle|\; |u|^2 < (\det(I - ZZ^*))^{\lambda_j}\right\}.$$

(4) The Classical domains of type IV: If $D = D^{\mathrm{IV}}_{n} := \left\{z \in \mathbb{C}^n \;\middle|\; 2|z|^2 < 1 + |zz^T|^2,\ |z| < 1\right\}$ $(n \ge 5)$, then $\gamma(D) = n$, $N_D(z,\bar{z}) = 1 - 2|z|^2 + |zz^T|^2$, and

$$H_D^{B^{q_0}}(\lambda_j) = \left\{(z,u) \in \mathbb{C}^{q_0} \times D^{\mathrm{IV}}_{n} \subset \mathbb{C}^{q_0} \times \mathbb{C}^{n} \;\middle|\; |u|^2 < \left(1 - 2|z|^2 + |zz^T|^2\right)^{\lambda_j}\right\}.$$

By [8], the origin-centered Kähler potential for the Cartan-Hartogs domain $H_D^{B^{q_0}}(\lambda_j)$ is defined by

$$\Phi(z,u) := -\log\left(N_D^{\lambda_j}(z,\bar{z}) - |u|^2\right). \tag{2.1}$$

Associated Kähler form $\omega(\lambda_j)$ on $H_D^{B^{q_0}}(\lambda_j)$ is given by

$$\omega(\lambda_j) := \frac{i}{2\pi}\partial\bar{\partial}\Phi. \tag{2.2}$$

The Kähler metric $g(\lambda_j)$ associated with $\omega(\lambda_j)$ on $H_D^{B^{q_0}}(\lambda_j)$ is defined by $ds^2 = \sum_{j,l=1}^{M} \frac{\partial^2\Phi}{\partial z_j \partial \bar{z}_l} dz_j \otimes d\bar{z}_l$, (where $M = q_0 + q$, $z = (z_1, \ldots, z_q)$, $u = (z_{q+1}, \ldots, z_M)$). For $q_0 = 1$, Loi and Zedda [8] investigated balanced metrics on Cartan-Hartogs domains.

Theorem 2.1. [12] *Let D be an irreducible bounded symmetric domain of dimension q and genus γ. Then the scalar curvature of a Cartan-Hartogs domain $\left(H_D^{B^{q_0}}(\lambda_j), g(\lambda_j)\right)$ is constant if and only if $g(\lambda_j)$ is Kähler-Einstein, that is, $\lambda_j = \frac{\gamma}{q+1}$.*

3. The weighted reproducing kernel of $\mathcal{H}_\beta$

For every $t > -1$, the Hua integral $\int N_D(z,\bar{z})^t dv(z)$ is evaluated as

$$\int_D N_D(z,\bar{z})^t dv(z) = \frac{\chi(0)}{\chi(t)} \int_D dv(z), \tag{3.1}$$

where $dv(z)$ is the Lebesgue measure on $\mathbb{C}^q$, and $\chi(t)$ denotes the Hua polynomial given by

$$\chi(t) := \prod_{j=1}^{r} \left(t + 1 + (j-1)\frac{a}{2}\right)_{1+b+(r-j)a}. \tag{3.2}$$

For any non-negative integer v, the rising factorial $(t)_v$ is given by

$$(t)_v := \frac{\Gamma(t+v)}{\Gamma(t)} = t(t+1)\dots(t+v-1). \tag{3.3}$$

Let $\mathcal{G}$ denote the connected component of the identity of the biholomorphism group of D, and $\mathcal{K}$, the stabilizer of the origin in $\mathcal{G}$. The natural action of $\mathcal{G}$ on the space $\mathcal{P}$ of holomorphic polynomials on $\mathbb{C}^q$, defined by $f \mapsto f \circ k$ for $k \in \mathcal{K}$, yields a Peter-Weyl decomposition

$$\mathcal{P} = \bigoplus_{\mu} \mathcal{P}_\mu,$$

where the decomposition is taken over all partitions, that is, $\mu = (\mu_1, \dots, \mu_r)$ is an r-tuple of non-negative integers with $\mu_1 \geq \dots \geq \mu_r \geq 0$, and each space $\mathcal{P}_\mu$ forms an irreducible representation space of $\mathcal{K}$. For each partition μ, the space $\mathcal{P}_\mu$ is contained in $\mathcal{P}_{|\mu|}$, where $|\mu|$ is the weight of μ defined by $|\mu| := \sum_{j=1}^{r} \mu_j$, and $\mathcal{P}_{|\mu|}$ denotes the space of homogeneous polynomials of degree $|\mu|$.

The Fock-Fischer inner product on the space $\mathcal{P}$ of holomorphic polynomials on $\mathbb{C}^q$ is defined by

$$\langle f, g \rangle_{\mathcal{F}} := \int_{\mathbb{C}^q} f(z)\overline{g(z)} d\mu_{\mathcal{F}}(z), \tag{3.4}$$

where

$$d\mu_{\mathcal{F}}(z) := \frac{1}{\pi^q} \exp\left(-|z|^2\right) dv(z). \tag{3.5}$$

For each partition μ, we denote by $K_{D,\mu}(z_1, \bar{z}_2)$ the reproducing Bergman kernel of $\mathcal{P}_{\mathbf{s}}$ with respect to (3.4).

Let $L^2_\alpha(\mathbb{C}^q, \rho_{\mathcal{F}})$ denote the weighted Hilbert space of holomorphic functions on $\mathbb{C}^q$ that are square-integrable with respect to $d\mu_{\mathcal{F}}$. The corresponding weighted reproducing Bergman kernel is given by

$$\begin{aligned} K(z_1, \bar{z}_2) &:= \sum_{\mathbf{s}} K_{D,\mu}(z_1, \bar{z}_2) \\ &= \exp\left(\langle z_1, z_2 \rangle\right). \end{aligned} \tag{3.6}$$

The Faraut-Korányi formula establishes a connection between the weighted reproducing Bergman kernels $K_{D,\mu}(z_1, \bar{z}_2)$ and the generic norm $N_D(z_1, \bar{z}_2)$ given by

$$\frac{1}{N_D(z_1,\bar{z}_2)^t}=\sum_{\mu}(t)_\mu K_{D,\mu}(z_1,\bar{z}_2), \tag{3.7}$$

the convergence of the series is uniform on compact subsets of $D\times D$ for each $t\in\mathbb{C}$, where $(t)_\mu$ is the generalized Pochhammer symbol given by

$$(t)_\mu:=\prod_{j=1}^{r}\left(t-\frac{j-1}{2}\times a\right)_{\mu_j}. \tag{3.8}$$

For more details (see [2, 4]).

Lemma 3.1. *Let D in $\mathbb{C}^q$ be a Cartan domain in its Harish-Chandra realization with the generic norm N_D and the genus γ. For $z_1\in D$, ψ be an automorphism of D such that $\psi(z_1)=0$. By [9], the function*

$$\Phi(z):=\frac{N_D^{\frac{\lambda_j}{2}}(z_1,\bar{z}_1)}{N_D^{\lambda_j}(z,\bar{z}_1)} \tag{3.9}$$

satisfies

$$|\Phi(z)|^2=\left(\frac{N_D(\psi(z),\overline{\psi(z)})}{N_D(z,\bar{z})}\right)^{\lambda_j}. \tag{3.10}$$

Let $F: H_D^{B^{q_0}}(\lambda_j)\longrightarrow H_D^{B^{q_0}}(\lambda_j)$ be defined by

$$(z,u)\longmapsto(\psi(z),\phi(z)u). \tag{3.11}$$

Consequently, F is an isometric isomorphism of $(H_D^{B^{q_0}}(\lambda_j),g(\lambda_j))$, i.e.,

$$\partial\bar{\partial}(\Psi(F(z,u)))=\partial\bar{\partial}(\Psi(z,u)). \tag{3.12}$$

Proof. By [9], since F is an automorphism of $H_D^{B^{q_0}}(\lambda_j)$,

$$N_D^{\lambda_j}(\psi(z),\overline{\psi(z)})=|J_{\mathbb{C}}\psi(z)|^{\frac{2\lambda_j}{\gamma}}N_D^{\lambda_j}(z,\bar{z}), \tag{3.13}$$

where $J_{\mathbb{C}}\psi(z)$ is the Complex Jacobian determinant of ψ.

By (3.10) and (3.13), we have

$$\begin{aligned}N_D^{\lambda_j}(\psi(z),\overline{\psi(z)})-\|\phi(z)u\|^2&=N_D^{\lambda_j}(\psi(z),\overline{\psi(z)})\left(1-\frac{\|u\|^2}{N_D^{\lambda_j}(z,\bar{z})}\right)\\&=|J_{\mathbb{C}}\psi(z)|^{\frac{2\lambda_j}{\gamma}}\left(N_D^{\lambda_j}(z,\bar{z})-\|u\|^2\right),\end{aligned}$$

this implies (3.12). ☐

Lemma 3.2. *Let D be an irreducible bounded symmetric domain with generic norm $N_D(z,\bar{\eta})$, dimension q, and genus γ. Then we have*

$$\det\left(\frac{\partial^2\Psi}{\partial z_j\partial\bar{z}_l}\right)_{j,l=1}^{d}(z,u)=\frac{\lambda_j^q C_D N_D^{\lambda_j}(z,\bar{z})^{(q+1)>\gamma}}{\left(N_D^{\lambda_j}(z,\bar{z})-\|u\|^2\right)^{q+1}}, \tag{3.14}$$

where $\Psi(z,u) = -\log\left(N_D^{\lambda_j}(z,\bar{z}) - \|u\|^2\right)$, $d = q + q_0$, $z = (z_1,\ldots,z_q)$, $u = (u_1,\ldots,u_{q_0}) = (z_{q+1},\ldots,z_d)$, *and*

$$C_D = \det\left(-\frac{\partial^2 \log N_D(z,\bar{z})}{\partial z_j \partial \bar{z}_l}\Bigg|_{z=0}\right).$$

Proof. It is well-known that

$$\frac{\left(\frac{i}{2\pi}\partial\bar{\partial}\Psi\right)^d}{d!} = \det\left(\frac{\partial^2\Psi}{\partial z_j\partial\bar{z}_l}\right)_{j,l=1}^d \frac{\omega_0^d}{d!}, \tag{3.15}$$

where $\omega_0 = \frac{i}{2\pi}\sum_{l=1}^d dz_l \wedge d\bar{z}_l$.

By (3.12) and (3.15), we have

$$\det\left(\frac{\partial^2\Psi(F)}{\partial z_j\partial\bar{z}_l}\right)_{j,l=1}^d = \det\left(\frac{\partial^2\Psi}{\partial z_j\partial\bar{z}_l}\right)_{j,l=1}^d. \tag{3.16}$$

As a consequence of the identity

$$\left(\frac{\partial^2\Psi(F)}{\partial z_j\partial\bar{z}_l}\right)_{j,l=1}^d (z,u) = \left(\frac{\partial F_l}{\partial z_j}\right)_{j,l=1}^d \left(\frac{\partial^2\Psi}{\partial z_j\partial\bar{z}_l}\right)_{j,l=1}^d (F(z,u)) \left(\frac{\partial \bar{F}_j}{\partial \bar{z}_l}\right)_{j,l=1}^d \tag{3.17}$$

By (3.16), we get

$$\det\left(\frac{\partial^2\Psi}{\partial z_j\partial\bar{z}_l}\right)_{j,l=1}^d (z,u) = |J_{\mathbb{C}}F(z,u)|^2 \det\left(\frac{\partial^2\Psi}{\partial z_j\partial\bar{z}_l}\right)_{j,l=1}^d (F(z,u)), \tag{3.18}$$

where $J_{\mathbb{C}}F(z,u) := \det\left(\frac{\partial F_l}{\partial z_j}\right)_{j,l=1}^d$ and

$$\left(\frac{\partial F_l}{\partial z_j}\right)_{j,l=1}^d = \begin{pmatrix} \frac{\partial F_1}{\partial z_1} & \cdots & \frac{\partial F_d}{\partial z_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_1}{\partial z_d} & \cdots & \frac{\partial F_d}{\partial z_d} \end{pmatrix}.$$

Evaluating at $(\tilde{z}_1,\tilde{u}_1) = F(z_1,u_1)$ with $N_D(0,\overline{0}) = 1$, we get:

$$|J_{\mathbb{C}}F(z_1,u_1)|^2 = \frac{1}{N_D(z_1,\bar{z}_1)^{\gamma+\lambda_j q_0}}. \tag{3.19}$$

Hence,

$$\det\left(\frac{\partial^2\Psi}{\partial z_j\partial\bar{z}_l}\right)_{j,l=1}^d (\tilde{z}_1,\tilde{u}_1) = \frac{1}{N_D(z_1,\bar{z}_1)^{\gamma+\lambda_j q_0}} \det\left(\frac{\partial^2\Psi}{\partial z_j\partial\bar{z}_l}\right)_{j,l=1}^d (0,\tilde{u}_1). \tag{3.20}$$

We now compute the block Hessian determinant at z=0:

$$\det\left(\frac{\partial^2\Psi}{\partial z_j\partial\bar{z}_l}\right)_{j,l=1}^d (0,u). \tag{3.21}$$

By

$$\begin{cases} N_D(0,0) = 1, \\ \frac{\partial \log N_D(z,\bar{z})}{\partial z_j}\Big|_{z=0} = \frac{\partial \log N_D(z,\bar{z})}{\partial \bar{z}_j}\Big|_{z=0} = 0 \quad (1 \le j \le q), \end{cases} \tag{3.22}$$

we have

$$\begin{cases} \left.\frac{\partial N_D^{\lambda_j}(z,\bar{z})}{\partial z_j}\right|_{z=0} = \frac{\partial}{\partial z_j}\left[e^{\lambda_j \log N_D(z,\bar{z})}\right]\Big|_{z=0} = \lambda_j N_D^{\lambda_j}(z,\bar{z})\left.\frac{\partial \log N_D(z,\bar{z})}{\partial z_j}\right|_{z=0} = 0, \\ \left.\frac{\partial N_D^{\lambda_j}(z,\bar{z})}{\partial \bar{z}_j}\right|_{z=0} = 0, \\ \left.\frac{\partial^2 N_D^{\lambda_j}(z,\bar{z})}{\partial z_j \partial \bar{z}_l}\right|_{z=0} = \lambda_j N_D^{\lambda_j}(z,\bar{z})\left.\frac{\partial^2 \log N_D(z,\bar{z})}{\partial z_j \partial \bar{z}_l}\right|_{z=0} + \lambda_j \partial N_D(z,\bar{z})^{\lambda_j}\frac{\partial \log N_D(z,\bar{z})}{\partial z_j}\cdot\left.\frac{\partial \log N_D^{\lambda_j}(z,\bar{z})}{\partial \bar{z}_l}\right|_{z=0} = -\lambda_j C_{jl}, \end{cases} \tag{3.23}$$

where $C_{jl} = -\left.\frac{\partial^2 \log N_D(z,\bar{z})}{\partial z_j \partial \bar{z}_l}\right|_{z=0}$. Hence,

$$\begin{cases} \frac{\partial^2 \Psi}{\partial z_j \partial \bar{z}_l}(0,u) = \lambda_j C_{jl}\frac{1}{1-\|u\|^2}, \quad (1 \le j,l \le q), \\ \frac{\partial^2 \Psi}{\partial u_j \partial \bar{u}_l}(0,u) = \frac{\delta_{jl}}{1-\|u\|^2} + \frac{\bar{u}_j u_l}{(1-\|u\|^2)^2}, \quad (1 \le j,l \le q_0), \\ \frac{\partial^2 \Psi}{\partial u_j \partial \bar{z}_l}(0,u) = 0, \quad (1 \le j \le q_0, 1 \le l \le q). \end{cases} \tag{3.24}$$

The above results may be expressed in block matrix form

$$\left(\frac{\partial^2 \Psi}{\partial z_j \partial \bar{z}_l}\right)_{j,l=1}^{d}(0,u) = \begin{pmatrix} \frac{\lambda_j}{1-\|u\|^2} C_q & 0 \\ 0 & \frac{1}{1-\|u\|^2} I_{q_0} + \frac{1}{(1-\|u\|^2)^2} u^* u \end{pmatrix}, \tag{3.25}$$

where I_{q_0} is the diagonal matrix $q_0 \times q_0$ with its diagonal elements 1, u^* denotes the conjugate transpose of the row vector $u = (u_1, \ldots, u_{q_0})$, and $C_q = (C_{jl})_{j,l=1}^q$.

By (3.25), we have

$$\det\left(\frac{\partial^2 \Psi}{\partial z_j \partial \bar{z}_l}\right)_{j,l=1}^{d}(0,u) = \frac{\lambda_j^q \det C_q}{(1-\|u\|^2)^{q+q_0+1}}. \tag{3.26}$$

Combining (3.20) and (3.26) we get (3.14). □

Theorem 3.3. *Let $H_D^{B^{q_0}}(\lambda_j)$ be a Cartan-Hartogs domain with the Kähler metric $g(\lambda_j)$. Consider $\beta > \max\left\{\frac{\gamma-1}{\lambda_j}, q+q_0\right\}$. We have the Bergman kernel $K_\beta(z,u;\bar{z},\bar{u})$ of the Hilbert space*

$$\mathcal{H}_\beta = \left\{ f \in Hol(H_D^{B^{q_0}}(\lambda_j)) : \int_{H_D^{B^{q_0}}(\lambda_j)} |f|^2 e^{-\beta\Psi}\frac{\omega(\lambda_j)^{q+q_0}}{(q+q_0)!} < \infty \right\}$$

can be expressed as

$$\begin{aligned} K_\beta(z,u;\bar{z},\bar{u}) = {} & \frac{\pi^{q+q_0}\chi_2(\beta-q-q_0-1)}{C_D \lambda_j^q (\chi_1\chi_2(0)) V(D) V(B^{q_0})}\left(\frac{1}{N_D^{\lambda_j}(z,\bar{z})}\right)^{\beta} \\ & \times \chi_1\left(\lambda_j\left(\beta + w\frac{d}{dw}\right) - \gamma\right)\left.\frac{1}{\left(1-\frac{w\|u\|^2}{N_D^{\lambda_j}(z,\bar{z})}\right)^{\beta+q}}\right|_{w=1}, \end{aligned} \tag{3.27}$$

where $C_D = -\det\left(\frac{\partial^2 \log N_D(z,\bar{z})}{\partial z_j \partial \bar{z}_l}\right)\Bigg|_{z=0}$, χ_1, χ_2 *and* $V(D), V(B^{q_0})$ *represent Hua polynomials and* $Vol(D)$ *and* $Vol(B^{q_0})$ *denote the Euclidean volumes of* D *and* B^{q_0}*, respectively.*

Proof. From (3.14), we define the inner product on $\mathcal{H}_\beta$ by

$$\langle f, g\rangle = \frac{\lambda_j^q C_D}{\pi^{q+q_0}} \int_{H_D^{B^{q_0}}(\lambda_j)} f(z,u)\overline{g(z,u)} \frac{N_D(z,\bar{z})^{\lambda_j(\beta-q_0)>\gamma}}{\left(1-\frac{\|u\|^2}{N_D^{\lambda_j}(z,\bar{z})}\right)^{\beta-(q+q_0+1)}} dv(z)dv(u),$$

where dv is the Euclidean measure.

For simplicity, we denote $D_1 = D$, $D_2 = B^{q_0}$. For a Cartan domain D_j $(1 \le j \le 2)$, let r_j, a_j, b_j, γ_j, d_j, ν_j, and N_j denote its rank, characteristic multiplicities, genus, dimension, generalized Pochhammer symbol, Hua polynomial, and generic norm of the Cartan domain, respectively.

For each j, $\mathcal{G}_j$ be the identity component of the biholomorphic self-map group of $D_j \subset \mathbb{C}^{q_j}$, and let $\mathcal{K}_j$ be the isotropy subgroup at the origin. For each $k = (k_1, k_2) \in \mathcal{K} := \mathcal{K}_1 \times \mathcal{K}_2$, the action is defined as

$$\pi_\mu(k)f(z,u) = f \circ k(z,u) := f(k_1 \circ z, k_2 \circ u)$$

of $\mathcal{K}$, then according to the Peter-Weyl Theorem, the space $\mathcal{P}$ of holomorphic polynomials on $\mathbb{C}^{q_1} \times \mathbb{C}^{q_2}$ can be decomposed as

$$\mathcal{P} = \bigoplus_{\mu,\nu} \mathcal{P}_{\mu(1)} \otimes \mathcal{P}_{\nu(2)}.$$

With each $\mathcal{P}_{\mu(j)}$ being an irreducible subspace that is invariant under the action of $\mathcal{K}_j$ on the space of holomorphic polynomials on $\mathbb{C}^{q_j}$ $(1 \le j \le 2)$. Under the action of $\mathcal{K}_1 \times \mathcal{K}_2$, the Hilbert space $\mathcal{H}_\beta$ remains invariant, that is,

$$\langle f\pi_\mu(k), g\pi_\mu(k)\rangle = \langle f, g\rangle \quad \text{for all } k \in \mathcal{K}_1 \times \mathcal{K}_2.$$

Consequently, the Hilbert space $\mathcal{H}_\beta$ admits a decomposition into irreducible $K_1 \times K_2$-invariant subspaces given by

$$\mathcal{H}_\beta = \bigoplus_{\mu,\nu} \mathcal{P}_{\mu(1)} \otimes \mathcal{P}_{\nu(2)},$$

where $\bigoplus$ denotes the orthogonal direct sum.

For each partition μ of length at most r_j, let $K_{\mu(j)}(z,\bar{u})$ denote the weighted reproducing kernel associated with the subspace $\mathcal{P}_{\mu(j)}$ with respect to the inner product defined in (3.4).

It follows from Schur's Lemma that there exists a positive constant $C_{\mu\nu}$ for which $C_{\mu\nu}K_{\mu(1)}K_{\nu(2)}(u,\bar{u})$ is the reproducing weighted Bergman kernel of the subspace $\mathcal{P}_{\mu(1)} \otimes \mathcal{P}_{\nu(2)}$ with respect to the inner product $\langle\cdot,\cdot\rangle$.

From the weighted reproducing Bergman kernel, we have

$$\dim \mathcal{P}_{\mu(1)} \dim \mathcal{P}_{\nu(2)}$$

$$= \frac{\lambda_j^q C_D}{\pi^{q+q_0}} \int_{H_D^{B^{q_0}}(\lambda_j)} C_{\mu\nu}K_{\mu(1)}(z,\bar{z})K_{\nu(2)}(u,\bar{u})N_D(z,\bar{z})^{\lambda_j(\beta-q_0)>\gamma}\left(1-\frac{\|u\|^2}{N_D(z,\bar{z})}\right)^{\beta>(q+q_0+1)} dv(z)dv(u).$$

Therefore, the weighted Bergman kernel of the Hilbert space H_β can be expressed as:

$$K_\beta(z,u;\bar z,\bar u)=\sum\frac{\dim P_{\mu(1)}\dim P_{\nu(2)}}{\langle K_{\mu(1)}(z,\bar z)K_{\nu(2)}(u,\bar u)\rangle}K_{\mu(1)}(z,\bar z)K_{\nu(2)}(u,\bar u), \tag{3.28}$$

where $\langle f\rangle$ denotes the integral of the integrand:

$$\frac{\lambda_j^q C_D}{\pi^{q+q_0}}\int_{H_D^{B^{q_0}}(\lambda_j)} f(z,u)N_D(z,\bar z)^{\lambda_j(\beta-q_0)>\gamma}\left(1-\frac{\|u\|^2}{N_D(z,\bar z)^{\lambda_j}}\right)^{\beta>(q+q_0+1)}dv(z)dv(u).$$

Assuming $\lambda_j\ \beta-\gamma>-1$ and $\beta-q-q_0-1>-1$, using the preceding results,

$$\int_D K_{\mu(1)}(z,\bar z)N_D(z,\bar z)^t dv(z)=\frac{\dim P_{\mu(1)}}{(r+t)_{\mu(1)}}\int_D N_D(z,\bar z)dv(z)$$

for $t>-1$ and (3.1), the Hua integral formula gives

$$\begin{aligned}&\frac{\lambda_j^q C_D}{\pi^{q+q_0}}\int_D K_{\mu_1}(z,\bar z)N_D(z,\bar z)^{\lambda_j(\beta+\nu)>\gamma}\,dv(z)\times\int_{B^{q_0}}K_{\nu_2}(u,\bar u)(1-\|u\|^2)^{\beta>(q+q_0+1)}\,dv(u)\\&\quad=\frac{\lambda_j^q C_D(\chi_1\chi_2(0))V(D)V(B^{q_0})}{\pi^{q+q_0}(\chi_1\chi_2(\lambda_j(\beta+\nu)-\gamma)(\beta-q-q_0-1))}\frac{\dim P_{\mu(1)}\dim P_{\nu(2)}}{(\lambda_j(\beta+\nu))_{\mu(1)}(\beta-q)_{\nu(2)}}.\end{aligned} \tag{3.29}$$

Using the preceding result (3.28), (3.29) and (3.7) we obtain

$$\begin{aligned}K_\beta(z,u;\bar z,\bar u)&=\sum e\chi(1)(\lambda_j(\beta+\nu)-\gamma)(\lambda_j(\beta+\nu))_{\mu(1)}(\beta-q)_{\nu(2)}K_{\mu(1)}(z,\bar z)K_{\nu(2)}(u,\bar u)\\
&=e\sum\chi(1)(\lambda_j(\beta+\nu)-\gamma)(\beta-q)_{\nu(2)}K_{\nu(2)}(u,\bar u)\frac{1}{N_D^{\lambda_j}(z,\bar z)^{(\beta+\nu)}}\\
&=\frac{e}{N_D(z,\bar z)^{\lambda_j\beta}}\sum\chi(1)(\lambda_j(\beta+\nu)-\gamma)(\beta-q)_{\nu(2)}K_{\nu(2)}\left(\frac{u}{N_D^{\lambda_j}(z,\bar z),\bar u}\right)\\
&=\frac{e}{N_D(z,\bar z)^{\lambda_j\beta}}\sum\chi(1)\left(\lambda_j\left(\beta+w\frac{d}{dw}\right)-\gamma\right)(\beta-q)_{\nu(2)}K_{\nu(2)}\left(\frac{wu}{N_D^{\lambda_j}(z,\bar z)},\bar u\right)\Bigg|_{w=1}\\
&=\frac{e}{N_D(z,\bar z)^{\lambda_j\beta}}\chi(1)\left(\lambda_j\left(\beta+w\frac{d}{dw}\right)-\gamma\right)\sum(\beta-q)_{\nu(2)}K_{\nu(2)}\left(\frac{wu}{N_D^{\lambda_j}(z,\bar z)},\bar u\right)\Bigg|_{w=1}\\
&=\frac{e}{N_D(z,\bar z)^{\lambda_j\beta}}\chi(1)\left(\lambda_j\left(\beta+w\frac{d}{dw}\right)-\gamma\right)\times\frac{1}{\left(1-\frac{w\|u\|^2}{N_D^{\lambda_j}(z,\bar z)}\right)^{\beta>q}}\Bigg|_{w=1},\end{aligned}$$

where

$$e=\frac{\pi^{q+q_0}\chi(2)(\beta-q-q_0-1)}{\lambda_j^q C_D(\chi(1)\chi(2)(0))V(D)V(B^{q_0})},$$

which concludes the proof. $\square$

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