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DERIVATION OF QUANTUM CODES FROM SPECIFIC RINGS

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Abstract. We study the construction of quantum error-correcting codes (QECCs) using the algebraic properties of cyclic codes defined over the product ring of the finite field of order p and a specific local chain ring. Our methodology relies on the analysis of Euclidean sums. This approach is used to construct generator polynomials and to calculate the dimension of these codes. A key part of our approach is the use of an isometric Gray mapping. Our technique preserves distances while adapting these classical structures to quantum systems. As a result, it generates new families of good quantum codes.

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1. Introduction

Quantum computing can solve problems that are impossible for conventional computers. It relies on quantum error-correcting codes (QECC) to be reliable. These codes protect highly fragile quantum data from external disturbances. The real challenge is that quantum physics prevents us from simply copying and pasting to save the information. QECCs must therefore employ intelligent detection and methods to address errors without compromising the integrity of the original data.

To protect data, these codes exploit two quantum phenomena: superposition and entanglement. In practice, they distribute the information across multiple physical qubits. The approach enables the non-invasive detection and correction of errors, preventing the direct observation of data that would otherwise be destroyed. Currently, there are several families of codes, each offering a different trade-off between error resilience and the required hardware.

Since their inception, QECCs have played an instrumental role in the advancement of quantum technologies. The field was initiated in 1995 with Shor's development [19] of a code to protect quantum memory. In 1996, Steane [20] had

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streamlined the detection and correction of these errors. Finally, in 1998, Calderbank et al. [5] achieved a new milestone in communication reliability using codes based on $GF(4)$.

Various studies have constructed quantum codes based on specific mathematical rings. Çalışkan et al. [6] laid the groundwork with cyclic codes over $\mathbb{F}_p \times (\mathbb{F}_p + v\mathbb{F}_p)$. Subsequently, Bag et al. [2], Ma et al. [16] and Ndiaye et al. [17] adapted these principles to the rings $\mathbb{F}_p[u]/\langle u^3 - u\rangle$, $\mathbb{F}_q + v\mathbb{F}_q + v^2\mathbb{F}_q$ and $F_{p^k} + vF_{p^k} + v^2F_{p^k} + \ldots + v^kF_{p^k}$ where $v^{r+1} = v$. To optimize the quality of these codes, Diao et al. [7] demonstrated the effectiveness of additive codes over $\mathbb{Z}_p \times (\mathbb{Z}_p + v\mathbb{Z}_p)$ where $v^2 = v$. Cyclic codes on local rings have attracted growing interest due to their diverse applications, including DNA coding [3]. Finally, recent work [9] has explored non-local rings to better understand their fundamental algebraic structure.

In this paper, we generalize the framework established in [9], extending the findings to broader classes of rings and investigating new code constructions with enhanced parameters. Specifically, the following tasks are achieved.

Our study utilizes the structure of the non-chain rings $\mathcal{R}_l = \mathbb{F}_p[u]/\langle u^l - u^{l-1}\rangle \simeq \mathbb{F}_p \times \mathbb{F}_p[v]/\langle v^{l-1}\rangle$. We employ an isometric Gray map $\Gamma : \mathcal{R}_l \to \mathbb{F}_p^l$, to construct cyclic codes adapted to quantum metrics, thereby overcoming the limitations of classical chain rings.

Instead of limiting itself to self-orthogonality, our article uses Euclidean sums $\mathcal{S}_\mathcal{C} = \mathcal{C}+\mathcal{C}^\perp$ to generate QECCs. The dimension k of the quantum code $[[ln, k, d]]_p$ is determined analytically by the degree of the gcd of the generator polynomials, providing fine control over the code rate.

This manuscript is structured as follows. Section 2 proceeds by detailing the fundamental concepts and examining the Gray map's application to linear codes over $\mathcal{R}_l$. In Section 3 the Euclidean sum structures for cyclic codes over $\mathcal{R}_l$ is introduced and their generator polynomials are established. Next, Section 4 utilizes these sums to construct QECCs, presenting new codes with optimized parameters. Section 5 provides numerical results and examples to illustrate our ideas. Finally, Section 6 concludes this study by summarizing our major contributions.

2. Preliminaries

Let $p \geq 3$ be a prime number and $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. Let n and k be positive integers such that $0 < k \leq n$. A code $\mathcal{C}$ of length n and dimension k over $\mathbb{F}_p$ is a subspace of $\mathbb{F}_p^n$ with dimension k. If $\mathcal{C}$ is an additive subgroup, it is called a *linear code*. The elements of $\mathcal{C}$ are called *codewords*. The *Hamming weight* $w_H(x)$ of an element $x \in \mathcal{C}$ is the number of its non-zero components. The distance d_H of the linear code is the minimum Hamming weight of nonzero codewords, that is

$$d_H = \min\{w_H(x) \ : \ x \in \mathcal{C}\backslash\{0\}\}.$$

2.1. The properties of the ring $\mathcal{R}_l$.

In this paper, we focus on linear codes over the ring $\mathcal{R}_l$ where $l \geq 2$ is an integer, and $\mathcal{R}_l$ is the set defined by:

$$\mathcal{R}_l = \mathbb{F}_p + u\mathbb{F}_p + u^2\mathbb{F}_p + \ldots + u^{l-1}\mathbb{F}_p$$

with $u^l = u^{l-1}$.

Any element $\alpha \in \mathcal{R}_l$ can be represented as a linear combination $\alpha = \lambda_0 + \lambda_1 u + \lambda_2 u^2 + \ldots + \lambda_{l-1} u^{l-1}$, where, for $i = 0, \ldots, l-1$, $\lambda_i \in \mathbb{F}_p$. The ring $\mathcal{R}_l$ is finite, commutative with an identity, and has characteristic p. It is a non-local as well as a non-chain ring.

Moreover, $\mathcal{R}_l$ has a one-to-one correspondence with the direct product of the rings $\mathcal{R}_{l,1} \times \mathcal{R}_{l,2}$, where $\mathcal{R}_{l,1} = \mathbb{F}_p$ and $\mathcal{R}_{l,2} = \mathbb{F}_p + v\mathbb{F}_p + \ldots + v^{l-2}\mathbb{F}_p$ with $v^{l-1} = 0$. In fact, consider the generator polynomial of the ideal defining the ring $\mathcal{R}_l$, denoted $g(u) = u^l - u^{l-1}$. In the polynomial ring $\mathbb{F}_p[u]$, this polynomial factors as $g(u) = u^{l-1}(u-1)$. The polynomials $g_1(u) = u - 1$ and $g_2(u) = u^{l-1}$ are coprime. Indeed, if a divisor $d(u)$ divides $u-1$, then $d(u)$ cannot divide u^{l-1} because $g_2(1) = 1^{l-1} \neq 0$ in $\mathbb{F}_p$. Thus, $\gcd(u-1, u^{l-1}) = 1$.
According to the Chinese Remainder Theorem (CRT) for rings, since the ideals $\langle u-1 \rangle$ and $\langle u^{l-1} \rangle$ are comaximal, we have the canonical isomorphism: $\mathcal{R}_l \cong \frac{\mathbb{F}_p[u]}{\langle u-1 \rangle} \times \frac{\mathbb{F}_p[u]}{\langle u^{l-1} \rangle}$.

To make this isomorphism explicit, we will determine the orthogonal idempotents in $\mathcal{R}_l$. According to Bézout's identity, there exist $a(u), b(u) \in \mathbb{F}_p[u]$ such that $a(u)(u-1) + b(u)u^{l-1} = 1$. So, the polynomial u^{l-1} satisfies the following congruences simultaneously: $u^{l-1} \equiv 1 \pmod{u-1}$ and $u^{l-1} \equiv 0 \pmod{u^{l-1}}$. We consider the elements $E_1(u) = u^{l-1}$ and $E_2(u) = 1 - u^{l-1}$. These elements verify the properties of idempotence and orthogonality. Indeed, we have $E_1(u)^2 = u^{(l-1)+(l-1)} = u^{2l-2} = u^{l-1} = E_1(u)$, and $E_2(u)^2 = 1 - 2u^{l-1} + u^{2l-2} = 1 - u^{l-1} = E_2(u)$. For the orthogonality, it is clear that $E_1(u)E_2(u) = u^{l-1} - u^{2l-2} = 0$, and by construction, we have $E_1(u) + E_2(u) = 1$.

Now, consider the mapping:

$$\begin{array}{rccc} \pi_1 : & \mathcal{R}_l & \longrightarrow & \mathcal{R}_{l,1} = \mathbb{F}_p \\ & \alpha = \lambda_0 + \lambda_1 u + \ldots + \lambda_{l-1}u^{l-1} & \longmapsto & \lambda_0 + \lambda_1 + \ldots + \lambda_{l-1} \end{array}$$

and

$$\begin{array}{rccc} \pi_2 : & \mathcal{R}_l & \longrightarrow & \mathcal{R}_{l,2} \\ & \alpha = \lambda_0 + \lambda_1 u + \ldots + \lambda_{l-1}u^{l-1} & \longmapsto & \lambda_0 + \lambda_1 v + \ldots + \lambda_{l-2}v^{l-2}, \end{array}$$

where $v^{l-1} = 0$. These applications π_1 and π_2 are surjective ring morphisms. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two rings defined by: $\mathcal{A}_1 = \{u^{l-1}.\alpha \mid \alpha \in \mathcal{R}_l\}$ and $\mathcal{A}_2 = \{(1-u^{l-1}).\alpha \mid \alpha \in \mathcal{R}_l\}$.
The following applications:

$$\begin{array}{rccc} \theta_1 : & \mathcal{A}_1 & \longrightarrow & \mathcal{R}_{l,1} \\ & u^{l-1}.\alpha & \longmapsto & \lambda_0 + \lambda_1 + \ldots + \lambda_{l-1}, \end{array}$$

and

$$\begin{array}{rccc} \theta_2 : & \mathcal{A}_2 & \longrightarrow & \mathcal{R}_{l,2} \\ & \lambda_0 + \lambda_1 u + \lambda_2 u^2 + \cdots + \lambda_{l-2}u^{l-2} - \left(\sum_{i=0}^{l-2} \lambda_i\right) u^{l-1} & \longmapsto & \lambda_0 + \lambda_1 v + \ldots + \lambda_{l-2}v^{l-2} \end{array}$$

are ring isomorphisms.
The fact that θ_1 and θ_2 are ring isomorphisms follows from straightforward algebraic verifications. The preservation of addition is immediate from their linear

definitions. The preservation of multiplication is verified by expanding the polynomial products and applying the defining ideal relations $u^k = u^{l-1}$ for $k \geq l-1$ in $\mathcal{A}_1$, and $v^{l-1} = 0$ in $\mathcal{A}_2$. Finally, bijectivity is established by noting that θ_1 maps onto the field $\mathbb{F}_p$ bijectively, and that the set $\{1, v, \ldots, v^{l-2}\}$ forms a basis for the $\mathbb{F}_p$-vector space $R_{l,2}$, which guarantees that θ_2 is both injective and surjective.
Next, consider the applications

$$\widetilde{\pi}_1 : \mathcal{R}_l^n \longrightarrow \mathcal{R}_{l,1}^n$$
$$(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \longrightarrow (\pi_1(\alpha_0), \pi_1(\alpha_1), \ldots, \pi_1(\alpha_{n-1}))$$

and

$$\widetilde{\pi}_2 : \mathcal{R}_l^n \longrightarrow \mathcal{R}_{l,2}^n$$
$$(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \longrightarrow (\pi_2(\alpha_0), \pi_2(\alpha_1), \ldots, \pi_2(\alpha_{n-1}))$$

It is easily seen that these maps are also surjective ring morphisms.
We consider also the following applications

$$\widetilde{\theta}_1 : \mathcal{A}_1^n \longrightarrow \mathcal{R}_{l,1}^n$$
$$(u^{l-1}\alpha_0, u^{l-1}\alpha_1, \ldots, u^{l-1}\alpha_{n-1}) \longrightarrow (\theta_1(u^{l-1}\alpha_0), \theta_1(u^{l-1}\alpha_1), \ldots, \theta_1(u^{l-1}\alpha_{n-1}))$$

and

$$\widetilde{\theta}_2 : \mathcal{A}_2^n \longrightarrow \mathcal{R}_{l,2}^n$$
$$(\beta_0, \beta_1, \ldots, \beta_{n-1}) \longrightarrow (\theta_2(\beta_0), \theta_2(\beta_1), \ldots, \theta_2(\beta_{n-1}))$$

where $\beta_i \in \mathcal{A}_2$.

Since θ_1 and θ_2 are ring isomorphisms, then $\widetilde{\theta}_1$ and $\widetilde{\theta}_2$ are also ring isomorphisms.

The properties of the local ring $\mathcal{R}_{l,2}$ have been explored in detail in [8], and in particular cases: $l = 3$ and p is a power of 2 (resp. a power of a prime number greater than or equal to 5), the isomorphism $\mathcal{R}_l \cong \mathcal{R}_{l,1} \times \mathcal{R}_{l,2}$ is studied in [22] (resp. [4]), and for $l = 4$ and $p \geq 3$ is investigated in [9]. From the above discussion, the following proposition introduces this correspondence for the general case:

Proposition 2.1. *Let p and l as given before. The non local ring $\mathcal{R}_l = \mathbb{F}_p[u]/\langle u^l - u^{l-1}\rangle$ is isomorphic to the direct product of rings $\mathcal{R}_{l,1} \times \mathcal{R}_{l,2}$, where $\mathcal{R}_{l,1} \simeq \mathbb{F}_p$ is a finite field and $\mathcal{R}_{l,2} = \mathbb{F}_p[v]/\langle v^{l-1}\rangle$ is a local ring.*
Explicitly, we have the isomorphism

$$\varphi : \mathcal{R}_l \longrightarrow \mathcal{R}_{l,1} \times \mathcal{R}_{l,2}$$
$$\alpha = \lambda_0 + \lambda_1 u + \cdots + \lambda_{l-1}u^{l-1} \longrightarrow (\pi_1(\alpha), \pi_2(\alpha))$$

Proposition 2.2. *Let $\alpha = \lambda_0 + \lambda_1 u + \cdots + \lambda_{l-1}u^{l-1} \in \mathcal{R}_l$. Then, α can be written as*

$$\alpha = \theta_2^{-1}(\pi_2(\alpha)) + \pi_1(\alpha) \cdot u^{l-1}.$$

Proof. We follow the same procedure given in Corollary 2 [21]. □

2.2. **Linear codes and Gray map.**

2.2.1. *Linear codes.* A linear code $\mathcal{C} \neq \emptyset$ of length n over $\mathcal{R}_l$ is an $\mathcal{R}_l$-submodule of $\mathcal{R}_l^n$. The components of $\mathcal{C}$ are called codewords. Let τ represent the operator that performs a cyclic shift from $\mathcal{R}_l^n$ to $\mathcal{R}_l^n$. The linear code $\mathcal{C}$ is referred to as a cyclic code over $\mathcal{R}_l$ with length n when the following condition holds: for any $\mathfrak{c} = (c_0, c_1, \ldots, c_{n-1}) \in \mathcal{C}$, one has

$$\tau(\mathfrak{c}) = (c_{n-1}, c_0, \ldots, c_{n-2}) \in \mathcal{C}.$$

There is a natural bijection between $\mathcal{R}_l^n$ and $\mathcal{R}_l[x]/\langle x^n - 1\rangle$. This is achieved by associating $\mathfrak{c} = (c_0, c_1, \ldots, c_{n-1}) \in \mathcal{R}_l^n$ to $\mathfrak{c}(x) = c_0 + c_1 x + \cdots + c_{n-1}x^{n-1} \in \mathcal{R}_l[x]/\langle x^n - 1\rangle$.

Theorem 2.3. *Consider $\mathfrak{c} = (c_0, c_1, \ldots, c_{n-1})$ a codeword in $\mathcal{C} \subset \mathcal{R}_l^n$, then*

$$\mathfrak{c} = \widetilde{\theta}_2^{-1}\left(\widetilde{\pi}_2(\mathfrak{c})\right) + \widetilde{\pi}_1(\mathfrak{c}) \cdot u^{l-1}.$$

Proof. From Proposition 2.2, we have that $c_i = \theta_2^{-1}(\pi_2(c_i)) + \pi_1(c_i)u^{l-1} \in \mathcal{R}_l$, for $i = 0, \ldots, n-1$. Hence

$$\begin{aligned}
\mathfrak{c} = (c_0, c_1, \ldots, c_{n-1}) &= \left(\theta_2^{-1}(\pi_2(c_0)) + \pi_1(c_0)u^{l-1}, \ldots, \theta_2^{-1}(\pi_2(c_{n-1})) + \pi_1(c_{n-1}) \cdot u^{l-1}\right) \\
&= \left(\theta_2^{-1}(\pi_2(c_0)), \theta_2^{-1}(\pi_2(c_1)), \ldots, \theta_2^{-1}(\pi_2(c_{n-1}))\right) \\
&= + \left(\pi_1(c_0) \cdot u^{l-1}, \pi_1(c_1) \cdot u^{l-1}, \ldots, \pi_1(c_{n-1}) \cdot u^{l-1}\right) \\
&= \widetilde{\theta}_2^{-1}\left(\widetilde{\pi}_2(c_0, c_1, \ldots, c_{n-1})\right) + \widetilde{\pi}_1(c_0, c_1, \ldots, c_{n-1}) \cdot u^{l-1} \\
&= \widetilde{\theta}_2^{-1}\left(\widetilde{\pi}_2(\mathfrak{c})\right) + \widetilde{\pi}_1(\mathfrak{c}) \cdot u^{l-1}.
\end{aligned}$$

□

Remark 2.4. Since $\mathcal{R}_l \simeq \mathcal{A}_1 \times \mathcal{A}_2 \simeq \mathcal{R}_{l,1} \times \mathcal{R}_{l,2}$, then any linear code $\mathcal{C}$ of length n over $\mathcal{R}_l$ can be represented in the form $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2)$ with $\mathcal{C}_i$ being a linear code of length n over $\mathcal{R}_{l,i}$ for $i \in \{1, 2\}$.
As a result, any codeword $\mathfrak{c} \in \mathcal{C}$ can be written as $\mathfrak{c} = (c_1, c_2)$, where $c_1 \in \mathcal{C}_1$, and $c_2 \in \mathcal{C}_2$, such that $\widetilde{\pi}_1(\mathfrak{c}) = c_1$ and $\widetilde{\pi}_2(\mathfrak{c}) = c_2$. Therefore,

$$\mathfrak{c} = \widetilde{\theta}_2^{-1}(c_2) + c_1 \cdot u^{l-1}.$$

Corollary 2.5. *Consider a linear code $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2) \subset \mathcal{R}_l^n$, then*

$$\mathcal{C} = \widetilde{\theta}_2^{-1}(\mathcal{C}_2) \oplus u^{l-1} \cdot \mathcal{C}_1.$$

Proof. The proof follows directly from Theorem 2.3 and Remark 2.4. □

Based on the facts in Remark 2.4, let $X, Y \in \mathcal{R}_l^n$ with $X = (X_1, Y_1)$ and $Y = (X_2, Y_2)$ where $X_1 = (x_1, x_2, \ldots, x_n) \in \mathcal{R}_{l,1}$, $X_2 = (x'_1, x'_2, \ldots, x'_n) \in \mathcal{R}_{l,2}$, $Y_1 = (y_1, y_2, \ldots, y_n) \in \mathcal{R}_{l,1}$ and $Y_2 = (y'_1, y'_2, \ldots, y'_n) \in \mathcal{R}_{l,2}$, the inner product of $X, Y \in \mathcal{R}_l^n$ is given by

$$\langle X, Y\rangle_{\mathcal{R}_l^n} = \left(\langle X_1, Y_1\rangle_{\mathcal{R}_{l,1}^n}, \langle X_2, Y_2\rangle_{\mathcal{R}_{l,2}^n}\right)$$

where, $\langle X_1, Y_1\rangle_{\mathcal{R}_{l,1}^n} = \sum_{i=1}^{n} x_i y_i$ is the inner product over $\mathcal{R}_{l,1}^n$ and $\langle X_2, Y_2\rangle_{\mathcal{R}_{l,2}^n} = \sum_{i=1}^{n} x'_i y'_i$ is the inner product over $\mathcal{R}_{l,2}^n$.

If $\langle X, Y\rangle_{\mathcal{R}_l^n} = 0$, then X and Y are orthogonal. This leads to define the dual code of $\mathcal{C}$ as

$$\mathcal{C}^\perp = \{X \in R_l^n \mid \langle X, Y\rangle_{\mathcal{R}_l^n} = 0 \text{ for all } Y \in \mathcal{C}\}.$$

The linear code $\mathcal{C}$ satisfies the dual-containing property if $\mathcal{C}^\perp \subset \mathcal{C}$. Conversely, $\mathcal{C}$ satisfies the self-orthogonal (resp. self-dual) property if $\mathcal{C} \subset \mathcal{C}^\perp$ (resp. $\mathcal{C} = \mathcal{C}^\perp$).

The following results will be useful for our construction.

Proposition 2.6. *Let $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2) \subseteq \mathcal{R}_l^n$ be a linear code. The dual code of $\mathcal{C}$ is $\mathcal{C}^\perp = (\mathcal{C}_1^\perp, \mathcal{C}_2^\perp)$. Furthermore, $\mathcal{C} \subseteq \mathcal{C}^\perp$ is equivalent to $\mathcal{C}_i \subseteq \mathcal{C}_i^\perp$ for $i = 1, 2$.*

Proof. Let $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2) \subseteq \mathcal{R}_l^n$ be a linear code, where $\mathcal{C}_1 \subseteq \mathbb{F}_p^n$ and $\mathcal{C}_2 \subseteq \mathcal{R}_{l,2}^n$ represent linear codes. Any codeword $\omega \in \mathcal{C}$ is in the form $\omega = (\nu, \delta_0 + v\delta_1 + \cdots + v^{l-2}\delta_{l-2}) \in \mathcal{C}_1 \times \mathcal{C}_2$. The dual code satisfies $\mathcal{C}^\perp = (\mathcal{C}_1^\perp, \mathcal{C}_2^\perp)$, where $\mathcal{C}_2^\perp$ is taken with respect to the Euclidean inner product $\langle\cdot,\cdot\rangle_{\mathcal{R}_{l,2}^n}$.

If $\mathcal{C}$ is self-orthogonal, then $\langle\omega, \omega\rangle = 0$, and

$$\Big(\nu \cdot \nu,\ \langle \delta_0 + v\delta_1 + \cdots + v^{l-2}\delta_{l-2},\ \delta_0 + v\delta_1 + \cdots + v^{l-2}\delta_{l-2}\rangle_{\mathcal{R}_{l,2}^n}\Big) = (0, 0).$$

This implies that $\nu \cdot \nu = 0$ in $\mathbb{F}_p$, and $\nu \in \mathcal{C}_1^\perp$. Hence $\mathcal{C}_1 \subset \mathcal{C}_1^\perp$. Also, we have $\langle \delta_0 + v\delta_1 + \cdots + v^{l-2}\delta_{l-2},\ \delta_0 + v\delta_1 + \cdots + v^{l-2}\delta_{l-2}\rangle_{\mathcal{R}_{l,2}^n} = 0$ in $\mathcal{R}_{l,2}$, that is $\delta_0 + v\delta_1 + \cdots + v^{l-2}\delta_{l-2} \in \mathcal{C}_2^\perp$. Hence $\mathcal{C}_2 \subset \mathcal{C}_2^\perp$. As a consequence, both $\mathcal{C}_1$ and $\mathcal{C}_2$ are self-orthogonal.

Conversely, if $\mathcal{C}_1$ and $\mathcal{C}_2$ are self-orthogonal, then for any $\omega = (\nu, \delta_0 + v\delta_1 + \cdots + v^{l-2}\delta_{l-2}) \in \mathcal{C}$ and any $\omega' = (\nu', \delta_0' + v\delta_1' + \cdots + v^{l-2}\delta_{l-2}') \in \mathcal{C}$, we have $\nu \cdot \nu' = 0$, and $\langle \delta_0 + v\delta_1 + \cdots + v^{l-2}\delta_{l-2},\ \delta_0' + v\delta_1' + \cdots + v^{l-2}\delta_{l-2}'\rangle_{\mathcal{R}_{l,2}^n} = 0$. Thus, $\langle\omega, \omega'\rangle = (0, 0)$, and $\mathcal{C} \subseteq \mathcal{C}^\perp$. This shows that $\mathcal{C}$ is self-orthogonal. □

Corollary 2.7. *Consider a linear code $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2) \subset \mathcal{R}_l^n$, then*

$$\mathcal{C}^\perp = \widetilde{\theta}_2^{-1}(\mathcal{C}_2^\perp) \oplus u^{l-1} \cdot \mathcal{C}_1^\perp.$$

Proof. Since the dual code of $\mathcal{C}$ is represented as $(\mathcal{C}_1^\perp, \mathcal{C}_2^\perp)$, such that $\mathcal{C}_1^\perp$ is a linear code over $\mathcal{R}_{l,1}^n$ and $\mathcal{C}_2^\perp$ is a linear code over $\mathcal{R}_{l,2}^n$. Then, for each element $\mathfrak{c}' = (c_1', c_2') \in (\mathcal{C}_1^\perp, \mathcal{C}_2^\perp)$ such that $\widetilde{\pi}_1(\mathfrak{c}') = c_1'$ and $\widetilde{\pi}_2(\mathfrak{c}') = c_2'$, it follows that $\mathfrak{c}' = \widetilde{\theta}_2^{-1}(c_2') + c_1' \cdot u^{l-1}$. By corollary 2.5, we obtain $\mathcal{C}^\perp = \widetilde{\theta}_2^{-1}(\mathcal{C}_2^\perp) \oplus u^{l-1} \cdot \mathcal{C}_1^\perp$. □

2.2.2. *Gray map.* The Gray map $\Gamma : \mathcal{R}_l \to \mathbb{F}_p^l$ is defined for an element $\alpha = (\lambda_0, \lambda_1, \ldots, \lambda_{l-1}) \in R_l$ by

$$\Gamma(\alpha) = \left(\lambda_0, \lambda_0 + \lambda_1, \lambda_0 + \lambda_1 + \lambda_2, \ldots, \sum_{i=0}^{l-1} \lambda_i\right).$$

This mapping can be represented by an invertible lower-triangular matrix over $\mathbb{F}_p$, ensuring Γ is an $\mathbb{F}_p$-linear bijection.

The map is naturally extended to $\Gamma_n : \mathcal{R}_l^n \to \mathbb{F}_p^{ln}$ by applying Γ coordinate-wise. For any element $\alpha \in \mathcal{R}_l^n$, the Gray weight is defined as $w_\Gamma(\alpha) = w_H(\Gamma_n(\alpha))$, and the Gray distance between $\alpha, \alpha' \in \mathcal{R}_l^n$ is defined by the Hamming distance of their images $d_\Gamma(\alpha, \alpha') = w_H(\Gamma_n(\alpha - \alpha'))$. By definition, Γ_n acts as a distance-preserving linear isometry.

Because of these isometric properties, the translation of linear codes over $\mathcal{R}_l$ to linear codes over $\mathbb{F}_p$ follows standard coding theory frameworks [15, 23]. We summarize this in the following lemma.

Lemma 2.8. *Consider a linear code $C \subseteq R_l^n$ with $|C| = p^k$ and minimum Gray distance d_Γ. Then, the image $\Gamma_n(C)$ is a linear code over $\mathbb{F}_p$ with parameters $[ln, k, d_H]$, where the minimum Hamming distance $d_H = d_\Gamma$.*

Remark 2.9. Let $\Gamma_1 : \mathcal{R}_{l,1} \to \mathbb{F}_p$ be the identity map and $\Gamma_2 : \mathcal{R}_{l,2} \to \mathbb{F}_p^{l-1}$ be the map defined by

$$\Gamma_2\left(\sum_{i=0}^{l-2}\lambda_i v^i\right) = (\lambda_0, \lambda_0+\lambda_1, \lambda_0+\lambda_1+\lambda_2, \ldots, \sum_{i=0}^{l-2}\lambda_i).$$

Then, the Gray map Γ can be decomposed as

$$\Gamma(\alpha) = \left(\Gamma_2\left(\sum_{i=0}^{l-2}\lambda_i v^i\right), \Gamma_1\left(\sum_{i=0}^{l-1}\lambda_i\right)\right),$$

for a given $\alpha = \lambda_0 + \lambda_1 u + \lambda_2 u^2 + \ldots + \lambda_{l-1}u^{l-1} \in \mathcal{R}_l$.

3. The algebraic form of Euclidean sums

The present section investigates the structure of cyclic codes over $\mathcal{R}_l$, and their Euclidean sums. We specifically determine the form of their generator polynomials. The Euclidean sum for a linear code $\mathcal{C}$ can be introduced by

$$\mathcal{S}_\mathcal{C} = \mathcal{C} + \mathcal{C}^\perp = \{c_1 + c_2 \mid c_1 \in \mathcal{C}, c_2 \in \mathcal{C}^\perp\}.$$

Notably, assuming $\mathcal{C}$ represents a cyclic code parameterized by length n across $\mathcal{R}_l$, hence $\mathcal{S}_\mathcal{C}$ preserves the cyclic structure, meaning it is also a cyclic code parameterized by length n on $\mathcal{R}_l$.

We first give some necessary results regarding cyclic codes over $\mathcal{R}_l$.

Lemma 3.1. *Consider a cyclic code $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2) \subseteq \mathcal{R}_l^n$, with $\gcd(n, p) = 1$. Then, there exist polynomials $\chi_0(x) \in \mathbb{F}_p[x]$, and $\chi_1(x), \chi_2(x), \ldots, \chi_{l-1}(x) \in \mathcal{R}_{l,2}[x]$, such that $\mathcal{C}_1 = \langle\chi_0(x)\rangle$ and $\mathcal{C}_2 = \langle\chi_1(x), v\chi_2(x), \ldots, v^{l-2}\chi_{l-1}(x)\rangle$, where $x^n - 1 = \chi_i(x)\phi_i(x)$ for $i = 0, 1, \ldots, l-1$, and $\chi_i(x) \mid \chi_{i-1}(x)$ for $i = 2, \ldots, l-1$. Moreover,*

(1) *$\mathcal{C} = \widetilde{\theta}_2^{-1}\left(\langle\chi_1(x), v\chi_2(x), \ldots, v^{l-2}\chi_{l-1}(x)\rangle\right) \oplus u^{l-1}\langle\chi_0(x)\rangle$ with $|\mathcal{C}| = p^{ln - \sum_{i=0}^{l-1} deg(\chi_i(x))}$ and $d_\Gamma(\mathcal{C}) = \min\{d_\Gamma(C_1), d_\Gamma(C_2)\} = \min_{0\le i\le l-1}\{d_H(\langle\chi_i(x)\rangle)\}$.*

(2) *$\mathcal{C}^\perp = \widetilde{\theta}_2^{-1}\left(\langle\phi_1^+(x), v\phi_2^+(x), \ldots, v^{l-2}\phi_{l-1}^+(x)\rangle\right) \oplus u^{l-1}\langle\phi_0^+(x)\rangle$, where ϕ_i^+ is the reciprocal polynomial of ϕ_i and defined by $\phi_i^+(x) = x^n\phi_i\left(\frac{1}{x}\right)$, for $0 \le i \le l-1$. Furthermore, $|\mathcal{C}^\perp| = p^{\sum_{i=0}^{l-1}\deg(\chi_i(x))}$.*

Proof. (1) According to the isomorphism established in Proposition 2.1, every cyclic code $\mathcal{C}$ over $\mathcal{R}_l$ corresponds to a pair $(\mathcal{C}_1, \mathcal{C}_2)$ where $\mathcal{C}_1$ is a cyclic code over $\mathbb{F}_p$ and $\mathcal{C}_2$ is a cyclic code over $\mathcal{R}_{l,2}$. The theoretical foundations on cyclic codes [15] guarantees the existence and uniqueness of a polynomial $\chi_0(x) \in \mathbb{F}_p[x]$ dividing $x^n - 1$ such that $\mathcal{C}_1 = \langle\chi_0(x)\rangle$. Similarly, since

$\gcd(n,p)=1$, according to [8] (Theorem 3.5), there exist unique polynomials $\chi_1(x),\ldots,\chi_{l-1}(x)\in\mathcal{R}_{l,2}[x]$ satisfying $\chi_{l-1}(x)\mid\cdots\mid\chi_1(x)\mid(x^n-1)$ such that $\mathcal{C}_2=\langle\chi_1(x),v\chi_2(x),\ldots,v^{l-2}\chi_{l-1}(x)\rangle$.
Applying the decomposition from Corollary 2.5, we obtain

$$\begin{aligned}\mathcal{C} &= \widetilde{\theta}_2^{-1}(\mathcal{C}_2)\oplus u^{l-1}\cdot\mathcal{C}_1\\ &= \widetilde{\theta}_2^{-1}\left(\langle\chi_1(x),v\chi_2(x),\ldots,v^{l-2}\chi_{l-1}(x)\rangle\right)\oplus u^{l-1}\langle\chi_0(x)\rangle\end{aligned}$$

As for the cardinality, since $\mathcal{C}\simeq\mathcal{C}_1\times\mathcal{C}_2$, we have $|\mathcal{C}|=|\mathcal{C}_1||\mathcal{C}_2|$. According to ([15], Theorem 4.2.4), it follows that $|\mathcal{C}_1|=p^{n-\deg(\chi_0(x))}$ and $|\mathcal{C}_2|=p^{(l-1)n-\sum_{i=1}^{l-1}\deg(\chi_i(x))}$. Therefore:

$$|\mathcal{C}|=p^{ln-\sum_{i=0}^{l-1}\deg(\chi_i(x))}.$$

Since $\mathcal{C}\simeq\mathcal{C}_1\times\mathcal{C}_2$, then $d_\Gamma(C)$ is determined by the minimum of the distances of its constituent ideals, which implies that

$$d_\Gamma(\mathcal{C})=\min\{d_\Gamma(\mathcal{C}_1),d_\Gamma(\mathcal{C}_2)\}.$$

Furthermore, the Gray map Γ_n defines an isometry between $(\mathcal{R}_l^n,d_\Gamma)$ and $(\mathbb{F}_p^{ln},d_H)$. Then, $d_\Gamma(\mathcal{C}_1)=d_H(\langle\chi_0(x)\rangle)$ and $d_\Gamma(\mathcal{C}_2)=\min_{1\le i\le l-1}\{d_H(\langle\chi_i(x)\rangle)\}$. Therefore:

$$d_\Gamma(\mathcal{C})=\min_{0\le i\le l-1}\{d_H(\langle\chi_i(x)\rangle)\}.$$

(2) Using a similar method, the proof of this part follows directly.

□

In the following, we provide the generator polynomials corresponding to $\mathcal{S}_\mathcal{C}$ related to cyclic codes characterized by length n across $\mathcal{R}_l$.

Theorem 3.2. *Consider a code $\mathcal{C}$ supposed to be cyclic and characterized by length n across $\mathcal{R}_l$, with $\gcd(n,p)=1$. Then,*

$$\begin{aligned}\mathcal{S}_\mathcal{C}=\widetilde{\theta}_2^{-1}\Big(\big\langle&\gcd\left(\chi_1(x),\phi_1^+(x)\right),v\gcd\left(\chi_2(x),\phi_2^+(x)\right),\\ &\ldots,v^{l-2}\gcd\left(\chi_{l-1}(x),\phi_{l-1}^+(x)\right)\big\rangle\Big)\oplus u^{l-1}\left\langle\gcd\left(\chi_0(x),\phi_0^+(x)\right)\right\rangle.\end{aligned}$$

with $\gcd(\chi_{l-1}(x),\phi_{l-1}^+(x))\mid\gcd(\chi_{l-2}(x),\phi_{l-2}^+(x))\mid\ldots\mid\gcd(\chi_2(x),\phi_2^+(x))\mid\gcd(\chi_1(x),\phi_1^+(x))$.
Furthermore, $|\mathcal{S}_\mathcal{C}|=p^{ln-\sum_{i=0}^{l-1}\deg(\gcd(\chi_i(x),\phi_i^+(x)))}$.

Proof. By the structural decomposition established in Remark 2.4, any cyclic code over $\mathcal{R}_l$ can be decomposed into components over $\mathbb{F}_p$ and $\mathcal{R}_{l,2}$.
Let $\mathcal{C}=(\mathcal{C}_1,\mathcal{C}_2)$ and $\mathcal{C}^\perp=(\mathcal{C}_1^\perp,\mathcal{C}_2^\perp)$.
By definition, the Euclidean sum codes of $\mathcal{C}_1$ and $\mathcal{C}_2$ are $\mathcal{S}_{\mathcal{C}_1}=\mathcal{C}_1+\mathcal{C}_1^\perp$ and $\mathcal{S}_{\mathcal{C}_2}=\mathcal{C}_2+\mathcal{C}_2^\perp$, respectively. Due to the linearity of the decomposition maps, the sum operates component-wise:

$$\mathcal{S}_\mathcal{C}=\mathcal{C}+\mathcal{C}^\perp=(\mathcal{C}_1+\mathcal{C}_1^\perp,\mathcal{C}_2+\mathcal{C}_2^\perp)=(\mathcal{S}_{\mathcal{C}_1},\mathcal{S}_{\mathcal{C}_2}).$$

From ([11], Theorem 4.3.7), the sum of two cyclic codes over the field $\mathbb{F}_p$ is generated by the greatest common divisor of their generator polynomials. Thus:

$$\mathcal{S}_{\mathcal{C}_1}=\langle\chi_0(x)\rangle+\langle\phi_0^+(x)\rangle=\langle\gcd(\chi_0(x),\phi_0^+(x))\rangle.$$

Similarly, over the local ring $\mathcal{R}_{l,2}$, the sum of the cyclic codes

$$\mathcal{C}_2 = \langle \chi_1(x), v\chi_2(x), \ldots, v^{l-2}\chi_{l-1}(x)\rangle, \text{ and}$$
$$\mathcal{C}_2^{\perp} = \langle \phi_1^+(x), v\phi_2^+(x), \ldots, v^{l-2}\phi_{l-1}^+(x)\rangle$$

is generated by the pairwise greatest common divisors of their torsion polynomials. Therefore:

$$\begin{aligned}\mathcal{S}_{\mathcal{C}_2} &= \langle \chi_1(x), v\chi_2(x), \ldots, v^{l-2}\chi_{l-1}(x)\rangle + \langle \phi_1^+(x), v\phi_2^+(x), \ldots, v^{l-2}\phi_{l-1}^+(x)\rangle \\ &= \left\langle \gcd(\chi_1(x), \phi_1^+(x)), v\gcd(\chi_2(x), \phi_2^+(x)), \ldots, v^{l-2}\gcd(\chi_{l-1}(x), \phi_{l-1}^+(x))\right\rangle.\end{aligned}$$

Since $\mathcal{S}_{\mathcal{C}}$ is a cyclic code, we apply Corollary 2.5 to reconstruct $\mathcal{S}_{\mathcal{C}}$ from its components $\mathcal{S}_{\mathcal{C}_1}$ and $\mathcal{S}_{\mathcal{C}_2}$, then, we obtain:

$$\begin{aligned}\mathcal{S}_{\mathcal{C}} &= \widetilde{\theta}_2^{-1}(\mathcal{S}_{\mathcal{C}_2}) \oplus u^{l-1}\cdot \mathcal{S}_{\mathcal{C}_1} \\ &= \widetilde{\theta}_2^{-1}\left(\left\langle \gcd(\chi_1(x), \phi_1^+(x)), \ldots, v^{l-2}\gcd(\chi_{l-1}(x), \phi_{l-1}^+(x))\right\rangle\right) \oplus u^{l-1}\left\langle \gcd(\chi_0(x), \phi_0^+(x))\right\rangle.\end{aligned}$$

Furthermore, since $\mathcal{R}_{l,2}$ is a finite chain ring, the cyclic codes $\mathcal{C}$ and $\mathcal{C}^{\perp}$ satisfy the divisibility chains $\chi_i(x) \mid \chi_{i-1}(x)$ and $\phi_i^+(x) \mid \phi_{i-1}^+(x)$ for $i = 2, \ldots, l-1$ as established by Lemma 3.1. Therefore, any common divisor of $\chi_i(x)$ and $\phi_i^+(x)$ naturally divides both $\chi_{i-1}(x)$ and $\phi_{i-1}^+(x)$. It follows directly that $\gcd(\chi_i(x), \phi_i^+(x)) \mid \gcd(\chi_{i-1}(x), \phi_{i-1}^+(x))$.
Finally, since $\mathcal{S}_{\mathcal{C}} \simeq \mathcal{S}_{\mathcal{C}_1} \times \mathcal{S}_{\mathcal{C}_2}$, and using the cardinality formulas for cyclic codes over $\mathbb{F}_p$ and $\mathcal{R}_{l,2}$, it follows that

$$\begin{aligned}|\mathcal{S}_{\mathcal{C}}| &= |\mathcal{S}_{\mathcal{C}_1}||\mathcal{S}_{\mathcal{C}_2}| \\ &= p^{n-\deg(\gcd(\chi_0(x), \phi_0^+(x)))} \times \prod_{i=1}^{l-1} p^{n-\deg(\gcd(\chi_i(x), \phi_i^+(x)))} \\ &= p^{ln-\sum_{i=0}^{l-1}\deg(\gcd(\chi_i(x), \phi_i^+(x)))}.\end{aligned}$$

This completes the proof. □

4. QECCs constructed using cyclic codes on $\mathcal{R}_l$ via Euclidean sums

This section is devoted to the construction of QECCs based on the Euclidean duals of cyclic codes over $\mathcal{R}_l$, using the CSS and Steane schemes to derive them from standard linear codes.

Theorem 4.1. [10] *Consider two linear codes $\mathcal{C}_1$ and $\mathcal{C}_2$ parameterized by $[n, k_1, d_1]_p$ along with $[n, k_2, d_2]_p$ respectively, with $\mathcal{C}_2^{\perp}$ a subset of $\mathcal{C}_1$. Then, a QECC with parameter $[[n, k_1 + k_2 - n, d]]_p$ exists, where $d = \min\{w_H(c) \mid c \in (\mathcal{C}_1 \setminus \mathcal{C}_2^{\perp}) \cup (\mathcal{C}_2 \setminus \mathcal{C}_1^{\perp})\} \geq \min\{d_1, d_2\}$. Specifically, if there exists a linear code $\mathcal{C}$ across $\mathbb{F}_p$ characterized by $[n, k, d]_p$ under the condition $\mathcal{C}^{\perp} \subseteq \mathcal{C}$, hence, the existence of a QECC parameterized by $[[n, 2k - n, d]]_p$, holds.*

Corollary 4.2. [20] *Let a code $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) be linear and characterized by $[n, k_1, d_1]_p$ (resp. $[n, k_2, d_2]_p$), under the conditions $\mathcal{C}_1^{\perp} \subseteq \mathcal{C}_1 \subseteq \mathcal{C}_2$ along with $k_2 \geq k_1 + 2$. Then, a QECC parameterized by $[[n, k_1 + k_2 - n, d \geq d']]_p$ can be found, where $d' = \min\{d_1, [\frac{p+1}{p}d_2]\}$.*

Within the next result, we identify characteristics concerning the Gray map along with dual codes for the ring $\mathcal{R}_l$.

Lemma 4.3. *Consider a linear code $\mathcal{C}$ over $\mathcal{R}_l$ parameterized by its length n. Hence, the following properties hold*

(1) $\Gamma_n(\mathcal{C}^\perp) = \Gamma_n(\mathcal{C})^\perp$.
(2) $\Gamma_n(\mathcal{S}_\mathcal{C})^\perp \subseteq \Gamma_n(\mathcal{S}_\mathcal{C})$.

Proof. (1) Let $a \in \mathcal{C}$ and $b \in \mathcal{C}^\perp$. By definition, $a \cdot b = 0$. Gray's map $\Gamma_n : \mathcal{R}_l^n \to \mathbb{F}_p^{ln}$ is a linear isometry preserving the scalar product. Thus, $a \cdot b = 0$ then, $\Gamma_n(a) \cdot \Gamma_n(b) = 0$. Therefore, $\forall\ \Gamma_n(a) \in \Gamma_n(\mathcal{C})$, we have $\Gamma_n(b) \in \Gamma_n(\mathcal{C})^\perp$. Hence the inclusion $\Gamma_n(\mathcal{C}^\perp) \subseteq \Gamma_n(\mathcal{C})^\perp$.

Since $\mathcal{R}_l$ is a finite ring, MacWilliams' theorem [23] gives $|\mathcal{C}| \cdot |\mathcal{C}^\perp| = |\mathcal{R}_l|^n = p^{ln}$. Since Γ_n is bijective, it preserves cardinals. Then, $|\Gamma_n(\mathcal{C})| = |\mathcal{C}|$ and $|\Gamma_n(\mathcal{C}^\perp)| = |\mathcal{C}^\perp|$. Over the field $\mathbb{F}_p$, $\Gamma_n(\mathcal{C})$ is a linear code of length ln, so $|\Gamma_n(\mathcal{C})| \cdot |\Gamma_n(\mathcal{C})^\perp| = p^{ln}$. By direct substitution, we get

$$|\Gamma_n(\mathcal{C})^\perp| = \frac{p^{ln}}{|\Gamma_n(\mathcal{C})|} = \frac{p^{ln}}{|\mathcal{C}|} = |\mathcal{C}^\perp| = |\Gamma_n(\mathcal{C}^\perp)|.$$

The inclusion $\Gamma_n(\mathcal{C}^\perp) \subseteq \Gamma_n(\mathcal{C})^\perp$ besides the equality of cardinals implies

$$\Gamma_n(\mathcal{C}^\perp) = \Gamma_n(\mathcal{C})^\perp.$$

(2) By the properties of the dual of a sum, $(\mathcal{S}_\mathcal{C})^\perp = (\mathcal{C} + \mathcal{C}^\perp)^\perp = \mathcal{C}^\perp \cap (\mathcal{C}^\perp)^\perp$. On the ring $\mathcal{R}_l$, duality is reflexive, then $(\mathcal{C}^\perp)^\perp = \mathcal{C}$, which gives $(\mathcal{S}_\mathcal{C})^\perp = \mathcal{C}^\perp \cap \mathcal{C}$. However, the intersection is trivially included in the sum $\mathcal{C}^\perp \cap \mathcal{C} \subseteq \mathcal{C} + \mathcal{C}^\perp$ then, $(\mathcal{S}_\mathcal{C})^\perp \subseteq \mathcal{S}_\mathcal{C}$. Since the application Γ_n preserves inclusions, we obtain $\Gamma_n((\mathcal{S}_\mathcal{C})^\perp) \subseteq \Gamma_n(\mathcal{S}_\mathcal{C})$. According to (1.), the operator Γ_n commutes with orthogonality, which implies $\Gamma_n((\mathcal{S}_\mathcal{C})^\perp) = \Gamma_n(\mathcal{S}_\mathcal{C})^\perp$. By substitution, we obtain

$$\Gamma_n(\mathcal{S}_\mathcal{C})^\perp \subseteq \Gamma_n(\mathcal{S}_\mathcal{C}).$$

□

Next, we present the fundamental result in this part.

Theorem 4.4. *Let $\mathcal{C} \subseteq \mathcal{R}_l^n$ be a cyclic code and $gcd(n, p) = 1$, such that $\mathcal{C} = \widetilde{\theta}_2^{-1}\left(\langle \chi_1(x), v\chi_2(x), \ldots, v^{l-2}\chi_{l-1}(x)\rangle\right) \oplus u^{l-1}\langle \chi_0(x)\rangle$ with $\chi_i(x)\phi_i(x) = x^n - 1$ within $\mathbb{F}_p[x]$ for $i = 0, 1, \ldots, l-1$, and $\chi_i(x) \mid \chi_{i-1}(x)$ and $\phi_i^+(x) \mid \phi_{i-1}^+(x)$ and $\gcd(\chi_i(x), \phi_i^+(x)) \mid \gcd(\chi_{i-1}(x), \phi_{i-1}^+(x))$ for $i = 2, 3, \ldots, l-1$.*

Then, an $[[ln, ln - 2\sum_{i=0}^{l-1} \deg(\gcd(\chi_i(x), \phi_i^+(x))), d]]_p$ QECC code can be found and d represents the minimum Hamming distance corresponding to the linear code $\Gamma(\mathcal{S}_\mathcal{C})$ across $\mathbb{F}_p$.

Proof. From the second point of Theorem 3.2 and Lemma 2.8, we can conclude that $\Gamma(\mathcal{S}_{\mathcal{C}})$ forms an $[ln, k, d]$ linear code on the field $\mathbb{F}_p$, such that

$$k = ln - \sum_{i=1}^{l} \deg(\gcd(\chi_i(x), \phi_i^+(x))),$$

with the minimum Hamming distance corresponding to $\Gamma(\mathcal{S}_{\mathcal{C}})$ on $\mathbb{F}_p$, is d. Therefore, the existence of a QECC parameterized by $[[ln, ln-2\sum_{i=1}^{l} \deg(\gcd(\chi_i(x), \phi_i^+(x))), d]]_p$ is ensured by using Theorem 4.1 and Lemma 4.3. □

Remark 4.5. It is clear that $\mathcal{S}_{\mathcal{C}} = \mathcal{C}$ is equivalent to $\mathcal{C}^{\perp}$ is contained within $\mathcal{C}$. Thus, it can be readily observed that $\mathcal{C}^{\perp} \subseteq \mathcal{C}$ if and only if $x^n - 1 \equiv 0 \pmod{\chi_i(x)\chi_i^+(x)}$, such that $i = 0, 1, \ldots, l-1$. From Lemma 2.8 besides the second assertion of Theorem 3.2, we hold that $\Gamma(\mathcal{S}_{\mathcal{C}}) = \Gamma(\mathcal{C})$ represent a linear code characterized by $[ln, k, d]$ over $\mathbb{F}_p$. Applying Theorem 4.1, It follows that an $[[ln, 2k - ln, d]]_p$ QECC exists.

4.1. **Another method to construct QECCs.** Let p and n are coprime and suppose that $p^t \equiv 1 \pmod{n}$, implying that $n \mid p^t - 1$ however $n \nmid p^{t-1} - 1$. Hence, an element $\mu \in \mathbb{F}_{p^t}$ exists, and satisfying $\mu^n = 1$. This is straightforward with the aim of verifying that μ^j, for $j \in \mathbb{Z}_n$, correspond to all zeros of the polynomial $x^n - 1$. Thus, we get the factorisation in $\mathbb{F}_{p^t}[x]$:

$$x^n - 1 = \prod_{j \in \mathbb{Z}_n} (x - \mu^j).$$

Now, assume $x^n - 1 = \Phi(x)\Theta(x)$. The polynomial $\Phi(x)$ is associated with the following characterization set:

$$\Delta(\Phi(x)) = \{j \in \mathbb{Z}_n \mid \Phi(\mu^j) = 0\}.$$

According to the description of the characterization set, obviously the degree of $\Phi(x)$ equals $|\Delta(\Phi(x))|$. If we take $\Phi(x) \in \mathbb{F}_{p^t}[x]$ generate a cyclic code $\mathcal{C} = \langle \Phi(x) \rangle$. Therefore, the characterization set corresponding to $\mathcal{C}$ is identical to $\Delta(\Phi(x))$.

Remark 4.6. In order to the purpose of extended analysis in this part, we refer to the presented findings given in [24].
Consider the factorization $x^n - 1 = \Phi(x)\Theta(x)\Psi(x) \in \mathbb{F}_{p^t}[x]$.
We have $\Delta(\gcd(\Phi(x), \Theta(x))) = \Delta(\Phi(x)) \cap \Delta(\Theta(x))$, and furthermore,

$$\deg(\gcd(\Phi(x), \Theta(x))) = |\Delta(\Phi(x)) \cap \Delta(\Theta(x))|.$$

Considering $x^n - 1 = \Phi(x)\Theta(x)\Psi(x) \in \mathbb{F}_{p^t}[x]$ together with a cyclic code $\mathcal{C} \subseteq \mathbb{F}_{p^t}^n$, formed by $\gcd(\Phi(x), \Theta(x))$, if we assume that $\Lambda = \{\vartheta \mid \vartheta = \pi, \pi+1, \ldots, \pi+\eta-1\}$ is a subset contained in $\Delta(\Phi(x)) \cap \Delta(\Theta(x))$, then the minimum distance associated with $\mathcal{C}$ is $\leq \eta$.
Let $\Phi(x) = (x - \mu^{\eta})(x - \mu^{\eta+1}) \ldots (x - \mu^{\eta+n-\epsilon-1}) \in \mathbb{F}_{p^t}[x]$ be a polynomial, where $\mu \in \mathbb{F}_{p^t}$ is a primitive element, $\eta \geq 0$, and $1 \leq \epsilon \leq p - 1$. Consequently, $\Delta(\Phi(x)) = \{\eta, \eta + 1, \ldots, \eta + n - \epsilon - 1\}$. Now suppose there exists another

polynomial $\Psi(x) \in \mathbb{F}_{p^t}[x]$ such that $x^n - 1 = \Psi(x)\Phi(x)$, the characteristic set associated with $\Psi^+(x)$ is then given by $\Delta(\Psi^+(x)) = \{n-\eta+1, n-\eta+2, \ldots, n-\eta+\epsilon\}$. This connection demonstrates how the composition of $\Phi(x)$ affects the domain corresponding to $\Psi^+(x)$ when they are combined to produce $x^n - 1$.

At this point, we turn our attention to another important result in this part that aids in the building of further novel QECCs.

Theorem 4.7. *Let n, η, and ϵ be defined so that $n = p^t - 1$, $\eta \geq 1$, and $2\eta - 1 < \epsilon \leq \frac{n+2\eta-1}{2}$. Thus, we obtain an $[[ln, ln - 2l\epsilon + 4l\eta - l, d]]_p$ QECC, where d meets the condition $d \geq \epsilon - 2\eta + 2$.*

Proof. Suppose $x^n - 1 = \Phi(x)\Psi(x)$, with $\Phi(x) = (x - \mu^\eta)(x - \mu^{\eta+1}) \ldots (x - \mu^{\eta+n-\epsilon-1})$, and $\mu \in \mathbb{F}_{p^t}$ represent a primitive element. Thus, by applying Remark 4.6, we conclude that $\Delta(\Psi^+(x)) = \{n-\eta+1, n-\eta+2, \ldots, n-\eta+\epsilon\}$. Additionally, one finds the claim $\epsilon > 2\eta - 1$, which indicates the fact $n + \eta - \epsilon - 1 \leq n - \eta < n - \eta + 1$. The first component of $\Delta(\Psi^+(x))$ is located immediately following the last component of $\Delta(\Phi(x))$.

Considering the fact $\eta \geq 1$, $\epsilon \geq 2\eta - 1 \geq \eta$, It is possible to express $\Delta(\Psi^+(x))$ by $\Delta(\Psi^+(x)) = \{-\eta + 1, -\eta + 2, \ldots, -1, 0, 1, \ldots, \epsilon - \eta\}$. Since $1 < \epsilon \leq \frac{n+2\eta-1}{2}$, we have $\Delta(\Psi(x)) \cap \Delta(\Psi^+(x)) = \{\eta, \eta + 1, \ldots, \epsilon - \eta\}$. Thus, according to Remark 4.6, it can be concluded that $\gcd(\Phi(x), \Psi^+(x)) = (x-\mu^\eta)(x-\mu^{\eta+1}) \ldots (x-\mu^{\epsilon-\eta})$. Therefore, $\deg(\gcd(\Phi(x), \Psi^+(x))) = \epsilon - 2\eta + 1$.

Examine a cyclic code $\mathcal{C}$ generated by $\widetilde{\theta}_2^{-1}\left(\langle \Phi(x), v\Phi(x), \ldots, v^{l-2}\Phi(x)\rangle\right) \oplus u^{l-1}\langle \Phi(x)\rangle$ parameterized by length n on $\mathcal{R}_l$. By applying Theorem 3.2 over $x^n - 1 = \Phi(x)\Psi(x)$, we obtain:

$$\mathcal{S}_\mathcal{C} = \widetilde{\theta}_2^{-1}\left(\left\langle \gcd\left(\Phi(x), \Psi^+(x)\right), v\gcd\left(\Phi(x), \Psi^+(x)\right), \right.\right.$$
$$\left.\left. \ldots, v^{l-2}\gcd\left(\Phi(x), \Psi^+(x)\right)\right\rangle\right) \oplus u^{l-1}\left\langle \gcd\left(\Phi(x), \Psi^+(x)\right)\right\rangle.$$

By using Remark 4.6, it follows $d_H(\gcd(\Phi(x), \Psi^+(x))) \geq \epsilon - 2\eta + 2$.

At this point, consider $\mathcal{C}_1 = \mathcal{S}_\mathcal{C}$ and $\mathcal{C}_2 = \widetilde{\theta}_2^{-1}\left(\langle w(x), vw(x), \ldots, v^{l-2}w(x)\rangle\right) \oplus u^{l-1}\langle w(x)\rangle$, where $w(x) = (x - \mu^\eta) \ldots (x - \mu^{\epsilon-\eta})$. It is evident that $\Gamma(\mathcal{C}_1)$ is an $[ln, ln-l\epsilon+2l\eta-l, \geq \epsilon-2\eta+2]_{p^t}$ linear code, and $\Gamma(\mathcal{C}_2)$ is an $[ln, ln-l\epsilon+2l\eta-l]_{p^t}$, linear code over $\mathbb{F}_{p^t}$. Obviously, $\Gamma(\mathcal{C}_1)^\perp \subseteq \Gamma(\mathcal{C}_2)$, and $ln - l\epsilon + 2l\eta > ln - l\epsilon + 2l\eta - l + 2$. Hence, by Corollary 4.2, we obtain a QECC parameterized by $[[ln, ln - 2l\epsilon + 4l\eta - l, d]]_p$ with $d \geq \epsilon - 2\eta + 2$. □

5. Applications

In this part, we aim to interpret the conclusions outlined previously by providing the following illustrations. The sequence $a_m a_{m-1} \ldots a_1 a_0$ represents the polynomial $a_m x^m + a_{m-1}x^{m-1} + \ldots + a_1 x + a_0$.

Example 5.1. Let $\mathcal{R}_5$ be the ring over $\mathbb{F}_5$. Consider a cyclic code of length $n = 24$. The polynomial $x^{24} - 1$ factors in $\mathbb{F}_5[x]$ as:

$$\begin{aligned} x^{24} - 1 \;=\; & (x+1)(x+2)(x+3)(x+4)(x^2+2)(x^2+3) \\ & (x^2+x+1)(x^2+x+2)(x^2+2x+3)(x^2+2x+4) \\ & (x^2+3x+3)(x^2+3x+4)(x^2+4x+1)(x^2+4x+2). \end{aligned}$$

Define the code $\mathcal{C} = \langle \chi_0(x), \chi_1(x), v\chi_2(x), v^2\chi_3(x) \rangle$, where

$$\begin{aligned}
\chi_0(x) &= x^6 + x^5 + 2x^4 + x^3 + 4x^2 + 4x + 3, \\
\chi_1(x) &= x^8 + 2x^7 + 2x^6 + 3x^5 + 4x^4 + 3x^3 + 2x^2 + 3, \\
\chi_2(x) &= x^6 + 4x^5 + 2x^4 + 3x^2 + 4x + 1, \text{ and} \\
\chi_3(x) &= x^4 + 4x^3 + 2x + 3.
\end{aligned}$$

The corresponding reciprocal polynomials $\phi_i^+(x)$ and the gcds characterizing the intersection of each component with its dual, $\gcd(\chi_i(x), \phi_i^+(x))$, are calculated as follows.
For $\phi_0^+(x) = x^6+3x^5+3x^4+2x^3+4x^2+2x+2$, then $\gcd(\chi_0(x), \phi_0^+(x)) = x^2+4x+1$.
For $\phi_1^+(x) = x^8+4x^6+x^5+3x^4+x^3+4x^2+4x+2$, thus $\gcd(\chi_1(x), \phi_1^+(x)) = x+4$.
For $\phi_2^+(x) = x^6 + 4x^5 + 2x^4 + 3x^2 + 4x + 1$, hence $\gcd(\chi_2(x), \phi_2^+(x)) = x + 4$.
For $\phi_3^+(x) = x^4 + 4x^3 + 3x + 2$, this yields $\gcd(\chi_3(x), \phi_3^+(x)) = x + 4$.

By Theorem 4.4, the QECC parameters are

$$[[ln, ln - 2\sum_{i=0}^{3} \deg(\gcd(\chi_i(x), \phi_i^+(x))), d]]_5 = [[96, 86, 5]]_5.$$

The minimum distance $d = 5$ is verified via the Gray map image $\Gamma(S_{\mathcal{C}})$. This code exceeds the parameters of comparable construction $[[96, 80, 5]]_5$ in [12] and in Grassl's table [10] $[[96, 86, 4]]_5$.

Example 5.2. Consider the following settings: let $p = 11$, $l = 3$, $\eta = 2$, and $\epsilon = 4$. We determine $n = p-1 = 10$. According to Theorem 4.7, we can construct QECCs characterized by $[[30, 27, \geq 2]]_{11}$.

TABLE 1. Novel QECCs built on $\mathcal{R}_l$

n	l	Generators χ_i	QECC	Existing QECCs
16	4	$\chi_0 = 1211011$, $\chi_1 = 122012022$ $\chi_2 = 11122002$, $\chi_3 = 1202111$	$[[64, 58, 4]]_3$	New
20	3	$\chi_0 = 111102$, $\chi_1 = 1222122$ $\chi_2 = 102121$	$[[60, 52, 4]]_3$	$[[60, 36, 4]]_3$ [2]
13	3	$\chi_0 = 1222$, $\chi_1 = 10221$, $\chi_2 = 1102$	$[[39, 34, 3]]_3$	$[[39, 21, 3]]_3$ [6]
16	3	$\chi_0 = 1202200012022$, $\chi_1 = 12112202022$ $\chi_2 = 1101112002$	$[[48, 14, 6]]_3$	New
22	4	$\chi_0 = 12034313024$, $\chi_1 = 1344402041421$ $\chi_2 = 122223413311$, $\chi_3 = 13024212034$	$[[88, 76, 7]]_5$	$[[88, 48, 6]]_5$ [1]
12	3	$\chi_0 = 1643$, $\chi_1 = 12145$, $\chi_2 = 1523$	$[[36, 31, 3]]_7$	$[[35, 21, 3]]_7$ [13]
8	5	$\chi_0 = 1334$, $\chi_1 = 1020403$, $\chi_2 = 131334$ $\chi_3 = 14031$, $\chi_4 = 1321$	$[[40, 26, 3]]_5$	$[[40, 24, 3]]_5$ [16]
6	3	$\chi_0 = 16(12)$, $\chi_1 = 1651$, $\chi_2 = 17(12)$	$[[18, 13, 3]]_{13}$	$[[18, 10, 3]]_{13}$ [16]
16	3	$\chi_0 = 108080(12)$, $\chi_1 = 170494068$ $\chi_2 = 185(12)8(12)(12)5$	$[[48, 44, 3]]_{13}$	New
13	4	$\chi_0 = 1w^{23}1$, $\chi_1 = 1w^{22}w^8w^{20}w^{10}4$ $\chi_2 = 1w^3w^{15}4$, $\chi_3 = 1w^{10}1$	$[[52, 28, 3]]_{25}$	New
22	3	$\chi_0 = 1022112121222012$, $\chi_1 = 12220100000211102$ $\chi_2 = 1021122222212011$	$[[66, 4, 10]]_9$	New

TABLE 2. Additional novel QECCs built on $\mathcal{R}_l$

p	l	η	ϵ	New Codes	Existing Codes
7	7	1	3	$[[42, 21, \geq 3]]_7$	$[[42, 12, \geq 4]]_7$ [7]
3^2	3	2	5	$[[24, 15, \geq 3]]_{3^2}$	$[[24, 21, \geq 2]]_{3^2}$ [18]
13	3	1	3	$[[36, 27, \geq 3]]_{13}$	$[[35, 27, \geq 3]]_{13}$ [14]
17	3	2	7	$[[48, 27, \geq 5]]_{17}$	$[[48, 26, \geq 5]]_{17}$ [13]
23	3	3	13	$[[66, 21, \geq 9]]_{23}$	$[[66, 60, \geq 4]]_{23}$ [13]
5^2	2	2	13	$[[48, 10, \geq 11]]_{5^2}$	$[[48, 42, \geq 3]]_{5^2}$ [18]
7^2	2	1	24	$[[96, 6, \geq 24]]_{7^2}$	$[[96, 74, \geq 7]]_{7^2}$ [18]
3^4	3	1	7	$[[240, 207, \geq 7]]_{3^4}$	$[[240, 225, \geq 3]]_{3^4}$ [18]
11^2	3	4	15	$[[360, 315, \geq 9]]_{11^2}$	$[[360, 345, \geq 4]]_{11^2}$ [18]
13^2	2	3	11	$[[336, 314, \geq 7]]_{13^2}$	$[[336, 330, \geq 3]]_{13^2}$ [18]

Table 1 lists several QECC codes derived from Theorem 4.4. These new codes offer significantly better parameters than those found in the current literature. Besides this, Table 2 presents another family of codes over different finite fields, obtained via Theorem 4.7.

6. CONCLUSION

Through this work, a method for constructing QECCs is explored. The given codes are obtained through the use of Euclidean sums corresponding to cyclic

codes across $\mathcal{R}_l$. Initially, this study introduces the algebraic properties of the ring $\mathcal{R}_l$. Then establishes the generator polynomial of cyclic codes and their Euclidean sums over this ring in Lemma 3.1 and Theorem 3.2. We explore the link connecting $\mathcal{S}_\mathcal{C}$ related to $\Gamma(\mathcal{C})$, parameterized by length ln in $\mathbb{F}_p$, with cyclic codes $\mathcal{C}$ characterized by length n across $\mathcal{R}_l$. Based on our findings, we demonstrate that such codes are suitable for effective application to construct QECCs parameterized by length ln over the field $\mathbb{F}_p$. These results are detailed in Theorem 4.4. Besides this, we explore another method of constructing such codes and given in Theorem 4.7. Furthermore, a list of newly constructed QECCs is provided in Tables 1 and 2. In the future, we will extend these results to determine LCD codes and skew cyclic codes. Additionally, evaluating the performance of these codes in post-quantum, code-based cryptosystems presents an exciting direction for future study.

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