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KIRCHHOFF-CHOQUARD PROBLEMS WITH DOUBLE-PHASE STRUCTURE

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Abstract. In this paper, we investigate a new class of non-local Kirchhoff-Choquard problems with a double-phase structure. The problem combines fractional operators with different growth behaviors and a non-local interaction of Choquard type. Our approach is based on an approximation procedure and the analysis of the associated energy functional through critical groups. By applying Morse theory, we establish the existence of at least one non-trivial weak solution. These results contribute to the study of non-local problems involving double-phase behavior, Kirchhoff-type operators, and Choquard interactions.

Keywords. Kirchhoff-Choquard problems, fractional double-phase operators, critical groups, non-trivaial solutions.
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1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be an open bounded set, λ is a real positive parameter. Consider the following Kirchhoff-Choquard type problems with the double-phase fractional structure:

$$(\mathcal{P}_L) \begin{cases} [\mathcal{M}([\mathfrak{u}]^p_{s,p})]^{p-1}(-\Delta)^s_p \mathfrak{u} + [\mathcal{M}([\mathfrak{u}]^q_{s,q})]^{q-1}(-\Delta)^s_q \mathfrak{u} = \lambda \left(\int_\Omega \frac{|\mathfrak{u}|^r}{|x-y|^\nu} dy \right) |\mathfrak{u}|^{r-2}\mathfrak{u} & \text{in} \quad \Omega\ , \\ \mathfrak{u} \, = \, 0 & \text{on } \mathbb{R}^N \backslash \Omega. \end{cases}$$

where $[\mathfrak{u}]_{s,p}$ is the Gagliardo seminorm (see section 2), $(-\Delta)^s_p$ is the fractional p-Laplace operator defined along a function $\phi \in C_0^\infty(\mathbb{R}^N)$ as

$$(-\Delta)^s_p \phi(x) = 2 \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N \backslash B_\epsilon(x)} \frac{|\phi(x)-\phi(y)|^{p-2}(\phi(x)-\phi(y))}{|x-y|^{N+sp}} dy$$

for $x \in \mathbb{R}^N$ and $B_\epsilon(x) := \{y \in \mathbb{R}^N : |x-y| < \epsilon\}$, see [10, 19], and the references therein for further details on the fractional p-Laplacian.

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(i) $\mathcal{M}:[0,\infty)\mapsto[0,\infty)$ is a continuous function and there exists a constant $\kappa>1$ such that $\mathcal{M}(t)\geq\kappa$ for $t\geq 0$.

(ii) $0<s<1<p<2r<q<p_s^*<\infty$ with $\frac{1}{p}+\frac{1}{q}=1$ and $q<p_{s,\nu}^*$ with $p_s^*=\dfrac{pN}{N-ps}$ and $p_{s,\nu}^*=\dfrac{(N-\frac{\nu}{2})p}{N-sp}$, $0<\nu<sp<sq<N$,

(iii) There exists $\sigma>0$ and $\theta\in(0,\lambda)$ such that $\left(\varpi^{p-1}+\varpi^{q-1}\right)\leq\dfrac{p\theta}{2r}.\dfrac{|t_1|^r|t_2|^r}{|x-y|^\nu}$ for all $(t_1,t_2)\in\mathbb{R}^2$, $|t_1|\leq\sigma$,$|t_2|\leq\sigma$, $\varpi=\max\limits_{0\leq t\leq 1}\mathcal{M}(t)$ and for all $(x,y)\in\mathbb{R}^{2N}$.

(iv) $F:\mathbb{R}\ni t\longmapsto F(t)=\left(\displaystyle\int_\Omega\frac{|t|^r}{|x-y|^\nu}\right)|t|^{r-2}t\in L^p(\Omega)$

(v) $\Omega\subseteq\mathbb{R}^N$ is a bounded smooth domain with Lebesgue measure $|\Omega|=1$.

The pioneering contributions of Zhikov [21, 22] were motivated by the modeling of strongly anisotropic materials within the framework of homogenization. The associated functionals also play a significant role in duality theory and in the study of the Lavrentiev phenomenon [22]. Within this framework, Zhikov introduced three distinct model functionals, primarily in connection with the Lavrentiev phenomenon, given as follows $\varphi_{p,q}(\mathfrak{u}):=\displaystyle\int_\Omega\left(|\nabla\mathfrak{u}|^p+a(x)|\nabla\mathfrak{u}|^q\right)dx$, and $1<p<q$. The combination of double-phase problems with fractional nonlocal operators, as considered in this paper, provides an effective framework for modeling various phenomena in physics without loss of information. In particular, in materials science especially in the study of phase transitions, this approach captures changes in microscopic particle organization that lead to distinct macroscopic properties, as well as the critical conditions under which such transitions occur.

The model introduced by Zhikov is not well suited to describe these phenomena, as it does not adequately account for the nature of phase changes. This motivates the investigation of double-phase problems enhanced by nonlocal fractional operators, which are more appropriate for such settings, we refer to [1, 4, 5] and references therein on non-local fractional operators.

Next, numerous intriguing studies have been conducted on the Kirchhoff-Choquard problems with fractional double phase structure subject to differents situations and boundary conditions.

In particular, when $p=2$, $1<r<\infty$ and $\mathcal{M}\equiv 1$ the problem $(\mathcal{P}_L)$ conduces to the following fractional Choquard-Laplacian equation:

$$(\mathcal{P}_1)\begin{cases}(-\Delta)^s\mathfrak{u}=\dfrac{1}{2}\lambda F(\mathfrak{u}) & \text{in}\quad\Omega\ ,\\ \mathfrak{u}\ =\ 0 & \text{on}\ \mathbb{R}^N\backslash\Omega.\end{cases}$$

This problem was studied by T. Mukherjee and K. Sreebadh in [14], where they established existence and multiplicity results via a variational approach.

For the case where $p = 2$, $1 < r < \infty$ and $s \to 1^-$ the equation of Choquard-Kirchhoff type:

$$(\mathcal{P}_2)\ \begin{cases} -\mathcal{M}\left(a + b\displaystyle\int_\Omega |\nabla \mathfrak{u}|^2 dx\right)\Delta\mathfrak{u} = \dfrac{1}{2}\lambda F(\mathfrak{u}) & \text{in } \ \Omega\ , \\ \mathfrak{u}\ =\ 0 & \text{in } \mathbb{R}^N\backslash\Omega, \end{cases}$$

with $a, b > 0$. Divya Goel, Sushmita Rawat and K. Sreenadh proved the existence of a positive high energy solution of a similar problem like the problem $(\mathcal{P}_2)$ in the general case with another Choquard term by using the Pohožaev manifold and linking theorem. Apart from this, they also studied the radial symmetry of solutions of the associated limit problem, see [9].
For the limit case of $(\mathcal{P}_L)$, namely, when $s \to 1^-$ and $1 < r < \infty$, the class of nonlocal problems of $(p, q)-$ Choquard-Kirchhoff type:

$$(\mathcal{P}_3)\ \begin{cases} -[\mathcal{M}(\displaystyle\int_\Omega |\nabla \mathfrak{u}|^p dx)]^{p-1}\Delta_p\mathfrak{u} - [\mathcal{M}(\displaystyle\int_\Omega |\nabla u|^q dx)]^{q-1}\Delta_q\mathfrak{u} = \lambda F(\mathfrak{u}) & \text{in } \ \Omega\ , \\ \mathfrak{u}\ =\ 0 & \text{in } \mathbb{R}^N\backslash\Omega, \end{cases}$$

which is an interesting generalization of fractional double-phase patterns for studying phase transition problems, see for more details [2].
The novelty of this work lies in establishing the existence of a nontrivial solution for a class of fractional double-phase Kirchhoff problems with nonlinear Choquard terms, obtained by computing the critical groups of the associated energy functional via variational methods combined with Morse theory. Readers may refer to [6, 13, 15, 16] and the references therein for further insights and details on algebraic topology, Morse's theory, and critical groups.

Definition 1.1. A function $\mathfrak{u} \in W_0^{s,p,q}(\Omega)$ is a weak solution of problem $(\mathcal{P}_L)$, if

$$\left[\mathcal{M}\left([\mathfrak{u}]_{s,p}^p\right)\right]^{p-1}\int_{\Omega\times\Omega}\frac{|\mathrm{U}(x,y)|^p\mathrm{U}(x,y)\Phi(x,y)}{|x-y|^{N+ps}}\,dxdy$$
$$+\left[\mathcal{M}\left([u]_{s,q}^q\right)\right]^{q-1}\int_{\Omega\times\Omega}\frac{|\mathrm{U}(x,y)|^q\mathrm{U}(x,y)\Phi(x,y)}{|x-y|^{N+qs}}\,dxdy = \lambda\int_\Omega F(\mathfrak{u})\phi dx$$

for any $\phi \in W_0^{s,p,q}(\Omega)$ (see Section 2), with $\mathrm{U}(x,y) = \mathfrak{u}(x) - \mathfrak{u}(y)$ and $\Phi(x,y) = \phi(x) - \phi(y)$.

Our main result is related to the investigation to problem $(\mathcal{P}_L)$, using Morse's theory combined with variational techniques.

Theorem 1.2. *Let (i)–(v) be holds true. Then, there exists $\overline{\lambda} > 0$ such that for any $\lambda \in (0, \overline{\lambda})$, the problem $(\mathcal{P}_L)$ admits at least a non-trivial weak solution.*

The paper is organized as follows. Section 2, recalls the relevant definitions and fundamental properties of the underlying function spaces $W_0^{s,p,q}(\Omega)$. In Section 3, we are checking some compactness properties and finaly, is devoted to the proofs of our main results 1.2.

2. Auxiliary results

We will begin by giving some basic results for our framework $W_0^{s,p,q}(\Omega)$, which will be used later. Define the fractional Sobolev spaces

$$W^{s,p}(\Omega) = \{\mathfrak{u} \in L^p(\Omega) : [u]_{s,p} < \infty\},$$

normed by

$$||\mathfrak{u}||_{s,p} := \left(||\mathfrak{u}||_p^p + [\mathfrak{u}]_{s,p}^p\right)^{\frac{1}{p}},$$

where, the semi-norm

$$[\mathfrak{u}]_{s,p} = \left(\int_\Omega \int_\Omega \frac{|\mathfrak{u}(x) - \mathfrak{u}(y)|^p}{|x-y|^{N+sp}} dxdy\right)^{\frac{1}{p}}.$$

For the problem $(\mathcal{P}_L)$, we will use the space defined by:

$$\mathcal{X} := W^{s,p}(\Omega) \cap W^{s,q}(\Omega),$$

endowed with the norm:

$$||\mathfrak{u}||_{\mathcal{X}} := ||\mathfrak{u}||_{s,p} + ||\mathfrak{u}||_{s,q}.$$

Note that $\mathcal{X}$ is a separable reflexive Banach space. In what follows, we will work in the closed linear subspace:

$$W_0^{s,p,q}(\Omega) := \left\{\mathfrak{u} \in \mathcal{X} : \mathfrak{u} = 0 \text{ a.e in } \mathbb{R}^N \setminus \Omega\right\},$$

equivalently renormed by setting:

$$||.|| = [.]_{s,p} + [.]_{s,q}.$$

Furthermore, the embedding $W_0^{s,p,q}(\Omega) \hookrightarrow L^\gamma(\Omega)$ is continuous for $\gamma \in [1, p_s^*]$ and compact for $\gamma \in [1, p_s^*)$, see [[5], Theorem 6.5 and corollary 7.2].
Next we consider the energy functional associated with the problem $(\mathcal{P}_L)$, $J_L : W_0^{s,p,q}(\Omega) \longrightarrow \mathbb{R}$ defined by:

$$J_L(\mathfrak{u}) = \frac{1}{p}\widehat{\mathcal{M}}_1(||\mathfrak{u}||^p) + \frac{1}{q}\widehat{\mathcal{M}}_2(||\mathfrak{u}||^q) - \frac{\lambda}{2r}\int_\Omega \int_\Omega \frac{|\mathfrak{u}|^r|\mathfrak{u}|^r}{|x-y|^\nu} dxdy$$

where $\widehat{\mathcal{M}}_1(t) = \displaystyle\int_0^t [\mathcal{M}(s)]^{p-1} ds$ and $\widehat{\mathcal{M}}_2(t) = \displaystyle\int_0^t [\mathcal{M}(s)]^{q-1} ds$.

Proposition 2.1. *(See [11]). Let $s_1, s_2 > 1$ and $0 < \mu < N$ with $\frac{1}{s_1} + \frac{1}{s_2} + \frac{\mu}{N} = 2$ and let $f \in L^{s_1}(\Omega)$ and $g \in L^{s_2}(\Omega)$. Then there exists a sharp constant $C(N, \mu, s_1, s_2)$ independent of f and g, such that*

$$\int_\Omega \int_\Omega \frac{f(x)g(y)}{|x-y|^\mu} dxdy \leq C(N, \mu, s_1, s_2)||f||_{s_1}||g||_{s_2}.$$

Remark 2.2. Since $p < q$, there exists a constant $C_p > 0$ such that $[\mathfrak{u}]_{s,p} \leq C_p[\mathfrak{u}]_{s,q}$ for all $\mathfrak{u} \in W_0^{s,p,q}(\Omega)$

Proposition 2.3. *(Simon's inequality)*

$$|\alpha-\beta|^p \leq \begin{cases} C_p\left(|\alpha|^{p-2}\alpha-|\beta|^{p-2}\beta\right)(\alpha-\beta), & p\geq 2,\\ C_p'\left[\left(|\alpha|^{p-2}\alpha-|\beta|^{p-2}\beta\right)(\alpha-\beta)\right]^{p/2}\left(|\alpha|^p+|\beta|^p\right)^{(2-p)/2}, & 1<p<2, \end{cases}$$

for all $(\alpha,\beta)\in\mathbb{R}^{2N}$, where C_p and C_p' are two positive constants depending only on p.

3. Compactness results

Define,

$$\mathcal{I}_L(\mathfrak{u})=\frac{1}{p}\widehat{\mathcal{M}}_1(||\mathfrak{u}||^p)+\frac{1}{q}\widehat{\mathcal{M}}_2(||\mathfrak{u}||^q) \quad \text{and} \quad \mathcal{H}_L(\mathfrak{u})=\frac{\lambda}{2r}\int_\Omega\int_\Omega\frac{|\mathfrak{u}|^r|\mathfrak{u}|^r}{|x-y|^\nu}dxdy.$$

We state these compactness lemmas, which will be useful in the proof of the main theorem.

Lemma 3.1. *Let (i) and (ii) holds true. Then, $J_L: W_0^{s,p,q}(\Omega)\longrightarrow\mathbb{R}$ defined by $J_L(\mathfrak{u})=\mathcal{I}_L(\mathfrak{u})-\mathcal{H}_L(\mathfrak{u})\in C^1(W_0^{s,p,q}(\Omega),\mathbb{R})$, and*

$$\langle\mathcal{I}_L'(\mathfrak{u}),\phi\rangle=[\mathcal{M}(||\mathfrak{u}||^p)]^{p-1}\int_\Omega\int_\Omega\frac{|U(x,y)|^{p-2}U(x,y)\Phi(x,y)}{|x-y|^{N+sp}}dxdy$$

$$+[\mathcal{M}(||\mathfrak{u}||^q)]^{q-1}\int_\Omega\int_\Omega\frac{|U(x,y)|^{q-2}U(x,y)\Phi(x,y)}{|x-y|^{N+sq}}dxdy$$

$$\langle\mathcal{H}_L'(\mathfrak{u}),\phi\rangle=\lambda\int_\Omega F(\mathfrak{u})\phi dx,$$

for all $\phi\in W_0^{s,p,q}(\Omega)$. In other hand $\mathcal{I}_L$ and $\mathcal{H}_L$ are respectively is weakly lower semi-continuous and sequentially weakly continuous in $W_0^{s,p,q}(\Omega)$.

Proof. see [18] and [20]. □

Definition 3.2. We say that J satisfies the (P.S) condition in $W_0^{s,p,q}(\Omega)$ if every sequence $\{\mathfrak{u}_n\}_n\subset W_0^{s,p,q}(\Omega)$ with $\{J(\mathfrak{u}_n)\}_n$ bounded and $J'(\mathfrak{u}_n)\longrightarrow 0$ admits a strongly convergent subsequence in $W_0^{s,p,q}(\Omega)$.

Lemma 3.3. *Suppose (i), (ii) and (iv) holds true. Then the functional J_L satisfies the (P.S) condition.*

Proof. Let $\{\mathfrak{u}_n\}_n\in W_0^{s,p,q}(\Omega)$ such that $\{J_L(\mathfrak{u}_n)\}_n$ is bounded and $J_L'(\mathfrak{u}_n)\longrightarrow 0$ as $n\longrightarrow\infty$.
We aim to prove that $\{\mathfrak{u}_n\}_n$ admits a strongly convergent subsequence in $W_0^{s,p,q}(\Omega)$.
Step 1: If $\{\mathfrak{u}_n\}_n$ is bounded in $W_0^{s,p,q}(\Omega)$.
By using the contradiction approach, we assume the step 1 is not hold, so there exists a subsequence denoted also by $\{\mathfrak{u}_n\}_{n\in\mathbb{N}}$ such that $||\mathfrak{u}_n||\longrightarrow\infty$. Now we write

$$J_L(\mathfrak{u}_n)=\frac{1}{p}\widehat{\mathcal{M}}_1(||\mathfrak{u}_n||^p)+\frac{1}{q}\widehat{\mathcal{M}}_2(||\mathfrak{u}_n||^q)-\frac{\lambda}{2r}\int_\Omega\int_\Omega\frac{|\mathfrak{u}_n|^r|\mathfrak{u}_n|^r}{|x-y|^\nu}dxdy,$$

$$\geq \frac{1}{q}\widehat{\mathcal{M}}_2(||\mathfrak{u}_n||^q) - \frac{\lambda}{2r}\int_\Omega\int_\Omega \frac{|\mathfrak{u}_n|^r|\mathfrak{u}_n|^r}{|x-y|^\nu}dxdy,$$

Using (i), we get

$$J_L(\mathfrak{u}_n) \geq \frac{\kappa^{q-1}}{q}||\mathfrak{u}_n||^q - \frac{\lambda}{2r}\int_\Omega\int_\Omega \frac{|\mathfrak{u}_n|^r|\mathfrak{u}_n|^r}{|x-y|^\nu}dxdy,$$

Applying Proposition 2.1 with $f = g = |\mathfrak{u}_n|^r$ and $s_1 = s_2 = \alpha > 1$, and assuming that $\alpha r < p^*$, we obtain

$$\int_\Omega\int_\Omega \frac{|\mathfrak{u}_n|^r|\mathfrak{u}_n|^r}{|x-y|^\nu}dxdy \leq C(N,\mu,\alpha)||\mathfrak{u}_n||^2_{L^\alpha} = C(N,\mu,\alpha)||\mathfrak{u}_n||^{2r}_{L^{\alpha r}},$$

By the the continuous embedding $W_0^{s,p,q}(\Omega) \hookrightarrow L^\gamma(\Omega)$ for $\gamma \in [1, p_s^*]$(see above), there exists a positive constant C_1 such that $||\mathfrak{u}_n||_{L^{\alpha r}} \leq C_1||\mathfrak{u}_n||$.

$$\int_\Omega\int_\Omega \frac{|\mathfrak{u}_n|^r|\mathfrak{u}_n|^r}{|x-y|^\nu}dxdy \leq C_1C(N,\mu,\alpha)||\mathfrak{u}_n||^{2r}, \tag{3.1}$$

Then

$$J_L(\mathfrak{u}_n) \geq \frac{\kappa^{q-1}}{q}||\mathfrak{u}_n||^q - \frac{\lambda}{2r}C_1C(N,\mu,\alpha)||\mathfrak{u}_n||^{2r}. \tag{3.2}$$

Since $\{J_L(\mathfrak{u}_n)\}_n$ is bounded and $2r < q$, we ensure that $n \longrightarrow \infty$ we have the desired contradiction, and hence step 1 follows. As a consequence, there exists a subsequence of $\{\mathfrak{u}_n\}_n$ denoted by the same notation $\{\mathfrak{u}_n\}_n$ such that $\mathfrak{u}_n \rightharpoonup u$ as $n \longrightarrow \infty$ in $W_0^{s,p,q}(\Omega)$ since $W_0^{s,p,q}(\Omega)$ is a reflexive space. Furthermore, we have

$$\mathfrak{u}_n \rightharpoonup \mathfrak{u} \text{ in } W_0^{s,p,q}(\Omega), \mathfrak{u}_n \to \mathfrak{u} \text{ a.e } \Omega, \text{ and } \mathfrak{u}_n \to \mathfrak{u} \text{ in } L^\sigma(\Omega), \forall\, 1 < \sigma < p_s^*. \tag{3.3}$$

Step 2: If $\mathfrak{u}_n \longrightarrow \mathfrak{u}$ as $n \longrightarrow \infty$ in $W_0^{s,p,q}(\Omega)$.
Denote by:

$$A^1_\mathfrak{u}(\mathrm{v}) = \int_{\Omega\times\Omega} \frac{|\mathrm{U}(x,y)|^{p-2}\mathrm{U}(x,y)\mathrm{V}(x,y)}{|x-y|^{N+sp}}dxdy$$

and

$$A^2_\mathfrak{u}(\mathrm{v}) = \int_{\Omega\times\Omega} \frac{|\mathrm{U}(x,y)|^{q-2}\mathrm{U}(x,y)\mathrm{V}(x,y)}{|x-y|^{N+sq}}dxdy$$

for all $(\mathfrak{u}, \mathrm{v}) \in (W_0^{s,p,q}(\Omega))^2$.
Clearly, according to Hölder inequality, $A^t_\mathfrak{u}$ $(t \in \{1,2\})$ is also continuous, given

$$\begin{cases} |A^1_\mathfrak{u}(\mathrm{v})| \leq ||\mathfrak{u}||^{p-1}||\mathrm{v}||, \\ |A^2_\mathfrak{u}(\mathrm{v})| \leq ||\mathfrak{u}||^{q-1}||\mathrm{v}||, \ \forall\, \mathrm{v} \in W_0^{s,p,q}(\Omega). \end{cases} \tag{3.4}$$

Moreover by (3.3) and the boundedness of $\mathcal{M}$ on closed interval give

$$\begin{cases} \left([\mathcal{M}(||\mathfrak{u}_n||^p)]^{p-1} - [\mathcal{M}(||\mathfrak{u}||^p)]^{p-1}\right) A^1_\mathfrak{u}(\mathfrak{u}_n - \mathfrak{u}) = o(1) \\ \left([\mathcal{M}(||\mathfrak{u}_n||^q)]^{q-1} - [\mathcal{M}(||\mathfrak{u}||^q)]^{q-1}\right) A^2_\mathfrak{u}(\mathfrak{u}_n - \mathfrak{u}) = o(1), \end{cases} \tag{3.5}$$

clearly, $\langle J'_L(\mathfrak{u}_n) - J'_L(\mathfrak{u}), \mathfrak{u}_n - \mathfrak{u}\rangle = o(1)$ as $n \longrightarrow \infty$, since $\mathfrak{u}_n \rightharpoonup \mathfrak{u}$ in $W_0^{s,p,q}(\Omega)$ and $J'_L(\mathfrak{u}_n) \longrightarrow 0$ in $W_0^{s,p,q,'}(\Omega)$.
Then we have as $n \to \infty$

$$
\begin{aligned}
o(1) &= [\mathcal{M}(||\mathfrak{u}_n||^p)]^{p-1}\, A^1_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u}) + [\mathcal{M}(||\mathfrak{u}_n||^q)]^{q-1}\, A^2_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u})\\
&+ \left([\mathcal{M}(||\mathfrak{u}_n||^p)]^{p-1} - [\mathcal{M}(||\mathfrak{u}||^p)]^{p-1}\right) A^1_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})\\
&+ ([\mathcal{M}(||\mathfrak{u}_n||^q)]^{q-1} - [\mathcal{M}(||\mathfrak{u}||^q)]^{q-1}) A^2_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})\\
&- [\mathcal{M}(||\mathfrak{u}_n||^p)]^{p-1}\, A^1_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u}) - [\mathcal{M}(||\mathfrak{u}_n||^q)]^{q-1}\, A^2_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})\\
&- \lambda\int_\Omega F(\mathfrak{u}_n)\mathfrak{u}_n dx - \lambda\int_\Omega F(\mathfrak{u})\mathfrak{u}_n dx + \lambda\int_\Omega F(\mathfrak{u}_n)\mathfrak{u} dx + \lambda\int_\Omega F(\mathfrak{u})\mathfrak{u}_n dx.
\end{aligned}
\tag{3.6}
$$

Now, from [[18], lemma 2.2] and [[8], lemma 2.3], we have

$$\int_\Omega F(\mathfrak{u}_n)\mathfrak{u}_n dx \longrightarrow \int_\Omega F(\mathfrak{u})\mathfrak{u} dx,$$

and

$$\int_\Omega F(\mathfrak{u}_n)\phi dx \longrightarrow \int_\Omega F(\mathfrak{u})\phi dx, \quad \forall \phi\in W_0^{s,p,q}(\Omega).$$

Then, for $\phi=\mathfrak{u}\in W_0^{s,p,q}(\Omega)$, we obtain in the last limit that

$$\int_\Omega F(\mathfrak{u}_n)\mathfrak{u} dx \longrightarrow \int_\Omega F(\mathfrak{u})\mathfrak{u} dx.$$

Now, by assumptions (ii), (iv) and (3.3) we have

$$\int_\Omega F(\mathfrak{u})\mathfrak{u}_n dx \longrightarrow \int_\Omega F(\mathfrak{u})\mathfrak{u} dx.$$

Hence, in (3.6), we give

$$
\begin{aligned}
o(1) &= \lim_{n\to\infty}([\mathcal{M}(||\mathfrak{u}_n||^p)]^{p-1}(A^1_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u}) - A^1_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})))\\
&+ \lim_{n\to\infty}([\mathcal{M}(||\mathfrak{u}_n||^q)]^{q-1}(A^2_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u}) - A^2_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u}))) + o(1),
\end{aligned}
$$

which gives otherwise

$$
\begin{aligned}
&\lim_{n\to\infty}([\mathcal{M}(||\mathfrak{u}_n||^p)]^{p-1}(A^1_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u}) - A^1_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})))\\
&+ \lim_{n\to\infty}([\mathcal{M}(||\mathfrak{u}_n||^q)]^{q-1}(A^2_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u}) - A^2_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u}))) = 0.
\end{aligned}
\tag{3.7}
$$

- Let $p\geq 2$, by Simon's inequality (2.3) and (3.7), we get

$$
\begin{aligned}
[\mathfrak{u}_n-\mathfrak{u}]^p_{s,p} &= \int_\Omega\int_\Omega \frac{|(\mathfrak{u}_n(x)-\mathfrak{u}_n(y)) - (\mathfrak{u}(x)-\mathfrak{u}(y))|^p}{|x-y|^{N+sp}} dxdy\\
&= \int_\Omega\int_\Omega \frac{|(U_n(x,y)) - U(x,y))|^p}{|x-y|^{N+sp}} dxdy
\end{aligned}
$$

$$\leq C_p\int_\Omega\int_\Omega \frac{[|(U_n(x,y))|^{p-2}U_n(x,y) - |(U(x,y))|^{p-2}U(x,y)](U_n(x,y)-U(x,y))}{|x-y|^{N+sp}} dxdy,$$

since

$$A^1_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u}) = \int_{\Omega\times\Omega} \frac{|U_n(x,y)|^{p-2}U_n(x,y)(U_n(x,y)) - U(x,y))}{|x-y|^{N+sp}} dxdy,$$

we have

$$[\mathfrak{u}_n-\mathfrak{u}]_{s,p}^p \leq C_p\left[A^1_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u})-A^1_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})\right].$$

The function $\phi(t)=|t|^{p-2}t$ is monotone increasing for $p>1$. Hence,

$$\left(|U_n|^{p-2}U_n-|U|^{p-2}U\right)(U_n-U)\geq 0.$$

Consequently,

$$A^1_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u})-A^1_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})\geq 0.$$

On the other hand, by the assumption (i), we have $\mathcal{M}(t)\geq\kappa>1$, and since $p>1$, we have

$$[\mathcal{M}(\|\mathfrak{u}_n\|^p)]^{p-1}\geq 1.$$

And, we can write

$$A^1_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u})-A^1_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})\leq[\mathcal{M}(\|\mathfrak{u}_n\|^p)]^{p-1}\left[A^1_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u})-A^1_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})\right],$$

then

$$[\mathfrak{u}_n-\mathfrak{u}]_{s,p}^p\leq C_p\,[\mathcal{M}(\|\mathfrak{u}_n\|^p)]^{p-1}\left[A^1_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u})-A^1_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})\right].$$

Now, by the same way for $q\geq 2$, we obtain

$$[\mathfrak{u}_n-\mathfrak{u}]_{s,q}^q\leq C_q\,[\mathcal{M}(\|\mathfrak{u}_n\|^q)]^{q-1}\left[A^2_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u})-A^2_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})\right]$$

and taking $C_{p,q}:=\max(C_p,C_q)>0$, we get

$$\begin{aligned}[\mathfrak{u}_n-\mathfrak{u}]_{s,p}^p+[\mathfrak{u}_n-\mathfrak{u}]_{s,q}^q\leq{}& C_{p,q}[(\mathcal{M}(||\mathfrak{u}_n||^p))^{p-1}(A^1_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u})-A^1_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u}))]\\ &+C_{p,q}[(\mathcal{M}(||\mathfrak{u}_n||^q))^{q-1}(A^2_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u})-A^2_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u}))]\end{aligned}$$

Thus

$$\begin{aligned}[\mathfrak{u}_n-\mathfrak{u}]_{s,p}\leq{}& C_{p,q}^{\frac{1}{p}}\Big((\mathcal{M}(||\mathfrak{u}_n||^p))^{p-1}(A^1_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u})-A^1_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u}))\\ &+(\mathcal{M}(||\mathfrak{u}_n||^q))^{q-1}(A^2_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u})-A^2_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u}))\Big)^{\frac{1}{p}}\\ ={}& o(1)\end{aligned}$$

and

$$\begin{aligned}[\mathfrak{u}_n-\mathfrak{u}]_{s,q}\leq{}& C_{p,q}^{\frac{1}{q}}\Big((\mathcal{M}(||\mathfrak{u}_n||^p))^{p-1}(A^1_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u})-A^1_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u}))\\ &+(\mathcal{M}(||\mathfrak{u}_n||^q))^{q-1}(A^2_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u})-A^2_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u}))\Big)^{\frac{1}{q}}\\ ={}& o(1)\end{aligned}$$

as $n \to \infty$.
Furthermore, we have

$$
\begin{aligned}
||\mathfrak{u}_n - \mathfrak{u}|| &= [\mathfrak{u}_n - \mathfrak{u}]_{s,p} + [\mathfrak{u}_n - \mathfrak{u}]_{s,q} \\
&\leq C_{p,q}^{\frac{1}{p}} \Big((\mathcal{M}(||\mathfrak{u}_n||^p))^{p-1} (A^1_{\mathfrak{u}_n}(\mathfrak{u}_n - \mathfrak{u}) - A^1_{\mathfrak{u}}(\mathfrak{u}_n - \mathfrak{u})) \\
&+ (\mathcal{M}(||\mathfrak{u}_n||^q))^{q-1} (A^2_{\mathfrak{u}_n}(\mathfrak{u}_n - \mathfrak{u}) - A^2_{\mathfrak{u}}(\mathfrak{u}_n - \mathfrak{u})) \Big)^{\frac{1}{p}} \\
&+ C_{p,q}^{\frac{1}{q}} \Big((\mathcal{M}(||\mathfrak{u}_n||^p))^{p-1} (A^1_{\mathfrak{u}_n}(\mathfrak{u}_n - \mathfrak{u}) - A^1_{\mathfrak{u}}(\mathfrak{u}_n - \mathfrak{u})) \\
&+ (\mathcal{M}(||\mathfrak{u}_n||^q))^{q-1} (A^2_{\mathfrak{u}_n}(\mathfrak{u}_n - \mathfrak{u}) - A^2_{\mathfrak{u}}(\mathfrak{u}_n - \mathfrak{u})) \Big)^{\frac{1}{q}} \\
&= o(1),
\end{aligned}
$$

which leads us to conclude that, $||\mathfrak{u}_n - \mathfrak{u}|| \to 0$.

- Now if $1 < p < q < 2$, we use (3.3) so there exist two positive constants β and β' such that for all $n \in \mathbb{N}$, we have

$$
[\mathfrak{u}_n]_{s,p} \leq \beta \text{ and } [\mathfrak{u}_n]_{s,q} \leq \beta'. \tag{3.8}
$$

Using the Simon's inequality, Hölder inquality and (3.7), we obtain

$$
[\mathfrak{u}_n - \mathfrak{u}]^p_{s,p} = \int_\Omega \int_\Omega \frac{|(U_n(x,y)) - U(x,y))|^p}{|x-y|^{N+sp}} dxdy
$$

$$
\leq C'_p \int_\Omega \int_\Omega \frac{\Big[\big(|U_n|^{p-2} U_n - |U|^{p-2} U \big) (U_n - U) \Big]^{p/2} \big(|U_n|^p + |U|^p \big)^{(2-p)/2}}{|x-y|^{N+sp}} dxdy.
$$

By Hölder inegality, we have

$$
\leq C'_p \Big(\int_\Omega \int_\Omega \frac{\big(|U_n|^{p-2} U_n - |U|^{p-2} U \big)(U_n - U)}{|x-y|^{N+sp}} dxdy \Big)^{\frac{p}{2}} \Big(\int_\Omega \int_\Omega \frac{\big(|U_n|^p + |U|^p \big)}{|x-y|^{N+sp}} dxdy \Big)^{\frac{2-p}{2}}
$$

$$
\leq C'_p \Big(\int_\Omega \int_\Omega \frac{\big(|U_n|^{p-2} U_n - |U|^{p-2} U \big)(U_n - U)}{|x-y|^{N+sp}} dxdy \Big)^{\frac{p}{2}} \Big([\mathfrak{u}_n]^p_{s,p} + [\mathfrak{u}]^p_{s,p} \Big)^{\frac{2-p}{2}},
$$

and by (3.8), ther exists a constant C''_p(depends only on p, β, u and C'_p) such that

$$
[\mathfrak{u}_n - \mathfrak{u}]^p_{s,p} \leq C''_p \Big(\int_\Omega \int_\Omega \frac{\big(|U_n|^{p-2} U_n - |U|^{p-2} U \big)(U_n - U)}{|x-y|^{N+sp}} dxdy \Big)^{\frac{p}{2}},
$$

and then

$$
[\mathfrak{u}_n - \mathfrak{u}]^p_{s,p} \leq C''_p [A^1_{\mathfrak{u}_n}(\mathfrak{u}_n - \mathfrak{u}) - A^1_{\mathfrak{u}}(\mathfrak{u}_n - \mathfrak{u})]^{\frac{p}{2}}
$$

Similarly, there exists a constant C''_q (depends only on q, β', u and C'_q) such that

$$
[\mathfrak{u}_n - \mathfrak{u}]^q_{s,q} \leq C''_q [A^2_{\mathfrak{u}_n}(\mathfrak{u}_n - \mathfrak{u}) - A^2_{\mathfrak{u}}(\mathfrak{u}_n - \mathfrak{u})]^{\frac{q}{2}}.
$$

Taking $C''_{p,q} = \max(C''_p, C''_q)$, we get

$$\begin{aligned}&[\mathfrak{u}_n - \mathfrak{u}]^p_{s,p} + [\mathfrak{u}_n - \mathfrak{u}]^q_{s,q}\\ &\leq C''_{p,q}\left(\left[A^1_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u}) - A^1_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})\right]^{\frac{p}{2}} + \left[A^2_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u}) - A^2_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})\right]^{\frac{q}{2}}\right).\end{aligned}$$

On the other hand, by applying the inequality $(x^{\frac{s}{2}}+y^{\frac{s}{2}}) \leq 2^{(1-\frac{s}{2})}(x+y)^{\frac{s}{2}}$ for all $(x,y) \in \mathbb{R}^2_+$ and for $1 < s < 2$, we discuss the following cases:

Case 1: If $\left(A^1_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u}) - A^1_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})\right) \leq 1$ and $\left(A^2_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u}) - A^2_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})\right) \leq 1$ for all $n \in \mathbb{N}$, we obtain

$$\begin{aligned}&\left[A^1_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u}) - A^1_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})\right]^{\frac{p}{2}} + \left[A^2_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u}) - A^2_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})\right]^{\frac{q}{2}}\\ &\leq \left[A^1_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u}) - A^1_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})\right]^{\frac{p}{2}} + \left[A^2_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u}) - A^2_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})\right]^{\frac{p}{2}}\\ &\leq 2^{\left(1-\frac{p}{2}\right)}\left(\left[A^1_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u}) - A^1_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})\right] + \left[A^2_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u}) - A^2_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})\right]\right)^{\frac{p}{2}}.\end{aligned}$$

Case 2: If $((A^1_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u}) - A^1_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})) \leq 1$ and $(A^2_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u}) - A^2_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})) \geq 1)$ for all $n \in \mathbb{N}$:
We use the following inequality: $a^p + b^q \leq 2(a^q + b^q)$ for $a \leq 1$, $b \geq 1$ and $p < q$ to have

$$\begin{aligned}&\left[A^1_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u}) - A^1_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})\right]^{\frac{p}{2}} + \left[A^2_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u}) - A^2_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})\right]^{\frac{q}{2}}\\ &\leq 2\left(\left[A^1_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u}) - A^1_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})\right]^{\frac{q}{2}} + \left[A^2_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u}) - A^2_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})\right]^{\frac{q}{2}}\right).\\ &\leq 2^{\left(1-\frac{q}{2}\right)}\left(\left[A^1_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u}) - A^1_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})\right] + \left[A^2_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u}) - A^2_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})\right]\right)^{\frac{q}{2}}.\end{aligned}$$

Case 3: If $((A^1_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u}) - A^1_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})) \leq 1$ and $(A^2_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u}) - A^2_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})) \geq 1)$ for all $n \in \mathbb{N}$:
We use also the inequality: $a^p + b^q \leq 2(a^p + b^p)$ for $a \geq 1$, $b \leq 1$ and $p < q$ to get

$$\begin{aligned}&\left[A^1_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u}) - A^1_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})\right]^{\frac{p}{2}} + \left[A^2_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u}) - A^2_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})\right]^{\frac{q}{2}}\\ &\leq 2\left(\left[A^1_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u}) - A^1_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})\right]^{\frac{p}{2}} + \left[A^2_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u}) - A^2_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})\right]^{\frac{p}{2}}\right).\\ &\leq 2^{\left(1-\frac{p}{2}\right)}\left(\left[A^1_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u}) - A^1_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})\right] + \left[A^2_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u}) - A^2_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})\right]\right)^{\frac{p}{2}}.\end{aligned}$$

Case 4: If $((A^1_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u}) - A^1_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})) \geq 1$ and $(A^2_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u}) - A^2_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})) \geq 1)$ for all $n \in \mathbb{N}$, we have also

$$\begin{aligned}&\left[A^1_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u}) - A^1_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})\right]^{\frac{p}{2}} + \left[A^2_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u}) - A^2_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})\right]^{\frac{q}{2}}\\ &\leq 2\left(\left[A^1_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u}) - A^1_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})\right]^{\frac{q}{2}} + \left[A^2_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u}) - A^2_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})\right]^{\frac{q}{2}}\right).\end{aligned}$$

Next, by assumption (i) and the same technic above, we get

$$\begin{cases}[\mathcal{M}(||\mathfrak{u}_n||^p)]^{p-1}\left(A^1_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u}) - A^1_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})\right) \geq \left(A^1_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u}) - A^1_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})\right) \geq 0\\ [\mathcal{M}(||\mathfrak{u}_n||^q)]^{p-1}\left(A^2_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u}) - A^2_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})\right) \geq \left(A^2_{\mathfrak{u}_n}(\mathfrak{u}_n-\mathfrak{u}) - A^2_{\mathfrak{u}}(\mathfrak{u}_n-\mathfrak{u})\right) \geq 0.\end{cases} \tag{3.9}$$

Then, using (3.7) and (3.9), we deduce that $||\mathfrak{u}_n - \mathfrak{u}|| \to 0$ when $n \to \infty$. $\square$

Lemma 3.4. *Assume (i) and (ii) holds true. Then, J_L is coercive.*

Proof. Taking $||u||$ large enough and using (3.2), we have

$$J_L(\mathfrak{u}) \geq \frac{\kappa^{q-1}}{q}||\mathfrak{u}||^q - \frac{\lambda}{2r}C_1 C(N,\mu,\alpha)||\mathfrak{u}||^{2r}.$$

Since $2r < q$, we have $||\mathfrak{u}||^q \geq ||\mathfrak{u}||^{2r}$, we obtain that

$$J_L(\mathfrak{u}) \geq [\frac{2\kappa^{q-1}}{q} - \frac{\lambda}{2r}C_1 C(N,\mu,\alpha)]||\mathfrak{u}||^{2r}$$

Taking $\lambda_1 = \dfrac{4r\kappa^{q-1}}{qC_1 C(N,\mu,\alpha)}$, then for any $\lambda \in (0,\lambda_1)$, J_L is coercive. Hence the result. $\square$

4. Proof of the main theorem 1.2

In this section, we recall some key concepts and results on critical groups. Let $\psi \in C^1(\mathcal{W},\mathbb{R})$, where satisfies the P.S condition, with $\mathcal{W}$ is a real Banach space . Let $\psi^c = \{\mathfrak{u} \in \mathcal{W} : \psi(\mathfrak{u}) \leq c\}$ and $\mathcal{K}_\psi = \{\mathfrak{u} \in \mathcal{W} : \psi^{'}(\mathfrak{u}) = 0\}$, where $c \in \mathbb{R}$. Then the critical groups of ψ at $\mathfrak{u}$ are defined by $C_n(\psi,\mathfrak{u}) = H_n(\psi^c \cap V, \psi^c \cap V \setminus \{\mathfrak{u}\})$ where $n \in \mathbb{N}$ and V is a neighbourhood of u such that $\mathcal{K}_\psi \cap V = \{\mathfrak{u}\}$, H_n is the singular relative homology with coefficient in an Abelian group G (see [17] for more details).

Theorem 4.1. *([12]) Let E a real Banach space and $\psi \in C^1(E,\mathbb{R})$ satisfy (PS) and be bounded below. If ψ has a homologically nontrivial critical point that is not a minimizer, then ψ has at least three critical points.*

Lemma 4.2. *([12]) Let ψ has a critical point $\mathfrak{u} = 0$ with $\psi(0) = 0$. Assume that ψ has a local linking at 0 with restpect to $\mathcal{W} = \mathcal{U} \bigoplus Z$, $n =$ dim $\mathcal{U}$ that is to say, there exists $\tau > 0$ such that*

$$\psi(\mathfrak{u}) = \begin{cases} \psi(\mathfrak{u}) \leq 0 & \textit{for } \mathfrak{u} \in \mathcal{U}, \|\mathfrak{u}\| \leq \tau, \\ \psi(\mathfrak{u}) > 0 & \textit{for } \mathfrak{u} \in Z, 0 < \|\mathfrak{u}\| \leq \tau. \end{cases}$$

Then, $C_n(\psi,0) \neq \Theta$ with Θ is the trivial homological group. so, 0 is a homologically nontrivial critical point of ψ.

Theorem 4.3. *Assume that (i) – (iii) and (v) holds true, then $C_1(J_L,0) \neq \Theta$.*

Proof. To apply Lemma 4.2, it suffices to verify that the functional J_L has a local linking at 0 with respect to $\mathcal{W} = \mathcal{U} \bigoplus Z$,where $\mathcal{U} =$ span$\{\phi\}$ with $\phi \in W_0^{s,p,q}(\Omega) \cap L^\infty(\Omega)$ such that $\|\phi\| = 1$ and Z completing $\mathcal{U}$ in $W_0^{s,p,q}(\Omega)$.

i) Since $\mathcal{U}$ has a one-dimensional, we have for $\sigma > 0$ there exists $\tau \in (0, 1]$ such that for $\mathfrak{u} \in \mathcal{U}$, $||\mathfrak{u}|| \leq \tau$ implies that $|\mathfrak{u}(x)| \leq \sigma$ for almost every $x \in \Omega$. For $\mathfrak{u} \in \mathcal{U}$ with $||\mathfrak{u}|| \leq \tau$, we have by (iii) and (v),

$$J_L(\mathfrak{u}) \leq \frac{1}{p}\left[\max_{0\leq t\leq\tau} \mathcal{M}(t)\right]^{p-1} ||\mathfrak{u}||^p + \frac{1}{q}\left[\max_{0\leq t\leq\tau} M(t)\right]^{q-1} ||\mathfrak{u}||^q - \frac{\lambda}{2r}\int_\Omega\int_\Omega \frac{|\mathfrak{u}|^r|\mathfrak{u}|^r}{|x-y|^\nu}dxdy$$

$$\leq \int_\Omega\int_\Omega \left[\frac{1}{p}(\varpi^{p-1} + \varpi^{q-1}) - \frac{\lambda}{2r}\frac{|\mathfrak{u}|^r|\mathfrak{u}|^r}{|x-y|^\nu}\right] dxdy \text{ (since } |\Omega| = 1)$$

$$\leq \int_\Omega\int_\Omega \left[\frac{1}{p}(\varpi^{p-1} + \varpi^{q-1}) - \frac{\theta}{2r}\frac{|\mathfrak{u}|^r|\mathfrak{u}|^r}{|x-y|^\nu}\right] dxdy \leq 0$$

ii) Now, for all $\mathfrak{u} \in Z$ with $0 < ||\mathfrak{u}|| \leq \tau$, we have from (3.1),

$$J_L(\mathfrak{u}) \geq \frac{\kappa^{p-1}}{p}||\mathfrak{u}||^p - \frac{\lambda}{2r}C_1C(N, \mu, \alpha)||\mathfrak{u}||^{2r}.$$

Since $p < 2r$ and $||\mathfrak{u}|| \leq \tau \leq 1$, then $||\mathfrak{u}||^{2r} \leq ||\mathfrak{u}||^p$. And then

$$J_L(\mathfrak{u}) \geq [\frac{\kappa^{p-1}}{p} - \frac{\lambda}{2r}C_1C(N, \mu, \alpha)]||\mathfrak{u}||^{2r},$$

thus, in order to ensure that $J_L(\mathfrak{u}) > 0$, it suffices to choose $\lambda \in (0, \lambda_2)$ such that $\lambda_2 = \dfrac{2r\kappa^{p-1}}{pC_1C(N, \mu, \alpha)}$.

□

Proof of theorem 1.2

We know that J_L satisfies the (PS) condition by the lemma 3.3 and also the functional J_L is weak lower semi-continuous by the lemma 3.1.
Taking $\overline{\lambda} = \min(\lambda_1, \lambda_2)$, then J_L is bounded from below by using the lemma 3.4. In other hand, we use the theorem 4.3 it follows that 0 is not a minimizer of J_L and is homologically nontrivial. Hence due to the theorem 4.1, we obtain the desired conclusion.

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