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SHARP WEIGHT RANGE FOR CRITICAL HARDY–RELLICH INEQUALITIES

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Abstract. We revisit a recently established family of critical Hardy–Rellich inequalities. Combining the Emden–Fowler transformation with the classical one-dimensional weighted Hardy inequality, we show that the weighted Hardy–Rellich inequality holds precisely when the weight exponent differs from the dimension, so that this single value is the unique critical one. We obtain explicit constants in every dimension greater than one, determine the sharp constant in the one-dimensional case, and extend the formulation involving the Laplacian to the full Muckenhoupt range. These results sharpen and unify the existing theory and answer in part an open problem raised in the recent literature.

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1. Introduction and main results

For $N \geq 1$ and $1 \leq p < N$, the classical Hardy inequality states that

$$\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p}\, dx \leq \left(\frac{p}{N-p}\right)^p \int_{\mathbb{R}^N} |\nabla u|^p \, dx$$

for all $u \in C_c^\infty(\mathbb{R}^N)$; under the additional restriction $u \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$ the inequality extends to all $p \neq N$, with the constant $\left(\frac{p}{|N-p|}\right)^p$. The endpoint case $p = N$ is well known to fail without a logarithmic correction.

The second-order analogue is the Rellich inequality

$$\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{2p}}\, dx \leq \left[\frac{p^2}{N(N-2p)(p-1)}\right]^p \int_{\mathbb{R}^N} |\Delta u|^p \, dx,$$

which holds for $N \geq 3$ and $1 < p < N/2$; see [13] for the original statement and [6] for the explicit constant, which is optimal, as the family $|x|^{-(N-2p)/p}$ shows. Analogues in L^p on general open subsets of $\mathbb{R}^N$, together with applications to eigenvalue bounds for biharmonic operators, are developed in [8]; refinements that separate the radial from the spherical contribution, and that do so at the

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level of identities rather than inequalities, are obtained in [11, 1], while the best constants in the associated Hardy–Rellich inequalities are determined in [15]. The admissible range of exponents is sharp at both ends: the constant degenerates as $p \to 1^+$ and again as $p \to (N/2)^-$, and at the endpoint $p = N/2$ no inequality of this form can hold. For $p = 2$ this is precisely the classical failure in dimension $N = 4$, already recorded by Rellich [14, §7], and the obstruction is the same one that defeats the Hardy inequality at $p = N$.

Hardy–Rellich type inequalities, which place a first-order quantity on the left and a second-order quantity on the right, were studied in [5, 15, 3] for $1 < p < N$. Again the case $p = N$ is excluded, and for the same structural reason as above: the failure of the first-order Hardy inequality at $p = N$ propagates to the second-order estimate. Weighted inequalities of this kind are also a basic ingredient in the variational analysis of singular elliptic equations, where weights that are powers of $|x|$, or of the distance to a submanifold, arise naturally; see [7] for a recent instance.

In [2] the author proves a striking critical-exponent Hardy–Rellich inequality which covers the previously inaccessible case $p = N$, at the cost of grouping the gradient and the lower-order term into a single quantity $\nabla(u/|x|)$. We recall the results of [2] that are relevant here. Theorem 1.1 below is [2, Theorem 1.1], Theorem 1.2 is [2, Theorem 1.2], and Corollary 1.3 is [2, Corollary 1.1].

Theorem 1.1. *For every $N \geq 1$ there exists $C_N > 0$ such that*

$$\int_{\mathbb{R}^N} \left| \nabla \left(\frac{u(x)}{|x|} \right) \right|^N dx \leq C_N \int_{\mathbb{R}^N} |\Delta u|^N \, dx, \qquad u \in C_c^\infty(\mathbb{R}^N \setminus \{0\}). \tag{1.1}$$

Theorem 1.2. *Let $N \geq 1$ and $a < 1$. There exists $C_{N,a} > 0$ such that*

$$\int_{\mathbb{R}^N} \left| \nabla \left(\frac{u(x)}{|x|} \right) \right|^N |x|^a \, dx \leq C_{N,a} \int_{\mathbb{R}^N} \left| D^2 u \right|^N |x|^a \, dx, \qquad u \in C_c^\infty(\mathbb{R}^N \setminus \{0\}). \tag{1.2}$$

Corollary 1.3. *For $-N < a < 1$ there exists $C_{N,a} > 0$ such that*

$$\int_{\mathbb{R}^N} \left| \nabla \left(\frac{u(x)}{|x|} \right) \right|^N |x|^a \, dx \leq C_{N,a} \int_{\mathbb{R}^N} |\Delta u|^N |x|^a \, dx, \qquad u \in C_c^\infty(\mathbb{R}^N \setminus \{0\}). \tag{1.3}$$

One of the questions left open in [2], namely Open Problem 1, concerns the determination of the optimal constants in (1.1) and (1.3).

The constants obtained in [2] are not explicit and, as we shall see, the range $a < 1$ in (1.2) is far from optimal. Both limitations have a common origin: an auxiliary boundedness result for a one-dimensional integral operator whose proof rests on Jensen's inequality. Jensen is sharp only for $p = 1$, and for $p > 1$ it is strictly weaker than the classical weighted Hardy inequality. The latter is available in every weighted Lebesgue exponent, produces an explicit constant, which is sharp for the one-dimensional problem, and accommodates a far larger family of weights.

Our contribution is to recast both the radial and the angular parts of the argument as instances of a single one-dimensional weighted Hardy inequality in the Emden–Fowler variable $t = \log r$. This removes the artificial restriction $a < 1$

and yields explicit constants. The improvement is summarised in the following statement.

Theorem 1.4. *Let $N \geq 2$ and $a \neq N$. Then for every $u \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$,*

$$\int_{\mathbb{R}^N} \left| \nabla \left(\tfrac{u(x)}{|x|} \right) \right|^N |x|^a \, dx \leq C_{N,a} \int_{\mathbb{R}^N} \left| D^2 u \right|^N |x|^a \, dx, \tag{1.4}$$

and for $a \neq 0$ one may take the explicit constant

$$C_{N,a} = 2^{N/2-1} \left[\left(\frac{N}{|a|} \right)^N + \left(\frac{N}{|N-a|} \right)^N \right]. \tag{1.5}$$

The remaining value $a = 0$ is covered by Theorem 1.1. For $N = 1$, the inequality holds for every $a \neq 1$ with the sharp constant $\big(1/|1-a|\big)$.

Theorem 1.5. *Let $N \geq 2$. The inequality (1.4) holds for every $a \neq N$ and fails for $a = N$. More precisely, at $a = N$ there is no constant $C > 0$ for which (1.4) holds for all $u \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$; the failure already occurs on the subspace of radial functions. Thus $a = N$ is the unique critical value of the weight exponent.*

Corollary 1.6. *Let $N \geq 2$ and $-N < a < N(N-1)$ with $a \neq N$. Then there exists a constant $C = C(N, a) > 0$ such that*

$$\int_{\mathbb{R}^N} \left| \nabla \left(\tfrac{u(x)}{|x|} \right) \right|^N |x|^a \, dx \leq C \int_{\mathbb{R}^N} |\Delta u|^N |x|^a \, dx, \qquad u \in C_c^\infty(\mathbb{R}^N \setminus \{0\}). \tag{1.6}$$

For $N = 2$ the admissible range reduces to $-2 < a < 2$.

A few remarks place these statements in context.

Remark 1.7. Theorem 1.4 strengthens (1.2) in two independent ways: the range $a < 1$ is replaced by the optimal range $a \neq N$, so that the inequality now holds for all $a > N$ as well and not merely below a threshold, and the constant is made explicit. Both gains come from the same source, namely the passage from a Jensen-type estimate to the sharp weighted Hardy inequality. The constant (1.5) is not claimed to be optimal: the factor $2^{N/2-1}$ arises from a convexity step that decouples the radial and angular contributions, and these are not in general extremised simultaneously. Determining the best constants in (1.1) and (1.3) remains Open Problem 1 of [2].

Remark 1.8. The explicit constant (1.5) degenerates as $a \to 0$, because the angular contribution is controlled through a weighted Hardy inequality whose constant blows up at the borderline weight exponent. This is a feature of the convexity-split method rather than of the inequality itself: at $a = 0$ the weight is trivial, the two sides of (1.4) transform identically under the dilation $u(x) \mapsto u(\lambda x)$, each scaling by λ^N, so that the quotient is scale invariant, and the inequality holds with a finite constant by Theorem 1.1. Consequently (1.4) is valid on the whole range $a \neq N$, with an explicit constant off the single point $a = 0$.

Remark 1.9. For $N = 1$ there is no angular component and the inequality reduces to the radial estimate. The relevant one-dimensional weighted Hardy inequality (Lemma 2.1 below) is valid at the exponent $p = N = 1$ as well, so no restriction beyond $a \neq 1$ is needed, and the resulting constant $1/|1-a|$ is sharp. This already extends [2, Theorem 1.2], stated there for $a < 1$, to all $a \neq 1$.

The paper is organized as follows. In Section 2, we recall the one-dimensional weighted Hardy inequality that underlies the main argument. In Section 3, we prove Theorem 1.4 and Corollary 1.6. Section 4 is devoted to the proof of Theorem 1.5. Finally, in Section 5, we discuss the constants appearing in our results and briefly address some remaining open problems.

2. A one-dimensional weighted Hardy inequality

The single analytic tool we use is the following classical inequality, stated in the power-weight form convenient for spherical coordinates.

Lemma 2.1. *Let $N \geq 1$ and let $\beta \in \mathbb{R}$ with $\beta \neq -1$. Then for every $\varphi \in C_c^\infty(0,\infty)$,*

$$\int_0^\infty r^\beta |\varphi(r)|^N \, dr \leq \left(\frac{N}{|\beta+1|}\right)^N \int_0^\infty r^{\beta+N} |\varphi'(r)|^N \, dr. \tag{2.1}$$

The constant $\big(N/|\beta+1|\big)^N$ is sharp.

Proof. Write $I = \int_0^\infty r^\beta |\varphi|^N \, dr$. Since $\beta \neq -1$, an integration by parts gives

$$I = -\frac{N}{\beta+1}\int_0^\infty r^{\beta+1} \, |\varphi|^{N-2} \varphi \, \varphi' \, dr \leq \frac{N}{|\beta+1|}\int_0^\infty r^{\beta+1} |\varphi|^{N-1} |\varphi'| \, dr.$$

Splitting the weight as $r^{\beta+1} = r^{\beta(N-1)/N} \, r^{(\beta+N)/N}$ and applying Hölder's inequality with exponents $N/(N-1)$ and N,

$$\int_0^\infty r^{\beta+1} |\varphi|^{N-1} |\varphi'| \, dr \leq \left(\int_0^\infty r^\beta |\varphi|^N \, dr\right)^{\frac{N-1}{N}} \left(\int_0^\infty r^{\beta+N} |\varphi'|^N \, dr\right)^{\frac{1}{N}} = I^{\frac{N-1}{N}} \, J^{\frac{1}{N}},$$

where $J = \int_0^\infty r^{\beta+N} |\varphi'|^N \, dr$. Hence $I \leq \frac{N}{|\beta+1|} I^{(N-1)/N} J^{1/N}$, and dividing by the finite quantity $I^{(N-1)/N}$ yields (2.1). The case $N = 1$ is the first display, the Hölder step being absent. Sharpness is classical; see [12, Chapter 1] or [10, §3]. □

Remark 2.2. In the Emden–Fowler variable $r = e^t$, writing $f(t) = \varphi(e^t)$ and $\mu = \beta + 1$, inequality (2.1) reads

$$\int_{\mathbb{R}} |f(t)|^N e^{\mu t} \, dt \leq \left(\frac{N}{|\mu|}\right)^N \int_{\mathbb{R}} |f'(t)|^N e^{\mu t} \, dt, \qquad \mu \neq 0,$$

which is the form in which it will be applied. It is precisely the vanishing of the exponential weight at $\mu = 0$, equivalently $\beta = -1$, that obstructs the endpoint, and this obstruction will reappear as the critical exponent $a = N$ for the radial part and as the degeneracy at $a = 0$ for the angular part.

We emphasise that Lemma 2.1 replaces the boundedness estimate for the operator $T_{a,b} f(x) = x^{-a} \displaystyle\int_0^x s^b f(s) \, ds$ used in [2]. The latter, being established through

Jensen's inequality, forced the restriction $a < 1$ and gave no explicit constant; the weighted Hardy inequality does neither.

3. Proof of the main inequalities

Throughout, $\nabla_{\mathbb{S}^{N-1}}$ denotes the spherical gradient on $\mathbb{S}^{N-1}$. For $x = r\omega$ with $r = |x| > 0$ and $\omega \in \mathbb{S}^{N-1}$, and for $u \in C^\infty(\mathbb{R}^N \setminus \{0\})$, the gradient decomposes as

$$\nabla u(r\omega) = \partial_r u(r\omega)\,\omega + \frac{1}{r}\nabla_{\mathbb{S}^{N-1}} u(r\omega), \tag{3.1}$$

the two summands being orthogonal since $\nabla_{\mathbb{S}^{N-1}} u(r\omega) \perp \omega$.

3.1. Two pointwise Hessian bounds.

We record the elementary pointwise estimates on which the proof rests.

Lemma 3.1. *For every $u \in C^\infty(\mathbb{R}^N \setminus \{0\})$, $r > 0$ and $\omega \in \mathbb{S}^{N-1}$,*

$$|\partial_{rr} u(r\omega)| \le \left|D^2 u(r\omega)\right|, \qquad \left|\partial_r\Big(\tfrac{1}{r}\nabla_{\mathbb{S}^{N-1}} u(r\omega)\Big)\right| \le \left|D^2 u(r\omega)\right|, \tag{3.2}$$

where $|D^2 u|$ denotes the Hilbert–Schmidt norm of the Hessian.

Proof. Differentiating $r \mapsto \nabla u(r\omega)$ along the fixed direction ω gives, by the chain rule, $\partial_r\big(\nabla u(r\omega)\big) = D^2 u(r\omega)\,\omega$. On the other hand, differentiating the decomposition (3.1) in r, with ω held fixed so that $\partial_r \omega = 0$, yields

$$\partial_r\big(\nabla u(r\omega)\big) = \partial_{rr} u(r\omega)\,\omega + \partial_r\Big(\tfrac{1}{r}\nabla_{\mathbb{S}^{N-1}} u(r\omega)\Big).$$

The first term is parallel to ω, while the second is tangential, because $\frac{1}{r}\nabla_{\mathbb{S}^{N-1}} u(r\omega) \perp \omega$ for every r, and differentiating a vector field that is everywhere orthogonal to the fixed vector ω produces a field still orthogonal to ω. These two components are therefore orthogonal, and Pythagoras' theorem gives

$$|\partial_{rr} u(r\omega)|^2 + \left|\partial_r\big(\tfrac{1}{r}\nabla_{\mathbb{S}^{N-1}} u(r\omega)\big)\right|^2 = \left|D^2 u(r\omega)\,\omega\right|^2 \le \left|D^2 u(r\omega)\right|^2,$$

the last inequality because $|D^2 u\,\omega| \le |D^2 u|$ for the unit vector ω. Both estimates in (3.2) follow. □

Remark 3.2. It is the *combination* $\partial_r\big(r^{-1}\nabla_{\mathbb{S}^{N-1}} u\big)$, and not $r^{-1}\partial_r \nabla_{\mathbb{S}^{N-1}} u$ on its own, that is dominated by $|D^2 u|$. Indeed $r^{-1}\partial_r \nabla_{\mathbb{S}^{N-1}} u = \partial_r\big(r^{-1}\nabla_{\mathbb{S}^{N-1}} u\big) + r^{-2}\nabla_{\mathbb{S}^{N-1}} u$, and the second summand carries the lower-order angular term, which is not controlled by the Hessian alone. This is why, in the angular estimate below, we work directly with the tangential field $P := r^{-1}\nabla_{\mathbb{S}^{N-1}} u$, whose radial derivative is exactly the quantity appearing in (3.2). The price is that the angular Hardy inequality is naturally weighted by r^{a-1} rather than r^{a-N-1}, which is the origin of the condition $a \neq 0$ in (1.5).

3.2. **The pointwise decomposition.** A direct computation from (3.1), using $\nabla(u/|x|) = \nabla u/r - u\,x/r^3$ and the orthogonality of the radial and tangential parts, gives

$$\left|\nabla\left(\tfrac{u(x)}{|x|}\right)\right|^2 = \frac{1}{r^4}\,|\nabla_{\mathbb{S}^{N-1}}u(r\omega)|^2 + \frac{1}{r^4}\big(r\partial_r u(r\omega) - u(r\omega)\big)^2. \tag{3.3}$$

From the elementary inequality $(A+B)^{N/2} \le 2^{N/2-1}(A^{N/2}+B^{N/2})$, valid for $N \ge 2$ and $A, B \ge 0$, we obtain

$$\left|\nabla\left(\tfrac{u}{|x|}\right)\right|^N \le 2^{N/2-1}\left[\frac{1}{r^{2N}}\,|\nabla_{\mathbb{S}^{N-1}}u|^N + \frac{1}{r^{2N}}\,|r\partial_r u - u|^N\right]. \tag{3.4}$$

We estimate the two contributions separately.

3.3. **The radial term.** Fix $\omega \in \mathbb{S}^{N-1}$ and set $g(r) := r\partial_r u(r\omega) - u(r\omega)$. Then $g \in C_c^\infty(0,\infty)$ and $g'(r) = r\,\partial_{rr}u(r\omega)$. Multiplying the radial part of (3.4) by $|x|^a = r^a$ and integrating in spherical coordinates,

$$\int_{\mathbb{R}^N} \frac{|r\partial_r u - u|^N}{r^{2N}}\,|x|^a\,dx = \int_{\mathbb{S}^{N-1}}\int_0^\infty r^{\,a-N-1}\,|g(r)|^N\,dr\,d\omega.$$

Applying Lemma 2.1 with $\beta = a - N - 1$, which is admissible precisely when $a \ne N$ and produces the constant $\big(N/|a-N|\big)^N$,

$$\begin{aligned}\int_0^\infty r^{\,a-N-1}\,|g|^N\,dr &\le \left(\frac{N}{|N-a|}\right)^N\int_0^\infty r^{\,a-1}\,|g'|^N\,dr\\ &= \left(\frac{N}{|N-a|}\right)^N\int_0^\infty r^{\,a+N-1}\,|\partial_{rr}u(r\omega)|^N\,dr,\end{aligned}$$

where we used $g' = r\,\partial_{rr}u$. Invoking the first bound in (3.2) and integrating over $\omega \in \mathbb{S}^{N-1}$,

$$\int_{\mathbb{R}^N} \frac{|r\partial_r u - u|^N}{r^{2N}}\,|x|^a\,dx \le \left(\frac{N}{|N-a|}\right)^N\int_{\mathbb{R}^N}\left|D^2u\right|^N|x|^a\,dx, \qquad a \ne N. \tag{3.5}$$

3.4. **The angular term.** Fix $\omega \in \mathbb{S}^{N-1}$ and set $P(r) := r^{-1}\nabla_{\mathbb{S}^{N-1}}u(r\omega)$, a smooth, compactly supported, $\mathbb{S}^{N-1}$-tangential vector field along $(0,\infty)$. By the second bound in (3.2), $|\partial_r P(r)| \le |D^2u(r\omega)|$. Changing to spherical coordinates,

$$\int_{\mathbb{R}^N} \frac{|\nabla_{\mathbb{S}^{N-1}}u|^N}{r^{2N}}\,|x|^a\,dx = \int_{\mathbb{S}^{N-1}}\int_0^\infty r^{\,a-1}\,|P(r)|^N\,dr\,d\omega,$$

since $r^{-2N}\,|\nabla_{\mathbb{S}^{N-1}}u|^N = r^{-N}\,|P|^N$ and the angular measure contributes the factor r^{N-1+a}.

To estimate the inner integral we apply Lemma 2.1 to $|P|$. As $|P|$ need not be smooth at its zeros, we regularise: for $\varepsilon > 0$ put

$$f_\varepsilon(r) := \big(|P(r)|^2 + \varepsilon^2\big)^{1/2} - \varepsilon.$$

Then $f_\varepsilon \in C_c^\infty(0,\infty)$, $f_\varepsilon \nearrow |P|$ pointwise as $\varepsilon \to 0^+$, and the chain rule together with the Cauchy–Schwarz inequality gives the pointwise bound

$$|f_\varepsilon'(r)| = \frac{\left|\langle P, \partial_r P\rangle\right|}{\left(|P|^2+\varepsilon^2\right)^{1/2}} \le |\partial_r P(r)| \le \left|D^2u(r\omega)\right|.$$

Applying Lemma 2.1 with $\beta = a-1$, admissible precisely when $a \neq 0$ and producing the constant $\left(N/|a|\right)^N$,

$$\int_0^\infty r^{\,a-1} |f_\varepsilon|^N \, dr \le \left(\frac{N}{|a|}\right)^N \int_0^\infty r^{\,a-1+N} |f_\varepsilon'|^N \, dr \le \left(\frac{N}{|a|}\right)^N \int_0^\infty r^{\,a+N-1} \left|D^2u(r\omega)\right|^N dr.$$

Letting $\varepsilon \to 0^+$ by monotone convergence on the left and integrating over $\omega \in \mathbb{S}^{N-1}$,

$$\int_{\mathbb{R}^N} \frac{|\nabla_{\mathbb{S}^{N-1}} u|^N}{r^{2N}} |x|^a \, dx \le \left(\frac{N}{|a|}\right)^N \int_{\mathbb{R}^N} \left|D^2u\right|^N |x|^a \, dx, \qquad a \neq 0. \tag{3.6}$$

3.5. **Conclusion of Theorem 1.4.** For $a \notin \{0, N\}$, combining (3.4) with the radial bound (3.5) and the angular bound (3.6) gives

$$\int_{\mathbb{R}^N} \left|\nabla\left(\tfrac{u}{|x|}\right)\right|^N |x|^a \, dx \le 2^{N/2-1} \left[\left(\frac{N}{|a|}\right)^N + \left(\frac{N}{|N-a|}\right)^N\right] \int_{\mathbb{R}^N} \left|D^2u\right|^N |x|^a \, dx,$$

which is (1.4)–(1.5). The remaining value $a=0$ is covered by Theorem 1.1, see Remark 1.8, so (1.4) holds throughout $a \neq N$. For $N=1$ there is no angular term, the convexity step is vacuous, and the radial estimate (3.5), valid at $p=N=1$ by Lemma 2.1, gives the inequality for every $a \neq 1$ with constant $1/|1-a|$. This proves Theorem 1.4. ☐

Remark 3.3. For radial u the angular term in (3.3) vanishes and the convexity step is unnecessary, so the argument yields the radial estimate

$$\int_{\mathbb{R}^N} \left|\nabla\left(\tfrac{u}{|x|}\right)\right|^N |x|^a \, dx \le \left(\frac{N}{|N-a|}\right)^N \int_{\mathbb{R}^N} |\partial_{rr} u|^N |x|^a \, dx, \qquad u \text{ radial}, \ a \neq N.$$

In the Emden–Fowler variable $r=e^t$, $v(t)=u(e^t)$, this is precisely Lemma 2.1 with $\mu = a-N$ applied to $w := v'-v$, since $r\partial_r u - u = v'-v$ and $r^2\partial_{rr}u = v''-v'$. The admissible profiles are here not all of $C_c^\infty(\mathbb{R})$ but only the image of $v \mapsto v'-v$, namely $\{w \in C_c^\infty(\mathbb{R}) : \int_{\mathbb{R}} w\, e^{-t}\, dt = 0\}$; nevertheless the constant $\left(N/|N-a|\right)^N$ remains sharp on the radial subspace. Indeed, truncating an antiderivative of the Hardy extremiser $e^{-\mu t/N}$ produces admissible functions $w_n = v_n' - v_n$ along which the ratio of the two sides of (2.1) tends to $\left(N/|N-a|\right)^N$; at $a=0$, where $\mu=-N$ resonates with the homogeneous solution of $v'-v=0$, one truncates the resonant antiderivative $t\,e^t$ instead, with the same limit.

3.6. **Proof of Corollary 1.6.** It suffices to combine Theorem 1.4 with the weighted Calderón–Zygmund inequality. The power weight $w(x) = |x|^a$ belongs to the Muckenhoupt class $A_N(\mathbb{R}^N)$ if and only if $-N < a < N(N-1)$; see [9, Ch. 7]. On that range, the Coifman–Fefferman theorem [4] ensures that Calderón–Zygmund operators, in particular the second-order Riesz transforms R_iR_j, through which $\partial_{ij}u = -R_iR_j\Delta u$, are bounded on $L^N(w\,dx)$. Hence

$$\int_{\mathbb{R}^N} \left|D^2u\right|^N |x|^a \, dx \le C(N,a) \int_{\mathbb{R}^N} |\Delta u|^N |x|^a \, dx, \qquad -N < a < N(N-1). \tag{3.7}$$

For $-N < a < N(N-1)$ with $a \ne N$, combining (1.4) and (3.7) gives (1.6). When $N = 2$ one has $N(N-1) = 2$, so the range is $-2 < a < 2$ and the constraint $a \ne N$ is automatic. $\square$

Remark 3.4. The exclusion $a = N$ in Corollary 1.6 is genuine for $N \ge 3$, where $N < N(N-1)$ and $a = N$ lies in the interior of the Muckenhoupt range: the weighted Calderón–Zygmund step is available there, but the left-hand inequality (1.4) itself fails, by Theorem 1.5.

4. Sharpness of the range: the critical exponent $a = N$

We prove that (1.4) fails at $a = N$, working on radial functions. By Theorem 1.4 it holds for every other value of a, so $a = N$ is the unique critical exponent.

Let u be radial, $u(x) = u(r)$ with $r = |x|$. In spherical coordinates, with $\omega_{N-1} = \left|\mathbb{S}^{N-1}\right|$ and using $\left|D^2u\right|^2 = |u''|^2 + (N-1)\,|u'/r|^2$ for radial u,

$$\int_{\mathbb{R}^N} \left|\nabla\left(\tfrac{u}{|x|}\right)\right|^N |x|^N \, dx = \omega_{N-1}\int_0^\infty \frac{|ru'-u|^N}{r^{2N}}\, r^{2N-1}\, dr = \omega_{N-1}\int_0^\infty \frac{|ru'-u|^N}{r}\, dr,$$

$$\int_{\mathbb{R}^N} \left|D^2u\right|^N |x|^N \, dx = \omega_{N-1}\int_0^\infty \left(|u''|^2 + (N-1)\,|u'/r|^2\right)^{N/2} r^{2N-1}\, dr.$$

Substituting $r = e^t$ and $v(t) = u(e^t)$, one has $ru' - u = v' - v$, $r^2u'' = v'' - v'$ and $u'/r = e^{-2t}v'$, so that

$$\int_0^\infty \frac{|ru'-u|^N}{r}\, dr = \int_{\mathbb{R}} |v'-v|^N \, dt,$$

$$\int_0^\infty \left(|u''|^2 + (N-1)\,|u'/r|^2\right)^{N/2} r^{2N-1}\, dr = \int_{\mathbb{R}} \left((v''-v')^2 + (N-1)(v')^2\right)^{N/2} dt.$$

Thus, for radial functions, the inequality (1.4) at $a = N$ is equivalent to

$$\int_{\mathbb{R}} |v'-v|^N \, dt \le C\int_{\mathbb{R}} \left((v''-v')^2 + (N-1)(v')^2\right)^{N/2} dt, \qquad v \in C_c^\infty(\mathbb{R}). \tag{4.1}$$

We construct a sequence violating (4.1). For $M > 2$ choose $\varphi_M \in C_c^\infty(\mathbb{R})$ with $\varphi_M \equiv 1$ on $[-M+1, M-1]$, supported in $[-M, M]$, with the transitions confined to $[-M, -M+1] \cup [M-1, M]$ and with $|\varphi_M'|$, $|\varphi_M''|$ bounded uniformly in M. Set

$v = \varphi_M$. On the plateau $[-M+1, M-1]$ one has $\varphi_M' = \varphi_M'' = 0$ and $\varphi_M = 1$, so the left integrand $|\varphi_M' - \varphi_M|^N$ equals 1 there; hence

$$\int_{\mathbb{R}} |\varphi_M' - \varphi_M|^N \, dt \geq 2M - 2.$$

On the right, the integrand is supported in the two transition intervals, of total length 2, and is bounded there by a constant $K = K(N)$ depending only on N and on the uniform bounds for φ_M', φ_M''; therefore

$$\int_{\mathbb{R}} \left((\varphi_M'' - \varphi_M')^2 + (N-1)(\varphi_M')^2\right)^{N/2} dt \leq 2K =: C_0,$$

with C_0 independent of M. Consequently the ratio of the two sides is at least $(2M-2)/C_0 \to \infty$ as $M \to \infty$, so no finite C can satisfy (4.1). The functions $u_M(x) = \varphi_M(\log|x|)$ are genuine radial elements of $C_c^\infty(\mathbb{R}^N \setminus \{0\})$, supported in the annulus $e^{-M} \leq |x| \leq e^M$, so (1.4) fails at $a = N$. This proves Theorem 1.5. ☐

Remark 4.1. The mechanism is transparent in the Emden–Fowler picture: at $a = N$ the exponential weight $e^{(a-N)t}$ becomes constant, the weighted Hardy inequality of Section 2 degenerates, and a function that is constant on a long interval makes the left side grow linearly in the length while leaving the right side bounded. This is the exact analogue of the failure of the classical Hardy inequality at $p = N$.

5. Concluding remarks

5.1. **On Open Problem 1 of [2].** Theorem 1.4 provides an *explicit* but not necessarily *optimal* constant in (1.2), namely $C_{N,a} = 2^{N/2-1}\left[(N/|a|)^N + (N/|N-a|)^N\right]$, for $N \geq 2$ and any $a \neq 0, N$. The corresponding statement in Corollary 1.6 carries an additional factor from the weighted Calderón–Zygmund inequality (3.7), whose sharp constant on $L^N(|x|^a\,dx)$ is not known in general. The full Open Problem 1, the determination of the best constants in (1.1) and (1.3), remains open. For *radial* functions, Remark 3.3 settles the corresponding question with the sharp constant

$$\sup_{u \text{ radial}} \frac{\int_{\mathbb{R}^N} |\nabla(u/|x|)|^N |x|^a \, dx}{\int_{\mathbb{R}^N} |\partial_{rr} u|^N |x|^a \, dx} = \left(\frac{N}{|N-a|}\right)^N, \qquad a \neq N.$$

5.2. **Sharp range and a critical-exponent analogy.** The role of $a = N$ as the unique critical endpoint in (1.4) parallels the role of $p = N$ as the critical exponent in the classical Hardy inequality. Both endpoints are characterised, through the Emden–Fowler reduction, by the vanishing of the exponential weight in the equivalent one-dimensional problem. Away from this single value the inequality holds in both directions $a < N$ and $a > N$.

5.3. **The fractional problems.** The fractional questions, Open Problems 2 and 3 of [2], remain open. The argument of Theorem 1.4 relies on the structure of ∂_{rr} as a genuine second derivative, through the radial identity $g' = r\,\partial_{rr}u$ and the Hessian decomposition of Lemma 3.1. A fractional analogue would require a nonlocal substitute for these identities, perhaps via the Caffarelli–Silvestre extension or the singular-integral representation of $(-\Delta)^s$. We hope to return to this elsewhere.

5.4. **The lower endpoint and the angular degeneracy.** The $|D^2u|$ form (1.4) holds for every $a \neq N$, with no lower bound on a; only the passage to the $|\Delta u|$ form (1.6) invokes the Muckenhoupt condition and hence requires $-N < a < N(N-1)$. Whether (1.6) persists for $a \leq -N$ or $a \geq N(N-1)$ appears to be open. We also recall, from Remark 1.8, that the explicit constant (1.5) degenerates at $a = 0$ although the inequality itself does not; producing an explicit constant that is uniform across a neighbourhood of $a = 0$, equivalently an angular estimate that does not pass through the borderline weight, would be of independent interest.

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References

1. Bez, N., Machihara, S., & Ozawa, T. (2023). Revisiting the Rellich inequality. Mathematische Zeitschrift, 303(2), Article 49. https://doi.org/10.1007/s00209-022-03203-4
2. Castro, H. (2025). A critical Hardy–Rellich inequality [Preprint]. arXiv. https://doi.org/10.48550/arXiv.2511.16537
3. Cazacu, C. (2020). A new proof of the Hardy–Rellich inequality in any dimension. Proceedings of the Royal Society of Edinburgh Section A: Mathematics, 150(6), 2894–2904. https://doi.org/10.1017/prm.2019.50
4. Coifman, R. R., & Fefferman, C. (1974). Weighted norm inequalities for maximal functions and singular integrals. Studia Mathematica, 51(3), 241–250. https://doi.org/10.4064/sm-51-3-241-250
5. Costa, D. G. (2009). On Hardy–Rellich type inequalities in $\mathbb{R}^N$. Applied Mathematics Letters, 22(6), 902–905. https://doi.org/10.1016/j.aml.2008.02.018
6. Davies, E. B., & Hinz, A. M. (1998). Explicit constants for Rellich inequalities in $L^p(\Omega)$. Mathematische Zeitschrift, 227(3), 511–523. https://doi.org/10.1007/PL00004389
7. Diatta, A., & Thiam, E. H. A. (2026). A doubly critical elliptic problem with submanifold singularities. Gulf Journal of Mathematics, 23(2). https://doi.org/10.56947/n8dw2419
8. Edmunds, D. E., & Evans, W. D. (2016). The Rellich inequality. Revista Matemática Complutense, 29(3), 511–530. https://doi.org/10.1007/s13163-016-0200-7
9. Grafakos, L. (2014). Modern Fourier analysis (3rd ed.), Graduate Texts in Mathematics, Vol. 250. Springer. https://doi.org/10.1007/978-1-4939-1230-8
10. Hardy, G. H., Littlewood, J. E., & Pólya, G. (1952). Inequalities (2nd ed.). Cambridge University Press.
11. Machihara, S., Ozawa, T., & Wadade, H. (2017). Remarks on the Rellich inequality. Mathematische Zeitschrift, 286(3–4), 1367–1373. https://doi.org/10.1007/s00209-016-1805-8
12. Opic, B., & Kufner, A. (1990). Hardy-type inequalities, Pitman Research Notes in Mathematics, Vol. 219. Longman.

13. Rellich, F. (1956). Halbbeschränkte Differentialoperatoren höherer Ordnung. In Proceedings of the International Congress of Mathematicians, Amsterdam 1954 (Vol. III, pp. 243–250). Noordhoff.
14. Rellich, F. (1969). Perturbation theory of eigenvalue problems. Gordon and Breach.
15. Tertikas, A., & Zographopoulos, N. B. (2007). Best constants in the Hardy–Rellich inequalities and related improvements. Advances in Mathematics, 209(2), 407–459. https://doi.org/10.1016/j.aim.2006.05.011

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