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BI-UNIVALENT FUNCTIONS ASSOCIATED WITH C-SHAPED DOMAINS

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ABSTRACT. Motivated by the geometric theory of analytic functions, this paper introduces two parameterized subclasses of bi-univalent functions through a subordination condition associated with a C-shaped domain. Sharp estimates for the initial Taylor coefficients are established, and corresponding Fekete–Szegö type inequalities are obtained. Furthermore, several special cases are discussed by assigning suitable values to the defining parameters, yielding a variety of new consequences for different families of bi-univalent functions. The obtained results not only unify and extend a number of existing coefficient estimates but also provide a flexible approach for investigating coefficient-related problems within the framework of geometric function theory.

Keywords. Analytic and bi-univalent functions; subordination principle; C-shaped mapping; initial coefficient estimates; Fekete–Szegö problem.
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1. Introduction

Coefficient problems play an important role in geometric function theory, where the initial Taylor coefficients of analytic and univalent functions provide essential information about their geometric properties. Among the classical coefficient problems, the Fekete–Szegö functional $a_3 - \eta a_2^2$, $\eta \in \mathbb{R}$, is particularly significant, as it describes the interaction between the second and third Taylor coefficients. The problem originated with Fekete and Szegö in 1933, who obtained the first sharp estimate for this functional in the class of normalized univalent functions. Since then, the Fekete–Szegö problem has been extensively studied for various subclasses of analytic and bi-univalent functions, including starlike, convex, Ma–Minda, Janowski, and operator-defined classes.

In recent years, the investigation of Fekete-Szegö inequalities has expanded significantly through the introduction of new subclasses associated with differential and integral operators, fractional calculus, q-calculus, and subordination to various geometric domains, see [11, 12]. These developments have produced sharper coefficient estimates and unified many previously known results, demonstrating

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that the Fekete–Szegö functional continues to serve as one of the principal tools for studying coefficient-related problems in geometric function theory, see for clarification [1, 4, 7, 14, 28].

Let $\mathbb{A}$ denote the family of all analytic functions defined in the open unit disk $\Theta = \{z \in \mathbb{C} : |z| < 1\}$, having the form

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m, \tag{1.1}$$

with the normalization conditions $f(0) = 0$ and $f'(0) = 1$.

We use the notation $\mathbb{S}$ for the class of all functions in $\mathbb{A}$ that are univalent in Θ. Basic concepts and well-established properties of analytic and univalent functions can be found in [6]. An important classical theorem in this field is the Koebe one-quarter theorem: for every $f \in \mathbb{S}$, the image of Θ under f necessarily contains a disk with radius not less than $\frac{1}{4}$.

As a consequence, every function $f \in \mathbb{S}$ admits an inverse function f^{-1}, defined on a disk $|\varpi| < r_0(f)$ with $r_0(f) \geq 0.25$, satisfying

$$f^{-1}(f(z)) = z, \quad z \in \Theta, \qquad f(f^{-1}(\varpi)) = \varpi, \quad |\varpi| < r_0(f).$$

The inverse function has the expansion

$$f^{-1}(\varpi) = \varpi - a_2\varpi^2 + (2a_2^2 - a_3)\varpi^3 - \cdots . \tag{1.2}$$

A function $f \in \mathbb{A}$ belongs to the class Σ of bi-univalent functions whenever both f and its inverse f^{-1} are univalent in Θ. The class Σ and its coefficient problems have been widely studied in recent literature [2, 8, 13].

The study of bi-univalent functions has attracted significant attention, especially following the work of Srivastava and coauthors [8], which stimulated extensive research in this direction. Several classical functions are known to belong to the class Σ, such as

$$\frac{z}{1-z}, \qquad -\log(1-z), \qquad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right).$$

In contrast, some standard univalent functions do not belong to Σ; for instance, the Koebe function, $z-\frac{z^2}{2}$, and $\frac{z}{1-z^2}$ are univalent in Θ but fail to be bi-univalent .

A substantial part of the literature has been devoted to deriving sharp or improved coefficient estimates for functions in Σ. Early results include Lewin [18], who obtained $|a_2| \leq 1.51$, followed by Brannan and Clunie [3], who conjectured $|a_2| \leq \sqrt{2}$, and Netanyahu [23], who established the upper bound $|a_2| \leq \frac{4}{3}$. Recent developments up to 2026 have further refined these estimates, where the latest results report that $|a_3| \leq 2.427\ldots$, $|a_4| \leq 3.461\ldots$, and $|a_5| \leq 4.993\ldots$ [24]. To refine these estimates, many subclasses of Σ, including starlike and convex bi-univalent functions, have been introduced and studied extensively [25, 5]. Among the most studied expressions in this context is the Fekete–Szegö functional $|a_3 - \eta a_2^2|$, which has been investigated for a wide range of function classes [10, 20, 26, 29].

Let $f, g \in \mathbb{A}$. The function f is said to be subordinate to g, denoted by $f(z) \prec g(z)$, whenever there exists a Schwarz function u, analytic in Θ, satisfying $u(0) = 0$ and $|u(z)| < 1$ for all $z \in \Theta$, such that

$$f(z) = g(u(z)).$$

Subordination is one of the basic concepts in geometric function theory and has been investigated in numerous works; see [9, 21]. If the function g is univalent in Θ, then the relation $f(z) \prec g(z)$ can be expressed equivalently by

$$f(0) = g(0) \quad \text{and} \quad f(\Theta) \subseteq g(\Theta).$$

Motivated by the recent development of analytic functions in geometric domains, several authors have introduced new frameworks to study coefficient problems under non-standard geometric configurations. In particular, Hadi et al. [15] investigated holomorphic functions subordinated to a C-shaped domain, introducing a novel setting that extends classical analytic function theory and enriches the study of geometric behavior of such functions.

$$\mathbb{S}^* = \left\{ f \in \mathbb{S} : \frac{zf'(z)}{f(z)} \prec 2 - \exp\left(\frac{\arctan(z)}{2}\right), \quad z \in \Theta \right\},$$

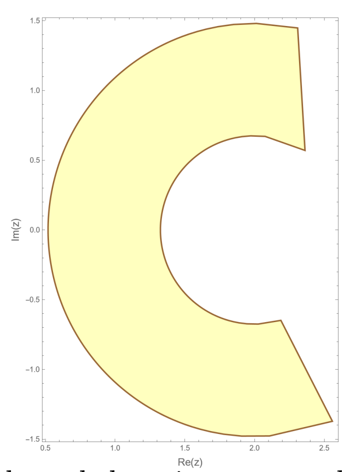

FIGURE 1. The C-shaped domain generated by the image transformation of the unit disk.

In this paper, we extend this idea by introducing and studying two new subclasses of bi-univalent functions defined in the same C-shaped domain, with the aim of obtaining new coefficient estimates and related functional inequalities.

2. Coefficient Estimates and Fekete–Szegö Inequality for the Class $D_\Sigma(\beta)$

This section presents several preliminary definitions and auxiliary results that will be used throughout the proofs of the main results.

Let $\mathbb{P}$ denote the family of analytic functions $c(z)$ within Θ, satisfying the normalization and positivity conditions

$$c(0) = 1, \qquad \Re\big(c(z)\big) > 0, \tag{2.1}$$

for all $z \in \Theta$.

Lemma 2.1. *(see [6]) If $c(z) \in \mathbb{P}$ has the expansion*

$$c(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots,$$

then its coefficients satisfy the sharp bound

$$|c_m| \le 2, \qquad m \in \mathbb{N} = \{1, 2, 3, \ldots\}. \tag{2.2}$$

Lemma 2.2. *(see [16]) if $m \in \mathbb{R}, L \in \mathbb{R}^+ \cup \{0\}$, and $n, c_1, c_2 \in \mathbb{C}$ with $n = a + ib$. If $\max\{|c_1|, |c_2|\} \le L$, then*

$$|(n+m)c_1 + (n-m)c_2| \le \begin{cases} 2(|a| + |b|)L, & |a| \ge |m|, \\ \\ 2(|m| + |b|)L, & |a| \le |m|, \end{cases} \tag{2.3}$$

Definition 2.3. Let f and $g = f^{-1}$ be given by (1.1) and (1.2), belong to the class $D_\Sigma(\beta)$ if the following subordination conditions are satisfied:

$$f'(z) + \beta z f''(z) \prec 2 - \exp\left(\frac{\arctan(z)}{2}\right),$$

and

$$g'(w) + \beta w g''(w) \prec 2 - \exp\left(\frac{\arctan(w)}{2}\right),$$

where $0 \le \beta \le 1$.

Theorem 2.4. *Assume that the function f, defined by (1.1), belongs to the class $D_\Sigma(\beta)$, where $0 \le \beta \le 1$. Then the initial coefficients satisfy the following bounds:*

$$|a_2| \le \frac{1}{\sqrt{6\,(1 + 2\beta + 2\beta^2)}},$$

and

$$|a_3| \le \frac{1}{6|1 + 2\beta|} + \frac{1}{16(1+\beta)^2}.$$

Proof. It is obvious that the expansion of the left-hand side of the class in Definition 2.3 gives

$$f'(z) + \beta z f''(z) = 1 + 2(1+\beta)a_2 z + 3(1+2\beta)a_3 z^2 + \cdots \tag{2.4}$$

$$g'(w) + \beta w g''(w) = 1 - 2(1+\beta)a_2 w + 3(1+2\beta)a_3 w^2 + \cdots \tag{2.5}$$

We consider the analytic functions

$$c(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \; s(w) = 1 + \sum_{n=1}^{\infty} s_n w^n \tag{2.6}$$

Define the functions

$$u(z) = \frac{c(z)-1}{c(z)+1} \quad \text{and} \quad v(w) = \frac{s(w)-1}{s(w)+1} \tag{2.7}$$

Next, consider the functions

$$\Phi(z) = 2 - \exp\left(\frac{\arctan\big(u(z)\big)}{2}\right). \tag{2.8}$$

and

$$\psi(z) = 2 - \exp\left(\frac{\arctan\big(v(w)\big)}{2}\right). \tag{2.9}$$

Expanding (2.8) and (2.9) into their Taylor series gives

$$\Phi(z) = 1 - \frac{c_1}{4}z + \frac{3c_1^2 - 8c_2}{32}z^2 + O(z^3). \tag{2.10}$$

$$\psi(z) = 1 - \frac{s_1}{4}z + \frac{3s_1^2 - 8s_2}{32}z^2 + O(z^3). \tag{2.11}$$

Then

$$2(1+\beta)a_2 = -\frac{c_1}{4} \tag{2.12}$$

$$3(1+2\beta)a_3 = \frac{1}{32}\left(3c_1^2 - 8c_2\right) \tag{2.13}$$

$$2(1+\beta)a_2 = \frac{s_1}{4} \tag{2.14}$$

$$3(1+2\beta)\left(2a_2^2 - a_3\right) = \frac{1}{32}\left(3s_1^2 - 8s_2\right) \tag{2.15}$$

From (2.12) and (2.14), we immediately obtain

$$c_1 = -s_1. \tag{2.16}$$

Furthermore,

$$128(1+\beta)^2 a_2^2 = c_1^2 + s_1^2,$$

equivalently,

$$a_2^2 = \frac{c_1^2 + s_1^2}{128(1+\beta)^2}. \tag{2.17}$$

From (2.13) and (2.15), by adding the left-hand sides and the right-hand sides, we obtain

$$3(1+2\beta)a_3 + 3(1+2\beta)\left(2a_2^2 - a_3\right) = \frac{1}{32}\left(3c_1^2 - 8c_2 + 3s_1^2 - 8s_2\right). \tag{2.18}$$

Simplifying, we get

$$6(1+2\beta)a_2^2 = \frac{1}{32}\left(3(c_1^2 + s_1^2) - 8(c_2 + s_2)\right). \tag{2.19}$$

From (2.17), we substitute the expression into (2.19). After simplification, we obtain

$$a_2^2 = \frac{c_2 + s_2}{24(1+2\beta+2\beta^2)}. \tag{2.20}$$

An application of Lemma 2.1 leads to the following estimate:

$$|a_2| \le \frac{1}{\sqrt{6(1+2\beta+2\beta^2)}}.$$

Subtracting (2.15) from (2.13), we obtain

$$3(1+2\beta)a_3 - 3(1+2\beta)\left(2a_2^2 - a_3\right) = \frac{1}{32}\left(3c_1^2 - 8c_2 - 3s_1^2 + 8s_2\right). \tag{2.21}$$

Simplifying and using (2.16), we obtain

$$a_3 - a_2^2 = \frac{-(c_2 - s_2)}{24(1+2\beta)}. \tag{2.22}$$

By using (2.17) and substituting it into (2.22), we obtain

$$a_3 = \frac{-(c_2 - s_2)}{24(1+2\beta)} + \frac{c_1^2 + s_1^2}{128(1+\beta)^2}. \tag{2.23}$$

Again applying Lemma 2.1, we infer that

$$|a_3| \le \frac{1}{6|1+2\beta|} + \frac{1}{16(1+\beta)^2}.$$

□

Corollary 2.5. *Suppose that $f(z)$ in (1.1) is a member of the class $D_\Sigma(0)$. Then*

$$|a_2| \le \frac{1}{\sqrt{6}},$$

and

$$|a_3| \le \frac{11}{48}.$$

Theorem 2.6. *Let the function $f \in D_\Sigma(\beta)$ and $\eta = \eta_1 + i\eta_2$ with $\eta_1, \eta_2 \in \mathbb{R}$, then*

$$|a_3 - \eta a_2^2| \le \begin{cases} \dfrac{1}{6}\left(\dfrac{1}{1+2\beta} + \dfrac{|\eta_2|}{1+2\beta+2\beta^2}\right), & 0 \le |1-\eta_1| \le \dfrac{1+2\beta+2\beta^2}{1+2\beta}, \\[2ex] \dfrac{|1-\eta_1| + |\eta_2|}{6(1+2\beta+2\beta^2)}, & |1-\eta_1| \ge \dfrac{1+2\beta+2\beta^2}{1+2\beta}. \end{cases} \tag{2.24}$$

Proof. From (2.22) we have

$$a_3 - a_2^2 = \frac{-(c_2 - s_2)}{24(1+2\beta)}. \tag{2.25}$$

Using (2.20) and (2.25), by simple calculation we get

$$\begin{aligned}a_3-\eta a_2^2 &= a_3-a_2^2+(1-\eta)a_2^2\\ &= \frac{-c_2+s_2}{24(1+2\beta)}+(1-\eta)\frac{c_2+s_2}{24(1+2\beta+2\beta^2)}\\ &= \left(F(\eta)+\frac{1}{24(1+2\beta)}\right)s_2+\left(F(\eta)-\frac{1}{24(1+2\beta)}\right)c_2,\end{aligned}$$

where

$$F(\eta)=\frac{1-\eta}{24(1+2\beta+2\beta^2)}.$$

Therefore Lemma 2.2 leads to

$$|a_3-\eta a_2^2|\leq\begin{cases}4\left(\dfrac{1}{24|1+2\beta|}+|\Im\{F(\eta)\}|\right), & 0\leq|\Re\{F(\eta)\}|\leq\dfrac{1}{24(1+2\beta)},\\ 4(|\Re\{F(\eta)\}|+|\Im\{F(\eta)\}|), & |\Re\{F(\eta)\}|\geq\dfrac{1}{24(1+2\beta)},\end{cases}$$

which completes the proof. □

Corollary 2.7. *Let the function $f\in D_\Sigma(0)$ and let $\eta=\eta_1+i\eta_2$ with $\eta_1,\eta_2\in\mathbb{R}$. Then*

$$|a_3-\eta a_2^2|\leq\begin{cases}\dfrac{1+|\eta_2|}{6}, & 0\leq|1-\eta_1|\leq 1,\\ \dfrac{|1-\eta_1|+|\eta_2|}{6}, & |1-\eta_1|\geq 1.\end{cases}$$

Corollary 2.8. *Let the function $f\in D_\Sigma(\beta)$ and let $\eta\in\mathbb{R}$. Then*

$$|a_3-\eta a_2^2|\leq\begin{cases}\dfrac{1}{6(1+2\beta)}, & 0\leq|1-\eta|\leq\dfrac{1+2\beta+2\beta^2}{1+2\beta},\\ \dfrac{|1-\eta|}{6(1+2\beta+2\beta^2)}, & |1-\eta|\geq\dfrac{1+2\beta+2\beta^2}{1+2\beta}.\end{cases}$$

3. Coefficient Estimates and Fekete–Szegö Inequality for the Class $\mathcal{K}_\Sigma(\beta,\vartheta)$

Definition 3.1. Let f and $g=f^{-1}$ be given by (1.1) and (1.2), belongs to the class $\mathcal{K}_\Sigma(\beta,\vartheta)$ whenever the following conditions are fulfilled:

$$\frac{[(zf'(z))']^\vartheta}{1+\beta(f'(z)-1)}\prec 2-\exp\left(\frac{\arctan(z)}{2}\right),$$

and

$$\frac{[(wg'(w))']^\vartheta}{1+\beta(g'(w)-1)}\prec 2-\exp\left(\frac{\arctan(w)}{2}\right),$$

where $\vartheta\geq 1$ and $0\leq\beta\leq 1$.

Theorem 3.2. *Let f, defined by (1.1), belong to the class $\mathcal{K}_\Sigma(\beta,\vartheta)$. Under this assumption, the following coefficient estimates hold:*

$$|a_2| \leq \frac{1}{\sqrt{32\vartheta^2 - 32\vartheta\beta + 6\vartheta + 4\beta^2 + 6\beta}}. \tag{3.1}$$

$$|a_3| \leq \frac{1}{6(3\vartheta-\beta)} + \frac{1}{4(4\vartheta-2\beta)^2}. \tag{3.2}$$

where $\vartheta \geq 1$ and $0 \leq \beta \leq 1$.

Proof. Let g be given by (1.2). Since f belongs to the class $\mathcal{K}_\Sigma(\beta,\vartheta)$, there are analytic functions $u, v : \mathbb{E} \to \mathbb{E}$ satisfying $u(0) = 0 = v(0)$ together with $|u(z)| < 1$ and $|v(z)| < 1$, such that

$$\frac{[(zf'(z))']^\vartheta}{1+\beta(f'(z)-1)} = 2 - \exp\left(\frac{\arctan\big(u(z)\big)}{2}\right). \tag{3.3}$$

$$\frac{[(wg'(w))']^\vartheta}{1+\beta(g'(w)-1)} = 2 - \exp\left(\frac{\arctan\big(v(w)\big)}{2}\right). \tag{3.4}$$

Expanding $\Phi(z)$ and $\psi(w)$ as power series, we obtain:

$$\Phi(z) = 2 - \exp\left(\frac{\arctan\big(u(z)\big)}{2}\right) = 1 - \frac{c_1}{4}z + \frac{3c_1^2 - 8c_2}{32}z^2 + \cdots. \tag{3.5}$$

$$\psi(w) = 2 - \exp\left(\frac{\arctan\big(v(w)\big)}{2}\right) = 1 - \frac{s_1}{4}w + \frac{3s_1^2 - 8s_2}{32}w^2 + \cdots. \tag{3.6}$$

Moreover, it follows that

$$\frac{[(zf'(z))']^\vartheta}{1+\beta(f'(z)-1)} = 1+(4\vartheta-2\beta)a_2z+\left[(8\vartheta^2 - 8\vartheta(\beta+1) + 4\beta^2)a_2^2 + (9\vartheta-3\beta)a_3\right]z^2+\cdots. \tag{3.7}$$

$$\frac{[(wg'(w))']^\vartheta}{1+\beta(g'(w)-1)} = 1-(4\vartheta-2\beta)a_2w+\left[(8\vartheta^2 - 8\vartheta\beta + 10\vartheta + 4\beta^2 - 6\beta)a_2^2 - 3(3\vartheta-\beta)a_3\right]w^2+\cdots. \tag{3.8}$$

Using (3.5), (3.6), (3.7), and (3.8), and comparing the coefficients of z and z^2, we get:

$$(4\vartheta-2\beta)a_2 = -\frac{c_1}{4}. \tag{3.9}$$

$$\big(8\vartheta^2 - 8\vartheta(\beta+1) + 4\beta^2\big)a_2^2 + (9\vartheta-3\beta)a_3 = \frac{1}{32}\left(3c_1^2 - 8c_2\right). \tag{3.10}$$

$$-(4\vartheta-2\beta)a_2 = -\frac{s_1}{4}. \tag{3.11}$$

$$\big(8\vartheta^2 - 8\vartheta\beta + 10\vartheta + 4\beta^2 - 6\beta\big)a_2^2 - 3(3\vartheta-\beta)a_3 = \frac{1}{32}\left(3s_1^2 - 8s_2\right). \tag{3.12}$$

From (3.9) and (3.11), we solve for a_2 as follows:

$$a_2 = -\frac{c_1}{4(4\vartheta-2\beta)} = \frac{s_1}{4(4\vartheta-2\beta)};$$

it follows that
$$c_1 = -s_1. \tag{3.13}$$

Thus,
$$a_2^2 = \frac{c_1^2 + s_1^2}{32(4\vartheta - 2\beta)^2}. \tag{3.14}$$

Adding (3.10) and (3.12) and using (3.14), we have:
$$\left(16\vartheta^2 - 16\vartheta\beta - 6\vartheta + 8\beta^2 - 6\beta\right)a_2^2 = \frac{1}{32}\left(3(c_1^2 + s_1^2) - 8(c_2 + s_2)\right). \tag{3.15}$$

$$\left(-32\vartheta^2 + 32\vartheta\beta - 6\vartheta - 4\beta^2 - 6\beta\right)a_2^2 = -\frac{1}{4}(c_2 + s_2). \tag{3.16}$$

Hence,
$$a_2^2 = \frac{c_2 + s_2}{4\left(32\vartheta^2 - 32\vartheta\beta + 6\vartheta + 4\beta^2 + 6\beta\right)}. \tag{3.17}$$

Applying Lemma 2.1 to (3.17), we get the desired inequality (3.1).

It remains to prove that the expression under the square root is always positive. Let
$$\Delta = 32\vartheta^2 - 32\vartheta\beta + 6\vartheta + 4\beta^2 + 6\beta.$$

Completing the square, we obtain
$$\Delta = 32\left(\vartheta - \frac{\beta}{2}\right)^2 + 6\vartheta + 2\beta(3 - 2\beta).$$

Since
$$\vartheta \geq 1 \quad \text{and} \quad 0 \leq \beta \leq 1,$$

we have
$$32\left(\vartheta - \frac{\beta}{2}\right)^2 \geq 0, \qquad 6\vartheta > 0, \qquad 2\beta(3 - 2\beta) \geq 0.$$

Therefore,
$$\Delta = 32\left(\vartheta - \frac{\beta}{2}\right)^2 + 6\vartheta + 2\beta(3 - 2\beta) > 0.$$

Hence,
$$32\vartheta^2 - 32\vartheta\beta + 6\vartheta + 4\beta^2 + 6\beta > 0,$$

which implies that
$$\sqrt{32\vartheta^2 - 32\vartheta\beta + 6\vartheta + 4\beta^2 + 6\beta}$$

is well-defined for all $\vartheta \geq 1$ and $0 \leq \beta \leq 1$.

Subtracting (3.10) from (3.12), and using (3.17), we obtain
$$6(\beta - 3\vartheta)(a_2^2 - a_3) = \frac{1}{4}(-c_2 + s_2). \tag{3.18}$$

$$a_3 = \frac{c_2 - s_2}{24(\beta - 3\vartheta)} + a_2^2. \tag{3.19}$$

Again, from (3.13), we have $c_1^2 = s_1^2$, and applying Lemma 2.1 then (3.19) yields the desired inequality. This completes the proof. □

Corollary 3.3. *Assume that the function f, defined by (1.1), belongs to the class $\mathcal{K}_\Sigma(1,\vartheta)$, where $\vartheta \geq 1$. Then*

$$|a_2| \leq \frac{1}{\sqrt{32\vartheta^2 - 26\vartheta + 10}},$$

and

$$|a_3| \leq \frac{1}{6(3\vartheta - 1)} + \frac{1}{16(2\vartheta - 1)^2}.$$

Corollary 3.4. *Assume that the function f, defined by (1.1), belongs to the class $\mathcal{K}_\Sigma(\beta,1)$, where $\vartheta \geq 1$. Then*

$$|a_2| \leq \frac{1}{\sqrt{38 - 26\beta + 4\beta^2}},$$

and

$$|a_3| \leq \frac{1}{6(3 - \beta)} + \frac{1}{16(2 - \beta)^2}.$$

Theorem 3.5. *Let the function $f \in \mathcal{K}_\Sigma(\beta,\vartheta)$ and $\eta = \eta_1 + i\eta_2$ with $\eta_1, \eta_2 \in \mathbb{R}$, then*

$$|a_3 - \eta a_2^2| \leq \begin{cases} \left(\frac{1}{6(3\vartheta-\beta)} + \frac{|\eta_2|}{32\vartheta^2 - 32\vartheta\beta + 6\vartheta + 4\beta^2 + 6\beta}\right), & 0 \leq |1-\eta_1| \leq \frac{32\vartheta^2 - 32\vartheta\beta + 6\vartheta + 4\beta^2 + 6\beta}{6(3\vartheta-\beta)}, \\ \frac{|1-\eta_1| + |\eta_2|}{(32\vartheta^2 - 32\vartheta\beta + 6\vartheta + 4\beta^2 + 6\beta)}, & |1-\eta_1| \geq \frac{32\vartheta^2 - 32\vartheta\beta + 6\vartheta + 4\beta^2 + 6\beta}{6(3\vartheta-\beta)}, \end{cases} \tag{3.20}$$

Proof. From (3.19) we have

$$a_3 - a_2^2 = \frac{c_2 - s_2}{24(\beta - 3\vartheta)}. \tag{3.21}$$

Using (3.17) and (3.21), by simple calculation we get

$$\begin{aligned} a_3 - \eta a_2^2 &= a_3 - a_2^2 + (1-\eta)a_2^2 \\ &= \frac{c_2 - s_2}{24(\beta - 3\vartheta)} + \frac{(1-\eta)(c_2 + s_2)}{4\left(32\vartheta^2 - 32\vartheta\beta + 6\vartheta + 4\beta^2 + 6\beta\right)} \\ &= \left(F(\eta) + \frac{1}{24(\beta - 3\vartheta)}\right) c_2 + \left(F(\eta) - \frac{1}{24(\beta - 3\vartheta)}\right) s_2 \end{aligned}$$

where

$$F(\eta) = \frac{1-\eta}{4\left(32\vartheta^2 - 32\vartheta\beta + 6\vartheta + 4\beta^2 + 6\beta\right)}.$$

Therefore Lemma 2.2 leads to

$$|a_3 - \eta a_2^2| \leq \begin{cases} 4\left(\frac{1}{24|\beta - 3\vartheta|} + |\Im\{F(\eta)\}|\right), & 0 \leq |\Re\{F(\eta)\}| \leq \frac{1}{24(\beta - 3\vartheta)}, \\ 4(|\Re\{F(\eta)\}| + |\Im\{F(\eta)\}|), & |\Re\{F(\eta)\}| \geq \frac{1}{24(\beta - 3\vartheta)}, \end{cases}$$

which completes the proof. □

Corollary 3.6. *Let the function $f \in \mathcal{K}_\Sigma(1, \vartheta)$ and $\eta = \eta_1 + i\eta_2$, where $\eta_1, \eta_2 \in \mathbb{R}$. Then*

$$|a_3 - \eta a_2^2| \leq \begin{cases} \dfrac{1}{6(3\vartheta - 1)} + \dfrac{|\eta_2|}{32\vartheta^2 - 26\vartheta + 10}, & 0 \leq |1 - \eta_1| \leq \dfrac{32\vartheta^2 - 26\vartheta + 10}{6(3\vartheta - 1)}, \\ \dfrac{|1 - \eta_1| + |\eta_2|}{32\vartheta^2 - 26\vartheta + 10}, & |1 - \eta_1| \geq \dfrac{32\vartheta^2 - 26\vartheta + 10}{6(3\vartheta - 1)}. \end{cases}$$

Corollary 3.7. *Let the function $f \in \mathcal{K}_\Sigma(\beta, 1)$ and $\eta = \eta_1 + i\eta_2$, where $\eta_1, \eta_2 \in \mathbb{R}$. Then*

$$|a_3 - \eta a_2^2| \leq \begin{cases} \dfrac{1}{6(3 - \beta)} + \dfrac{|\eta_2|}{38 - 26\beta + 4\beta^2}, & 0 \leq |1 - \eta_1| \leq \dfrac{38 - 26\beta + 4\beta^2}{6(3 - \beta)}, \\ \dfrac{|1 - \eta_1| + |\eta_2|}{38 - 26\beta + 4\beta^2}, & |1 - \eta_1| \geq \dfrac{38 - 26\beta + 4\beta^2}{6(3 - \beta)}. \end{cases}$$

Corollary 3.8. *Let the function $f \in \mathcal{K}_\Sigma(\beta, \vartheta)$ and let $\eta \in \mathbb{R}$. Then*

$$|a_3 - \eta a_2^2| \leq \begin{cases} \dfrac{1}{6(3\vartheta - \beta)}, & 0 \leq |1 - \eta| \leq \dfrac{32\vartheta^2 - 32\vartheta\beta + 6\vartheta + 4\beta^2 + 6\beta}{6(3\vartheta - \beta)}, \\ \dfrac{|1 - \eta|}{32\vartheta^2 - 32\vartheta\beta + 6\vartheta + 4\beta^2 + 6\beta}, & |1 - \eta| \geq \dfrac{32\vartheta^2 - 32\vartheta\beta + 6\vartheta + 4\beta^2 + 6\beta}{6(3\vartheta - \beta)}. \end{cases}$$

4. Conclusions

This research has defined two new classes of bi-univalent functions related to a geometric domain of the shape C, and investigates some of their coefficient properties. Limits for the first coefficients are established, and the generalized Fekete-Szegö inequality for these two classes have been derived. The results yielded several known outcomes as special cases when appropriate values are chosen for the parameters, in addition to new results not reported in previous studies. These findings have demonstrated that the two proposed classes represent a suitable extension of several classes studied in analytic function theory and can be used to investigate other coefficient problems for similar classes in future work.

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