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NONLOCAL KDV DYNAMICS WITH FRACTIONAL BESSEL SLOPE

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Abstract. We study a one-dimensional nonlocal dispersive equation combining Airy dispersion, Helmholtz averaging, and a slope filtered by a fractional Bessel operator. For fractional orders from zero to one half, the nonlinear vector field is locally Lipschitz on the natural energy space, including the endpoint case. A contraction argument based on the Airy group gives local well-posedness, uniqueness, and locally Lipschitz dependence on the initial data. Conservation of a coercive quadratic energy is justified for mild solutions by resolvent regularization. Consequently, every real-valued energy-space solution exists globally.

Keywords. Nonlocal KdV equation, Bessel operator, well-posedness, conservation law, dispersive equation
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1. Introduction

The Korteweg–de Vries equation

$$u_t + u_{xxx} + uu_x = 0 \tag{1.1}$$

is the standard unidirectional approximation for weakly nonlinear, weakly dispersive long waves. Its cubic Airy phase describes the leading dispersive correction to nondispersive transport, while the quadratic term represents the first nonlinear modification of the wave speed; see Korteweg and de Vries [11]. The modern low-regularity theory is strongly influenced by the bilinear analysis of Kenig, Ponce and Vega [8].

The continuing relevance of KdV-type dynamics is also reflected in recent contributions. Taboye and Ennouari [15] studied stabilization of a linear KdV equation with delayed boundary feedback and internal saturation, while Khadraoui, Khadraoui and Ould-Hammouda [9] established existence and energy decay for a coupled modified KdV system with memory. In contrast, the equation considered here is conservative and spatially nonlocal, and our focus is the construction of a global flow in the natural energy space.

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The Camassa–Holm equation provides a complementary long-wave description in which nonlinear dispersion is mediated by the Helmholtz inverse. In nonlocal form it reads

$$u_t + uu_x + \partial_x(1-\partial_x^2)^{-1}\left(u^2 + \frac{1}{2}u_x^2\right) = 0. \tag{1.2}$$

It was introduced as an integrable shallow-water equation with peaked solitary waves [2]. The Dullin–Gottwald–Holm equation subsequently combined linear KdV dispersion with nonlinear Camassa–Holm dispersion [4].

A second physical source of nonlocal dispersive equations is continuum mechanics with an internal length scale. In integral-type nonlocal elasticity, the stress at a point depends on a spatial average of the strain in a neighborhood rather than only on the local strain. Helmholtz kernels provide a differential realization of this averaging and lead naturally to dispersive wave equations [3]. Long-wave reductions of nonlocally and nonlinearly elastic media have also produced classical and fractional Camassa–Holm equations [5]. These works suggest that a filtered spatial gradient can be a meaningful constitutive variable in a one-dimensional wave model.

The Cauchy theory and wave dynamics of fractional long-wave models have received renewed attention. Besov-space well-posedness and analyticity for fractional and generalized fractional Camassa–Holm equations were established in [6, 12]. For a generalized fractional Benjamin–Bona–Mahony equation, local existence, conserved quantities and solitary waves were studied in [13], while spectral stability of solitary waves was investigated in [14]. Numerical evidence for solitary waves, stability and dispersive-shock behavior in fractional Camassa–Holm equations was reported in [10]. The model considered here differs from those equations because the fractional Bessel operator enters a variational nonlinear slope interaction, whereas the linear part retains the Airy dispersion.

To make the internal length explicit, let $\ell > 0$ and define

$$J_{\alpha,\ell} = (1-\ell^2\partial_x^2)^{-(1-\alpha)/2}, \qquad \mathcal{D}_{\alpha,\ell} = \partial_x J_{\alpha,\ell}.$$

The quantity $\mathcal{D}_{\alpha,\ell}u$ may be interpreted as a strain or velocity gradient measured after spatial averaging over a scale comparable with ℓ. Its Fourier symbol is

$$i\xi(1+\ell^2\xi^2)^{(\alpha-1)/2}.$$

It behaves as the ordinary derivative for $|\xi|\ell \ll 1$, whereas for $|\xi|\ell \gg 1$ its order is α. Thus ℓ represents an internal or correlation length and α controls the sensitivity of the nonlinear response to short-wave gradients. After nondimensionalization we set $\ell = 1$ and write

$$\mathcal{D}_\alpha = \partial_x(1-\partial_x^2)^{(\alpha-1)/2}, \qquad 0 \le \alpha \le \frac{1}{2}. \tag{1.3}$$

Its symbol is

$$d_\alpha(\xi) = i\xi\langle\xi\rangle^{\alpha-1}, \qquad \langle\xi\rangle = (1+\xi^2)^{1/2}. \tag{1.4}$$

We consider a scalar field $u = u(x,t)$ representing a longitudinal velocity, displacement amplitude, or free-surface amplitude in a unidirectional regime. The nonlinear response couples the amplitude with the filtered slope through the cubic interaction

$$\mathcal{V}_\alpha(u) = \int_{\mathbb{R}} \left[\frac{1}{2}u^3 + \frac{1}{2}u(\mathcal{D}_\alpha u)^2\right] dx. \tag{1.5}$$

Its variational derivative is

$$\mathcal{G}_\alpha(u) = \frac{3}{2}u^2 + \frac{1}{2}(\mathcal{D}_\alpha u)^2 - \mathcal{D}_\alpha(u\mathcal{D}_\alpha u). \tag{1.6}$$

The evolution equation studied in this paper is

$$\begin{cases} u_t + u_{xxx} + \mathcal{K}\partial_x \mathcal{G}_\alpha(u) = 0, & (x,t) \in \mathbb{R}\times\mathbb{R}, \\ u(x,0) = u_0(x), \end{cases} \tag{1.7}$$

where

$$\mathcal{K} = (1-\partial_x^2)^{-1}.$$

The operator $\mathcal{K}$ averages the nonlinear force with the exponentially decaying kernel $G(x) = \frac{1}{2}e^{-|x|}$, while $\mathcal{D}_\alpha$ filters the slope before it enters the nonlinear interaction. Equation (1.7) is therefore a phenomenological nonlocal long-wave model combining Airy dispersion, a finite interaction length and a scale-dependent nonlinear gradient. We do not claim that it is the unique asymptotic reduction of a specific continuum system. Rather, it isolates structural ingredients that occur separately in established models.

At the formal value $\alpha = 1$, one has $\mathcal{D}_1 = \partial_x$ and

$$\mathcal{G}_1(u) = \frac{3}{2}u^2 - \frac{1}{2}u_x^2 - uu_{xx},$$

so that

$$\mathcal{K}\partial_x\mathcal{G}_1(u) = uu_x + \mathcal{K}\partial_x\left(u^2 + \frac{1}{2}u_x^2\right). \tag{1.8}$$

Thus, the formal member $\alpha = 1$ combines the nonlocal Camassa–Holm nonlinearity with Airy dispersion. The present analysis is restricted to $0 \le \alpha \le 1/2$, where the high-frequency attenuation is strong enough for the nonlinear vector field to be locally Lipschitz on $H^1(\mathbb{R})$.

The main result is the following.

Theorem 1.1. *Let $0 \le \alpha \le 1/2$ and let $u_0 \in H^1(\mathbb{R})$ be real-valued. Then (1.7) has a unique global real-valued solution*

$$u \in C(\mathbb{R}; H^1(\mathbb{R})) \cap C^1(\mathbb{R}; H^{-2}(\mathbb{R})).$$

For every $T > 0$, the data-to-solution map is locally Lipschitz from $H^1(\mathbb{R})$ into $C([-T,T]; H^1(\mathbb{R}))$. Moreover,

$$\|u(t)\|_{H^1} = \|u_0\|_{H^1}, \qquad t \in \mathbb{R}. \tag{1.9}$$

The endpoint $\alpha = 1/2$ requires special care. Since $H^{1/2}(\mathbb{R})$ is not an algebra, the square $(\mathcal{D}_\alpha u)^2$ is estimated in L^2 by the embedding $H^{1/4}(\mathbb{R}) \hookrightarrow L^4(\mathbb{R})$. The remaining amplitude–slope term is controlled by the multiplier property $H^1 \cdot H^\alpha \subset H^\alpha$.

The proof has two ingredients. First, the nonlinear vector field $\mathcal{N}_\alpha(u) = \mathcal{K}\partial_x\mathcal{G}_\alpha(u)$ is locally Lipschitz from H^1 into itself. The Airy group then yields local well-posedness and a blow-up alternative by the contraction principle. Second, the flow conserves the positive quadratic quantity

$$\mathcal{E}(u) = \frac{1}{2}\int_{\mathbb{R}} (u^2 + u_x^2)\, dx. \tag{1.10}$$

The conservation law is first proved for smooth solutions and then justified for mild H^1 solutions by resolvent regularization. Since $\mathcal{E}$ controls the complete H^1 norm, the local solution extends globally.

2. Physical and variational motivation

2.1. **A nonlocal constitutive picture.** A local one-dimensional constitutive law relates stress to strain evaluated at the same point. In a nonlocal medium, the effective strain may instead be represented by a spatial average

$$\varepsilon_{\text{eff}} = G_\ell * \varepsilon,$$

where G_ℓ is concentrated on a neighborhood of size ℓ. For the Helmholtz kernel,

$$G_\ell(x) = \frac{1}{2\ell} e^{-|x|/\ell}, \qquad (1 - \ell^2\partial_x^2)^{-1} f = G_\ell * f.$$

Such kernels occur naturally in differential formulations of nonlocal elasticity and introduce both a finite interaction length and dispersive wave propagation [3]. In (1.7) the same mechanism is used twice: the nonlinear force is averaged by $\mathcal{K}$, and the gradient entering the constitutive interaction is replaced by $\mathcal{D}_\alpha u$.

The interpretation is especially transparent when u is a longitudinal velocity or displacement amplitude. The ordinary slope u_x measures local strain, whereas

$$\mathcal{D}_\alpha u = \partial_x (1 - \partial_x^2)^{-(1-\alpha)/2} u$$

measures the derivative after Bessel averaging. It retains the long-wave gradient while suppressing very short spatial oscillations. Fractional powers of nonlocal kernels have appeared in derivations of fractional Camassa–Holm equations for elastic waves [5]; equation (1.7) uses a related filtered-gradient mechanism in a semilinear setting.

2.2. **Long-wave dispersion.** The term u_{xxx} is the linear dispersive correction characteristic of a unidirectional long-wave approximation. Linearizing at the rest state gives the Fourier phase

$$\omega(\xi) = -\xi^3,$$

so phase and group velocities depend on wavelength. Restoring dimensional constants would lead to a family of the form

$$u_t + c_0 u_x + \beta u_{xxx} + \gamma (1 - \ell^2 \partial_x^2)^{-1} \partial_x \mathcal{G}_{\alpha,\ell}(u) = 0,$$

with wave speed c_0, dispersive coefficient β, nonlinear coefficient γ and internal length ℓ. A Galilean change of variables, scaling of x, t, u and normalization of ℓ reduce this family to (1.7) when the relevant coefficients are nonzero and have compatible signs.

2.3. **Variational interaction and Hamiltonian structure.** The cubic functional (1.5) couples the wave amplitude with the energy of the filtered slope. Its translation invariance is compatible with a homogeneous medium, and its variational derivative gives the nonlinear force in (1.7).

Proposition 2.1. *Let $0 \leq \alpha \leq 1/2$ and let $u, h \in \mathcal{S}(\mathbb{R})$ be real-valued. Then*

$$D\mathcal{V}_\alpha(u)[h] = \int_{\mathbb{R}} \mathcal{G}_\alpha(u) h \, dx,$$

where $\mathcal{G}_\alpha$ is given by (1.6).

Proof. Differentiating (1.5) at u in the direction h gives

$$D\mathcal{V}_\alpha(u)[h] = \int_{\mathbb{R}} \left[\frac{3}{2}u^2 h + \frac{1}{2}h(\mathcal{D}_\alpha u)^2 + u(\mathcal{D}_\alpha u)(\mathcal{D}_\alpha h)\right] dx.$$

The symbol d_α in (1.4) is purely imaginary and odd. Hence $\mathcal{D}_\alpha^* = -\mathcal{D}_\alpha$ on L^2. Therefore,

$$\int_{\mathbb{R}} u(\mathcal{D}_\alpha u)(\mathcal{D}_\alpha h) \, dx = -\int_{\mathbb{R}} \mathcal{D}_\alpha(u\mathcal{D}_\alpha u) h \, dx,$$

which proves the formula. □

For smooth functions define

$$\mathcal{H}_\alpha(u) = -\frac{1}{2}\int_{\mathbb{R}} (u_x^2 + u_{xx}^2) \, dx + \mathcal{V}_\alpha(u). \tag{2.1}$$

Since

$$\frac{\delta}{\delta u}\left[-\frac{1}{2}\int (u_x^2 + u_{xx}^2) dx\right] = (1 - \partial_x^2) u_{xx},$$

equation (1.7) can be written formally as

$$u_t = -\mathcal{K}\partial_x \frac{\delta \mathcal{H}_\alpha}{\delta u}.$$

Thus the Airy dispersion and nonlinear filtered-gradient response are incorporated in a single conservative flow.

3. Preliminary estimates

We use the Fourier transform

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x) \, dx$$

and write $\Lambda = (1 - \partial_x^2)^{1/2}$. The Sobolev norm is

$$\|f\|_{H^s} = \left\| \langle \xi \rangle^s \widehat{f}(\xi) \right\|_{L^2_\xi}.$$

Throughout the paper, constants may depend on α but not on the functions being estimated.

Lemma 3.1. *Let $0 \le \alpha \le 1/2$. Then*

$$\|\mathcal{D}_\alpha f\|_{H^{1-\alpha}} \le \|f\|_{H^1}\,, \tag{3.1}$$

$$\|\mathcal{K}\partial_x f\|_{H^1} \le \|f\|_{L^2}\,, \tag{3.2}$$

$$\|\mathcal{K}\partial_x \mathcal{D}_\alpha f\|_{H^1} \le \|f\|_{H^\alpha}\,. \tag{3.3}$$

Moreover, $\mathcal{D}_\alpha$ maps real-valued functions into real-valued functions, commutes with ∂_x and is skew-adjoint on L^2.

Proof. The first estimate follows from

$$\langle\xi\rangle^{1-\alpha}|d_\alpha(\xi)| = |\xi| \le \langle\xi\rangle.$$

For (3.2),

$$\langle\xi\rangle\frac{|\xi|}{1+\xi^2} \le 1.$$

Finally, the symbol of $\mathcal{K}\partial_x\mathcal{D}_\alpha$ is $-\xi^2\langle\xi\rangle^{\alpha-3}$ and

$$\langle\xi\rangle|\xi|^2\langle\xi\rangle^{\alpha-3} \le \langle\xi\rangle^\alpha.$$

The remaining assertions follow because d_α is odd and purely imaginary. ☐

We shall use the one-dimensional embeddings

$$H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R}), \qquad H^{1/4}(\mathbb{R}) \hookrightarrow L^4(\mathbb{R}). \tag{3.4}$$

We also use a standard Sobolev multiplier property; see [1, 7].

Lemma 3.2. *For every $0 \le \sigma \le 1$ there exists $C_\sigma > 0$ such that*

$$\|fg\|_{H^\sigma} \le C_\sigma \|f\|_{H^1} \|g\|_{H^\sigma} \tag{3.5}$$

for all $f \in H^1(\mathbb{R})$ and $g \in H^\sigma(\mathbb{R})$.

Proof. For $\sigma = 0$, the assertion follows from $H^1 \hookrightarrow L^\infty$. For $\sigma = 1$, it follows from the product rule and the same embedding. Intermediate cases follow by complex interpolation of the multiplication operator $g \mapsto fg$ between L^2 and H^1. ☐

Lemma 3.3. *If $0 \le \alpha \le 1/2$ and $u \in H^1(\mathbb{R})$, then*

$$\|\mathcal{D}_\alpha u\|_{L^4} \le C_\alpha \|u\|_{H^1}\,. \tag{3.6}$$

Proof. By (3.1), $\mathcal{D}_\alpha u \in H^{1-\alpha}$. Since $1-\alpha \ge 1/2 > 1/4$, the second embedding in (3.4) gives the result. ☐

4. The nonlinear mapping and local well-posedness

Define

$$\mathcal{N}_\alpha(u) := \mathcal{K}\partial_x\left[\frac{3}{2}u^2 + \frac{1}{2}(\mathcal{D}_\alpha u)^2 - \mathcal{D}_\alpha(u\mathcal{D}_\alpha u)\right]. \tag{4.1}$$

The next proposition is the key analytic estimate.

Proposition 4.1. *Let $0 \le \alpha \le 1/2$. There exists $C_\alpha > 0$ such that, for all $u, v \in H^1(\mathbb{R})$,*

$$\|\mathcal{G}_\alpha(u) - \mathcal{G}_\alpha(v)\|_{L^2} \le C_\alpha\big(\|u\|_{H^1} + \|v\|_{H^1}\big) \|u - v\|_{H^1}, \tag{4.2}$$

and consequently

$$\|\mathcal{N}_\alpha(u) - \mathcal{N}_\alpha(v)\|_{H^1} \le C_\alpha\big(\|u\|_{H^1} + \|v\|_{H^1}\big) \|u - v\|_{H^1}. \tag{4.3}$$

In particular,

$$\|\mathcal{N}_\alpha(u)\|_{H^1} \le C_\alpha \|u\|_{H^1}^2. \tag{4.4}$$

Proof. We estimate the three terms in $\mathcal{G}_\alpha(u) - \mathcal{G}_\alpha(v)$. First, by $H^1 \hookrightarrow L^\infty$,

$$\begin{aligned}\left\|u^2 - v^2\right\|_{L^2} &\le \|(u - v)(u + v)\|_{L^2} \\ &\le C\big(\|u\|_{H^1} + \|v\|_{H^1}\big) \|u - v\|_{H^1}.\end{aligned}$$

Second,

$$(\mathcal{D}_\alpha u)^2 - (\mathcal{D}_\alpha v)^2 = \mathcal{D}_\alpha(u - v)(\mathcal{D}_\alpha u + \mathcal{D}_\alpha v).$$

Hölder's inequality and Lemma 3.3 give

$$\begin{aligned}\left\|(\mathcal{D}_\alpha u)^2 - (\mathcal{D}_\alpha v)^2\right\|_{L^2} &\le \|\mathcal{D}_\alpha(u - v)\|_{L^4} \left(\|\mathcal{D}_\alpha u\|_{L^4} + \|\mathcal{D}_\alpha v\|_{L^4}\right) \\ &\le C_\alpha\big(\|u\|_{H^1} + \|v\|_{H^1}\big) \|u - v\|_{H^1}.\end{aligned}$$

This remains valid at $\alpha = 1/2$ and does not use the false endpoint assertion that $H^{1/2}$ is an algebra.

For the variational term, write

$$u\mathcal{D}_\alpha u - v\mathcal{D}_\alpha v = (u - v)\mathcal{D}_\alpha u + v\mathcal{D}_\alpha(u - v).$$

Since $1 - \alpha \ge \alpha$, estimate (3.1) gives

$$\|\mathcal{D}_\alpha w\|_{H^\alpha} \le C_\alpha \|w\|_{H^1}.$$

By Lemma 3.2,

$$\|u\mathcal{D}_\alpha u - v\mathcal{D}_\alpha v\|_{H^\alpha} \le C_\alpha\big(\|u\|_{H^1} + \|v\|_{H^1}\big) \|u - v\|_{H^1}.$$

Since $\mathcal{D}_\alpha : H^\alpha \to L^2$ is bounded, these estimates prove (4.2). Finally, (3.2) implies

$$\|\mathcal{N}_\alpha(u) - \mathcal{N}_\alpha(v)\|_{H^1} \le \|\mathcal{G}_\alpha(u) - \mathcal{G}_\alpha(v)\|_{L^2},$$

which gives (4.3). Taking $v = 0$ yields (4.4). □

Let

$$S(t) = e^{-t\partial_x^3}$$

be the Airy group. Since its Fourier multiplier has modulus one,

$$\|S(t)f\|_{H^1} = \|f\|_{H^1}. \tag{4.5}$$

The mild formulation of (1.7) is

$$u(t) = S(t)u_0 - \int_0^t S(t - \tau)\mathcal{N}_\alpha(u(\tau))\, d\tau. \tag{4.6}$$

Theorem 4.2. *Let $0 \le \alpha \le 1/2$ and $u_0 \in H^1(\mathbb{R})$. There exists $T = T_\alpha(\|u_0\|_{H^1}) > 0$ and a unique solution*

$$u \in C([-T,T]; H^1(\mathbb{R})) \cap C^1([-T,T]; H^{-2}(\mathbb{R}))$$

of (1.7). The data-to-solution map is locally Lipschitz. One may choose

$$T \ge \frac{c_\alpha}{1 + \|u_0\|_{H^1}}, \tag{4.7}$$

where $c_\alpha > 0$ depends only on α.

Proof. Let $X_T = C([-T,T]; H^1)$ with its supremum norm and define

$$\Phi(u)(t) = S(t)u_0 - \int_0^t S(t-\tau)\mathcal{N}_\alpha(u(\tau))\, d\tau.$$

By (4.5) and (4.4),

$$\|\Phi(u)\|_{X_T} \le \|u_0\|_{H^1} + C_\alpha T \|u\|_{X_T}^2 . \tag{4.8}$$

Similarly, (4.3) gives

$$\|\Phi(u) - \Phi(v)\|_{X_T} \le C_\alpha T\big(\|u\|_{X_T} + \|v\|_{X_T}\big) \|u - v\|_{X_T} . \tag{4.9}$$

Take $R = 2\|u_0\|_{H^1} + 1$. If T is chosen so that $C_\alpha T R \le 1/8$, then (4.8) and (4.9) show that Φ is a strict contraction on the closed ball of radius R in X_T. This proves existence, uniqueness and (4.7).

The same difference estimate gives locally Lipschitz dependence on the initial datum. Finally,

$$u_t = -u_{xxx} - \mathcal{N}_\alpha(u) \in C([-T,T]; H^{-2}),$$

which yields the asserted time regularity. Since all operators preserve real-valuedness, real initial data generate real solutions. □

Corollary 4.3. *For each $u_0 \in H^1(\mathbb{R})$ there exists a unique maximal solution on an interval $(-T_-, T_+)$. If $T_+ < \infty$, then*

$$\limsup_{t \uparrow T_+} \|u(t)\|_{H^1} = \infty,$$

and the analogous statement holds backward in time.

5. The conserved quadratic energy

Set

$$\mathcal{A} = 1 - \partial_x^2, \qquad \mathcal{K} = \mathcal{A}^{-1},$$

and define

$$\mathcal{E}(u) = \frac{1}{2}\langle \mathcal{A}u, u\rangle = \frac{1}{2}\int_{\mathbb{R}} (u^2 + u_x^2)\, dx. \tag{5.1}$$

5.1. **The formal cancellation.**

Proposition 5.1. *Let u be a sufficiently smooth, decaying real-valued solution of (1.7). Then*

$$\frac{d}{dt}\mathcal{E}(u(t)) = 0.$$

Proof. Using (1.7), the self-adjointness of $\mathcal{A}$ and $\mathcal{A}\mathcal{K} = I$, we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(u(t)) &= \langle \mathcal{A}u, u_t\rangle \\ &= -\langle \mathcal{A}u, u_{xxx}\rangle - \langle \mathcal{A}u, \mathcal{K}\partial_x \mathcal{G}_\alpha(u)\rangle \\ &= \int_{\mathbb{R}} u_x \mathcal{G}_\alpha(u)\, dx. \end{aligned} \tag{5.2}$$

The Airy contribution vanishes because

$$\langle \mathcal{A}u, u_{xxx}\rangle = \int u u_{xxx}\, dx - \int u_{xx} u_{xxx}\, dx = 0.$$

Substituting (1.6) into (5.2),

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(u(t)) = \frac{3}{2}\int u^2 u_x\, dx + \frac{1}{2}\int u_x (\mathcal{D}_\alpha u)^2\, dx \\ - \int u_x \mathcal{D}_\alpha (u \mathcal{D}_\alpha u)\, dx.\end{aligned}$$

The first integral is zero. For the last one, skew-adjointness and commutation with ∂_x give

$$\begin{aligned} -\int u_x \mathcal{D}_\alpha(u\mathcal{D}_\alpha u)\, dx &= \int \mathcal{D}_\alpha(u_x) u \mathcal{D}_\alpha u\, dx \\ &= \int u\, \partial_x(\mathcal{D}_\alpha u)\, \mathcal{D}_\alpha u\, dx \\ &= -\frac{1}{2}\int u_x (\mathcal{D}_\alpha u)^2\, dx. \end{aligned}$$

The remaining two terms cancel. □

The cancellation is also a consequence of translation invariance:

$$0 = \frac{d}{dy}\mathcal{V}_\alpha(u(\cdot + y))\Big|_{y=0} = \int_{\mathbb{R}} \mathcal{G}_\alpha(u) u_x\, dx.$$

5.2. **Justification at H^1 regularity.** The preceding computation contains products meaningful for H^1 functions, but u_t need only belong to H^{-2}. We therefore justify conservation through resolvent regularization. For $\delta > 0$, let

$$J_\delta = (1 - \delta \partial_x^2)^{-1}. \tag{5.3}$$

The operators J_δ are self-adjoint contractions on every H^s, commute with all Fourier multipliers in the equation and converge strongly to the identity as $\delta \downarrow 0$.

Lemma 5.2. *For every real-valued $u \in H^1(\mathbb{R})$,*

$$\langle u_x, \mathcal{G}_\alpha(u)\rangle_{L^2} = 0. \tag{5.4}$$

Proof. By (4.2), $\mathcal{G}_\alpha : H^1 \to L^2$ is continuous. Choose real-valued $u_n \in \mathcal{S}(\mathbb{R})$ with $u_n \to u$ in H^1. Then

$$\mathcal{G}_\alpha(u_n) \longrightarrow \mathcal{G}_\alpha(u) \quad \text{in } L^2.$$

For every n, Proposition 5.1 gives $\langle (u_n)_x, \mathcal{G}_\alpha(u_n)\rangle = 0$. Passing to the limit in $L^2 \times L^2$ proves (5.4). □

Theorem 5.3. *Let $u \in C([-T,T]; H^1)$ be the mild solution given by Theorem 4.2. Then*

$$\mathcal{E}(u(t)) = \mathcal{E}(u_0), \qquad |t| \le T. \tag{5.5}$$

Proof. Set $u_\delta = J_\delta u$ and

$$\mathcal{E}_\delta(t) = \frac{1}{2} \|u_\delta(t)\|_{H^1}^2 .$$

Since $u_t \in C([-T,T]; H^{-2})$ and J_δ gains two derivatives, u_δ is differentiable as an L^2-valued function. Using the commutation properties of J_δ, $\mathcal{A}$, $\mathcal{K}$ and ∂_x, we find

$$\begin{aligned} \frac{d}{dt}\mathcal{E}_\delta(t) &= \langle \mathcal{A} J_\delta u, J_\delta u_t\rangle \\ &= \langle \partial_x J_\delta u, J_\delta \mathcal{G}_\alpha(u)\rangle . \end{aligned} \tag{5.6}$$

The regularized Airy term vanishes by skew-adjointness. Integrating (5.6) from 0 to t gives

$$\mathcal{E}_\delta(t) - \mathcal{E}_\delta(0) = \int_0^t \langle J_\delta u_x(\tau), J_\delta \mathcal{G}_\alpha(u(\tau))\rangle \, d\tau. \tag{5.7}$$

Because $u \in C([-T,T]; H^1)$ and $\mathcal{G}_\alpha : H^1 \to L^2$ is continuous, the sets

$$\{u_x(t) : |t| \le T\}, \qquad \{\mathcal{G}_\alpha(u(t)) : |t| \le T\}$$

are compact in L^2. Strong convergence $J_\delta \to I$ is therefore uniform on these compact sets. Hence the integrand in (5.7) converges uniformly to $\langle u_x(\tau), \mathcal{G}_\alpha(u(\tau))\rangle = 0$ by Lemma 5.2. Also, $\mathcal{E}_\delta(t) \to \mathcal{E}(u(t))$ uniformly on $[-T,T]$. Passing to the limit $\delta \downarrow 0$ proves (5.5). □

6. Global well-posedness

Proof of Theorem 1.1. Let $(-T_-, T_+)$ be the maximal lifespan supplied by Corollary 4.3. Conservation of the quadratic energy gives

$$\|u(t)\|_{H^1} = \|u_0\|_{H^1}$$

throughout the maximal interval. The blow-up alternative excludes finite T_+ and finite T_-, so $T_+ = T_- = \infty$.

It remains to record global Lipschitz dependence on bounded time intervals. Let u and v be global solutions with initial data u_0, v_0 in a fixed ball of H^1. From (4.6) and (4.3),

$$\begin{aligned} \|u(t) - v(t)\|_{H^1} &\le \|u_0 - v_0\|_{H^1} \\ &\qquad + C_\alpha \int_0^{|t|} \left(\|u(\tau)\|_{H^1} + \|v(\tau)\|_{H^1}\right) \|u(\tau) - v(\tau)\|_{H^1} \, d\tau. \end{aligned}$$

The conserved norms and Grönwall's inequality imply, for $|t| \leq T$,

$$\|u(t) - v(t)\|_{H^1} \leq \exp(C_\alpha T(\|u_0\|_{H^1} + \|v_0\|_{H^1})) \|u_0 - v_0\|_{H^1}.$$

This proves the claimed locally Lipschitz property of the global flow. □

7. Conclusions and further perspectives

We introduced and analyzed the nonlocal dispersive equation

$$u_t + u_{xxx} + \mathcal{K}\partial_x \left[\frac{3}{2}u^2 + \frac{1}{2}(\mathcal{D}_\alpha u)^2 - \mathcal{D}_\alpha(u\mathcal{D}_\alpha u)\right] = 0, \qquad 0 \leq \alpha \leq \frac{1}{2}. \tag{7.1}$$

The model combines Airy dispersion, Helmholtz–Bessel averaging of the nonlinear force and a fractional filtered slope whose high-frequency order is α. Although (7.1) is proposed as a phenomenological model rather than as the unique asymptotic reduction of a particular continuum system, it provides a conservative and analytically tractable framework in which these mechanisms interact.

The first contribution is a complete local Cauchy theory in the natural energy space $H^1(\mathbb{R})$. The nonlinear vector field is locally Lipschitz from $H^1(\mathbb{R})$ into itself, so the Duhamel formulation associated with the Airy group can be solved by a direct contraction argument. This yields existence, uniqueness, persistence in H^1 and locally Lipschitz dependence on the initial datum.

The scope is deliberately focused on establishing a complete Cauchy theory at the natural energy regularity. More refined dynamical questions, including solitary waves, stability and scattering, require tools beyond the elementary Sobolev framework used here and are separated from the present global well-posedness result.

A relevant feature is the inclusion of the endpoint $\alpha = 1/2$. At this value one cannot use an algebra property of $H^{1/2}(\mathbb{R})$. Instead, the quadratic filtered-slope term is controlled through

$$\|\mathcal{D}_\alpha u\|_{L^4} \leq C \|u\|_{H^1},$$

which follows from the regularity supplied by the Bessel multiplier and the one-dimensional Sobolev embedding into $L^4(\mathbb{R})$. This endpoint argument is an essential analytical ingredient.

The second contribution is rigorous conservation of the coercive quadratic functional

$$\mathcal{E}(u(t)) = \frac{1}{2}\int_{\mathbb{R}} \left(u(x,t)^2 + u_x(x,t)^2\right) dx.$$

For smooth solutions, conservation follows from skew-adjointness of $\mathcal{D}_\alpha$, commutation with ∂_x and translation invariance of the cubic interaction. Since an H^1 solution satisfies the equation only in a negative Sobolev space, the formal calculation is not sufficient at the natural regularity. Resolvent regularization and continuity of $\mathcal{G}_\alpha : H^1(\mathbb{R}) \to L^2(\mathbb{R})$ justify the identity for mild solutions.

Coercivity of $\mathcal{E}$ rules out the blow-up alternative. Consequently, the Cauchy problem is globally well posed for every real-valued initial datum in $H^1(\mathbb{R})$, without a smallness assumption. The same estimates give a quantitative Lipschitz bound for the flow on bounded time intervals and bounded subsets of $H^1(\mathbb{R})$.

The restriction $0 \leq \alpha \leq 1/2$ should be interpreted as a threshold of the present H^1 argument, not necessarily as an optimal threshold for the equation. For $\alpha > 1/2$, the filtered-slope interaction loses too much regularity for the elementary Sobolev product estimates used here. It remains open whether Airy dispersion can compensate for this loss through Bourgain spaces, refined bilinear estimates or paradifferential methods.

Further questions include persistence and global bounds in higher Sobolev spaces; existence, decay and variational characterization of traveling waves; spectral and orbital stability of solitary waves; dependence of the flow on α and on an explicit internal length scale; and numerical analysis of wave speed, dispersive spreading and coherent-structure interactions. The present results provide a rigorous foundation for these developments.

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