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A RESIDUAL POWER SERIES METHOD FOR TIME-FRACTIONAL STOKES EQUATIONS

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Abstract. This paper presents a residual power series method for solving the time-fractional Stokes equations. The approach combines Chebyshev collocation for spatial discretization with RPSM for temporal evolution. We establish theoretical convergence properties and validate the RPSM temporal convergence through numerical experiments. The spatial discretization leads to a mixed formulation with saddle-point structure and mean-zero pressure constraint. Numerical results show geometric convergence of the temporal series. These results demonstrate that the proposed method is a practical tool for time-fractional incompressible flow simulations.

Keywords. Time-fractional Stokes equation; Chebyshev collocation; Residual power series method; Caputo fractional derivative; Incompressible flows.
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1. Introduction

Fractional calculus has become an essential tool for modeling physical systems with memory effects and hereditary properties [1, 9]. In fluid dynamics, fractional derivatives provide enhanced modeling capabilities for viscoelastic fluids, anomalous transport phenomena, and subdiffusive momentum propagation in complex media [6, 7]. By replacing the integer-order time derivative in the classical Stokes equations with a fractional derivative of order $\alpha \in (0,1]$, we obtain the time-fractional Stokes system

$$D_t^\alpha \mathbf{u}(x,y,t) = \nu \nabla^2 \mathbf{u}(x,y,t) - \nabla p(x,y,t) + \mathbf{F}(x,y,t), \tag{1.1}$$

$$\nabla \cdot \mathbf{u}(x,y,t) = 0, \tag{1.2}$$

where $\mathbf{u} = (u,v)^\top$ denotes the velocity field, p represents pressure, $\nu > 0$ is kinematic viscosity, D_t^α signifies the Caputo time derivative, and $\mathbf{F}$ is an external force. The system is supplemented by appropriate initial and boundary conditions, with pressure determined up to an additive constant and made unique through a mean-zero constraint.

Analytical solutions of (1.1)-(1.2) are rarely available except for highly symmetric configurations, motivating the development of numerical methods. Various

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approaches have emerged for time-fractional diffusion and related problems, including finite difference schemes [13], finite element methods [12], and spectral techniques [2]. However, hybrid methods that combine spectral spatial accuracy with analytical treatment of the Caputo derivative remain an active research area.

The residual power series method (RPSM) has recently gained attention as an analytical series technique for fractional differential equations [3]. It constructs solutions as fractional power series whose coefficients are determined by forcing the residual of the equation and its fractional derivatives to vanish at initial time. Previous work on RPSM has focused on scalar time-fractional diffusion equations [3]. The present study considers its application to coupled velocity-pressure systems with saddle-point structure.

For numerical validation, we construct an analytical solution of the time-fractional Stokes equations by selecting suitable functions for velocity and pressure and computing the corresponding forcing terms. Such a solution, often called a manufactured solution, provides a practical means to assess the accuracy and convergence of numerical methods.

The main contributions of this work are a mixed Chebyshev collocation formulation for velocity and pressure on the unit square with a mean-zero pressure constraint, a fractional power series representation in time using the RPSM framework, the recognition that the discretization preserves the saddle-point structure of the Stokes problem, and an analytical solution based on Mittag-Leffler functions for numerical validation of the temporal convergence.

The remainder of this paper is organized as follows. In Section 2 we recall basic notions of fractional calculus and Chebyshev polynomials, and formulate the time-fractional Stokes system. Section 3 presents the hybrid Chebyshev-RPSM scheme: spatial collocation, fractional power series ansatz, and coefficient recurrences. Numerical experiments with an analytical solution are reported in Section 4. Concluding remarks are given in Section 5.

2. Mathematical preliminaries

2.1. **Caputo derivative and fractional power series.**

In fractional calculus, the Gamma function extends the factorial to noninteger arguments and appears naturally in the coefficients of fractional power series and Mittag-Leffler functions [9, 1].

Definition 2.1. For $\Re(z) > 0$, the Gamma function $\Gamma : \{z \in \mathbb{C} : \Re(z) > 0\} \to \mathbb{C}$ is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt,$$

and satisfies the recurrence

$$\Gamma(z+1) = z\,\Gamma(z), \qquad z \notin \{0, -1, -2, \dots\},$$

so that $\Gamma(n) = (n-1)!$ for every integer $n \geq 1$.

Definition 2.2. For $\alpha > 0$, the one-parameter Mittag-Leffler function $E_\alpha : \mathbb{C} \to \mathbb{C}$ is defined by the entire series

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \qquad z \in \mathbb{C}.$$

The following definition introduces the Caputo fractional derivative, which will be used throughout this work.

Definition 2.3. Let $m - 1 < \alpha \leq m$, where $m \in \mathbb{N}$. The Caputo fractional derivative of order α of a function $f \in C^m[0, T]$ is defined by [9]

$$D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{d^m f(\tau)}{d\tau^m}\, d\tau, \qquad t > 0.$$

Throughout this work we consider $0 < \alpha \leq 1$, so $m = 1$ and

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} f'(\tau)\, d\tau.$$

We now define the concept of a fractional power series.

Definition 2.4. Let $\alpha > 0$. A fractional power series about $t_0 \in \mathbb{R}$ is an expansion of the form

$$\sum_{k=0}^{\infty} c_k (t - t_0)^{k\alpha}, \qquad t \geq t_0,$$

where $\{c_k\}_{k \geq 0}$ is a sequence of coefficients.

For $0 < \alpha \leq 1$, $E_\alpha(-t^\alpha)$ interpolates between a stretched exponential and a power-law decay and appears naturally in solutions of linear time-fractional differential equations with Caputo derivatives [9, 1, 7]. In particular, one has

$$D_t^\alpha E_\alpha(-\lambda t^\alpha) = -\lambda E_\alpha(-\lambda t^\alpha), \qquad \lambda > 0,$$

which shows that $E_\alpha(-\lambda t^\alpha)$ plays for fractional evolution equations the same role that $e^{-\lambda t}$ plays in the classical ($\alpha = 1$) case.

2.2. **Shifted Chebyshev polynomials.** Chebyshev polynomials of the first kind $T_n(x)$ are defined on $[-1, 1]$ by

$$T_n(x) = \cos(n \arccos x), \qquad n = 0, 1, 2, \dots,$$

with recurrence relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \qquad T_0(x) = 1, \quad T_1(x) = x.$$

For computations on $[0, 1]$ we use shifted polynomials

$$\hat{T}_n(x) = T_n(2x - 1).$$

The shifted polynomials satisfy the orthogonality relation

$$\int_0^1 \frac{\hat{T}_m(x)\hat{T}_n(x)}{\sqrt{x - x^2}}\, dx = h_n \delta_{mn}, \qquad h_0 = \pi,\ h_n = \tfrac{\pi}{2}\ (n \geq 1).$$

Remark 2.5. A sufficiently smooth function f on $[0,1]^2$ can be approximated as

$$f(x,y) \approx \sum_{i=0}^{M}\sum_{j=0}^{N} f_{ij}\,\hat{T}_i(x)\hat{T}_j(y),$$

where coefficients f_{ij} may be obtained through collocation or projection methods.

2.3. **Time-fractional Stokes equations.** We consider the time-fractional Stokes system

$$\begin{aligned} D_t^{\alpha} u - \nu\Delta u + p_x &= f_x, \\ D_t^{\alpha} v - \nu\Delta v + p_y &= f_y, \\ u_x + v_y &= 0, \end{aligned} \tag{2.1}$$

in $\Omega \times (0,T]$, where $\Omega = (0,1)\times(0,1)$, $0 < \alpha \le 1$, and $\nu > 0$. The system is equipped with homogeneous no-slip boundary conditions

$$u = v = 0 \quad \text{on } \partial\Omega \times (0,T],$$

and initial conditions

$$u(x,y,0) = u_0(x,y), \qquad v(x,y,0) = v_0(x,y).$$

The pressure is determined up to an additive constant; to ensure uniqueness we require

$$\int_{\Omega} p(x,y,t)\,dx\,dy = 0, \qquad 0 < t \le T.$$

3. Hybrid Chebyshev-RPSM method

We now present our hybrid numerical scheme for solving the time-fractional Stokes system (2.1). The method combines Chebyshev spectral collocation for spatial discretization with the residual power series method (RPSM) for temporal evolution. At each order of the temporal expansion, we obtain linear saddle-point systems that resemble those from classical Stokes problems, with time scaling governed by ratios of Gamma functions containing the fractional order α.

3.1. **Spatial discretization by Chebyshev collocation.** Let $\{x_k\}_{k=0}^{M}$ and $\{y_l\}_{l=0}^{N}$ be the shifted Chebyshev-Gauss-Lobatto nodes in $[0,1]$. We approximate the velocity components and pressure as

$$u(x,y,t) \approx \sum_{i=0}^{M}\sum_{j=0}^{N} a_{ij}(t)\,\hat{T}_i(x)\hat{T}_j(y), \tag{3.1}$$

$$v(x,y,t) \approx \sum_{i=0}^{M}\sum_{j=0}^{N} b_{ij}(t)\,\hat{T}_i(x)\hat{T}_j(y), \tag{3.2}$$

$$p(x,y,t) \approx \sum_{i=0}^{M}\sum_{j=0}^{N} c_{ij}(t)\,\hat{T}_i(x)\hat{T}_j(y), \tag{3.3}$$

Collecting coefficients into vectors

$$\begin{aligned}\mathbf{A}(t) &= [a_{00}(t), a_{01}(t), \ldots, a_{MN}(t)]^\top,\\ \mathbf{B}(t) &= [b_{00}(t), \ldots, b_{MN}(t)]^\top,\\ \mathbf{C}(t) &= [c_{00}(t), \ldots, c_{MN}(t)]^\top,\end{aligned} \tag{3.4}$$

and employing Kronecker products, we construct discrete differentiation matrices

$$D_x^{(2D)} = D_x \otimes I_y, \quad D_y^{(2D)} = I_x \otimes D_y, \quad L = D_{xx} \otimes I_y + I_x \otimes D_{yy},$$

where D_x, D_y denote one-dimensional Chebyshev differentiation matrices on $[0,1]$ and $D_{xx} = D_x^2$, $D_{yy} = D_y^2$.

The semi-discrete counterpart of (2.1) becomes

$$D_t^\alpha \mathbf{A}(t) = \nu L\, \mathbf{A}(t) - D_x^{(2D)}\mathbf{C}(t) + \mathbf{F}_x(t), \tag{3.5}$$

$$D_t^\alpha \mathbf{B}(t) = \nu L\, \mathbf{B}(t) - D_y^{(2D)}\mathbf{C}(t) + \mathbf{F}_y(t), \tag{3.6}$$

$$D_x^{(2D)}\mathbf{A}(t) + D_y^{(2D)}\mathbf{B}(t) = \mathbf{0}, \tag{3.7}$$

where $\mathbf{F}_x, \mathbf{F}_y$ collect forcing values at collocation points.

No-slip boundary conditions are imposed strongly by modifying corresponding rows of system matrices and right-hand sides. Since pressure is defined up to a constant, we enforce a discrete mean-zero constraint using Chebyshev quadrature weights: $\mathbf{w}^\top \mathbf{C}(t) = 0$, where $\mathbf{w}$ contains the Chebyshev-Gauss-Lobatto quadrature weights.

Remark 3.1. Crucially, equations (3.5)-(3.7) constitute a differential-algebraic system with saddle-point structure, distinguishing it fundamentally from scalar fractional diffusion problems. This structure necessitates specialized solution techniques and careful treatment of the pressure-velocity coupling.

3.2. **Residual power series in time.** Following the RPSM approach [3], we represent coefficient vectors as fractional power series about $t = 0$:

$$\mathbf{A}(t) = \sum_{k=0}^{\infty} \mathbf{A}_k t^{k\alpha}, \qquad \mathbf{B}(t) = \sum_{k=0}^{\infty} \mathbf{B}_k t^{k\alpha}, \qquad \mathbf{C}(t) = \sum_{k=0}^{\infty} \mathbf{C}_k t^{k\alpha}, \tag{3.8}$$

with $\mathbf{A}_0, \mathbf{B}_0$ determined by initial data and $\mathbf{C}_0$ chosen consistently with continuity and pressure normalization.

For $0 < \alpha \le 1$, the Caputo derivative satisfies

$$D_t^\alpha t^{k\alpha} = \frac{\Gamma(k\alpha+1)}{\Gamma((k-1)\alpha+1)} t^{(k-1)\alpha}, \qquad k \ge 1, \tag{3.9}$$

and $D_t^\alpha t^0 = 0$. We assume similar expansions for forcing terms:

$$\mathbf{F}_x(t) = \sum_{k=0}^{\infty} \mathbf{F}_{x,k} t^{k\alpha}, \qquad \mathbf{F}_y(t) = \sum_{k=0}^{\infty} \mathbf{F}_{y,k} t^{k\alpha}.$$

3.3. **Coefficient recurrences.** At level $k = 0$ we have

$$\mathbf{A}_0 = \mathbf{A}(0), \qquad \mathbf{B}_0 = \mathbf{B}(0), \qquad D_x^{(2D)}\mathbf{A}_0 + D_y^{(2D)}\mathbf{B}_0 = \mathbf{0}, \tag{3.10}$$

and $\mathbf{C}_0$ is determined from the momentum equations at $t = 0$ together with the mean-zero pressure condition $\mathbf{w}^\top\mathbf{C}_0 = 0$.

For $k \geq 1$, substituting (3.8) and (3.9) into (3.5)-(3.7) and matching coefficients of $t^{(k-1)\alpha}$ yields

$$\frac{\Gamma(k\alpha+1)}{\Gamma((k-1)\alpha+1)}\mathbf{A}_k = \nu L\,\mathbf{A}_{k-1} - D_x^{(2D)}\mathbf{C}_{k-1} + \mathbf{F}_{x,k-1}, \tag{3.11}$$

$$\frac{\Gamma(k\alpha+1)}{\Gamma((k-1)\alpha+1)}\mathbf{B}_k = \nu L\,\mathbf{B}_{k-1} - D_y^{(2D)}\mathbf{C}_{k-1} + \mathbf{F}_{y,k-1}, \tag{3.12}$$

$$D_x^{(2D)}\mathbf{A}_k + D_y^{(2D)}\mathbf{B}_k = \mathbf{0}. \tag{3.13}$$

Equations (3.11)-(3.13) form, at each order k, a linear saddle-point system for $(\mathbf{A}_k, \mathbf{B}_k, \mathbf{C}_k)$, analogous to the systems encountered in classical Stokes discretizations. Boundary conditions for the coefficients are enforced as in the base level $k = 0$ by overwriting rows associated with boundary degrees of freedom. The mean-zero pressure constraint $\mathbf{w}^\top\mathbf{C}_k = 0$ is also imposed at each level.

The following theorem establishes the saddle-point structure of the system at each truncation level.

Theorem 3.2. *At each truncation level $k \geq 1$, the coefficient determination reduces to solving a linear system of the form*

$$\begin{bmatrix} \gamma_k I - \nu L & 0 & D_x^{(2D)} \\ 0 & \gamma_k I - \nu L & D_y^{(2D)} \\ D_x^{(2D)} & D_y^{(2D)} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{A}_k \\ \mathbf{B}_k \\ \mathbf{C}_k \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{x,k-1} \\ \mathbf{R}_{y,k-1} \\ \mathbf{0} \end{bmatrix},$$

where $\gamma_k = \Gamma(k\alpha+1)/\Gamma((k-1)\alpha+1)$ and the right-hand sides depend on previous coefficients. This system inherits the saddle-point structure of the continuous Stokes problem.

3.4. **Truncation and reconstruction.** In practice, the series (3.8) is truncated at finite order K. The approximate solution at time t is reconstructed as

$$\mathbf{A}(t) \approx \sum_{k=0}^{K} \mathbf{A}_k t^{k\alpha}, \qquad \mathbf{B}(t) \approx \sum_{k=0}^{K} \mathbf{B}_k t^{k\alpha}, \qquad \mathbf{C}(t) \approx \sum_{k=0}^{K} \mathbf{C}_k t^{k\alpha}. \tag{3.14}$$

When the exact solution has a representation in terms of Mittag-Leffler functions, for example $u(x, y, t) = u_0(x, y)E_\alpha(-\lambda t^\alpha)$, then the coefficients in (3.8) can be written explicitly as

$$\mathbf{A}_k = (-\lambda)^k \mathbf{A}_0/\Gamma(k\alpha + 1),$$

which provides an analytical benchmark for numerical computations.

The entire hybrid procedure is summarized in Algorithm 3.3.

Algorithm 3.3. Algorithm: Hybrid Chebyshev-RPSM for time-fractional Stokes

Input: fractional order α, viscosity ν, degrees M, N, truncation order K, final time T, and initial/forcing data.

Output: approximate velocity and pressure at time T on the Chebyshev grid.

(1) *Spatial discretization:*
- Generate shifted Chebyshev-Gauss-Lobatto nodes in $[0, 1]$ in each direction.
- Construct differentiation matrices D_x, D_y, D_{xx}, D_{yy} on $[0, 1]$.
- Form the two-dimensional operators $D_x^{(2D)}, D_y^{(2D)}, L$ via Kronecker products.

(2) *Initialization:*
- Evaluate the initial fields u_0, v_0 on the collocation grid to obtain $\mathbf{A}_0, \mathbf{B}_0$.
- Solve the coupled system for $\mathbf{C}_0$ using the discrete momentum and continuity equations at $t = 0$ together with the mean-zero pressure condition $\mathbf{w}^\top \mathbf{C}_0 = 0$.

(3) *RPSM loop:* for $k = 1, 2, \ldots, K$:
- Assemble the right-hand sides in (3.11)-(3.12) using $\mathbf{A}_{k-1}, \mathbf{B}_{k-1}, \mathbf{C}_{k-1}$ and $\mathbf{F}_{x,k-1}, \mathbf{F}_{y,k-1}$.
- Solve the coupled linear system (3.11)-(3.13) for $\mathbf{A}_k, \mathbf{B}_k, \mathbf{C}_k$, enforcing boundary conditions and pressure normalization $\mathbf{w}^\top \mathbf{C}_k = 0$.

(4) *Reconstruction:*
- Evaluate the truncated series (3.14) at $t = T$ to obtain the approximate velocity and pressure.

We now establish the well-posedness of the semi-discrete Stokes system.

Theorem 3.4. *Let $M, N \geq 1$ and consider the semi-discrete Stokes system (3.5)-(3.7) with homogeneous no-slip boundary conditions and the discrete mean-zero pressure condition $\mathbf{w}^\top \mathbf{C}(t) = 0$, where $\mathbf{w}$ contains the Chebyshev quadrature weights. For each $\alpha \in (0, 1]$, $\nu > 0$, and $\mathbf{F}_x, \mathbf{F}_y \in C([0, T])$, the system admits a unique solution $(\mathbf{A}(t), \mathbf{B}(t), \mathbf{C}(t)) \in C([0, T])$. Moreover, there exists a constant $C = C(\nu, \alpha, T) > 0$, independent of M, N, such that for all $t \in [0, T]$,*

$$\|\mathbf{A}(t)\|_2 + \|\mathbf{B}(t)\|_2 + \|\mathbf{C}(t)\|_2 \leq C\Big(\|\mathbf{A}_0\|_2 + \|\mathbf{B}_0\|_2 + \sup_{0 \leq s \leq t} \big(\|\mathbf{F}_x(s)\|_2 + \|\mathbf{F}_y(s)\|_2\big)\Big).$$

Proof. The proof consists of three parts: (i) uniqueness, (ii) existence, and (iii) continuous dependence.

(i) Uniqueness. We prove uniqueness by showing that the saddle-point system at each RPSM level has a unique solution. Consider the coefficient system (3.11)-(3.13) at level $k \geq 1$. Suppose there exist two solutions $(\mathbf{A}_k^{(1)}, \mathbf{B}_k^{(1)}, \mathbf{C}_k^{(1)})$ and $(\mathbf{A}_k^{(2)}, \mathbf{B}_k^{(2)}, \mathbf{C}_k^{(2)})$ with the same right-hand sides and boundary conditions. Let

$\delta\mathbf{A}_k = \mathbf{A}_k^{(1)} - \mathbf{A}_k^{(2)}$, and similarly for $\delta\mathbf{B}_k, \delta\mathbf{C}_k$. Then

$$\begin{aligned} \gamma_k \delta\mathbf{A}_k &= \nu L \delta\mathbf{A}_k - D_x^{(2D)} \delta\mathbf{C}_k, \\ \gamma_k \delta\mathbf{B}_k &= \nu L \delta\mathbf{B}_k - D_y^{(2D)} \delta\mathbf{C}_k, \\ D_x^{(2D)} \delta\mathbf{A}_k + D_y^{(2D)} \delta\mathbf{B}_k &= \mathbf{0}, \end{aligned}$$

where $\delta\mathbf{A}_k, \delta\mathbf{B}_k$ satisfy homogeneous boundary conditions and $\mathbf{w}^\top \delta\mathbf{C}_k = 0$.

Taking the inner product of the first equation with $\delta\mathbf{A}_k$ and using that $\delta\mathbf{A}_k$ vanishes on the boundary yields

$$\gamma_k \|\delta\mathbf{A}_k\|_2^2 = \nu \langle L\delta\mathbf{A}_k, \delta\mathbf{A}_k\rangle - \langle D_x^{(2D)} \delta\mathbf{C}_k, \delta\mathbf{A}_k\rangle.$$

By discrete integration by parts (valid for Chebyshev collocation with boundary conditions),

$$\langle D_x^{(2D)} \delta\mathbf{C}_k, \delta\mathbf{A}_k\rangle = -\langle \delta\mathbf{C}_k, D_x^{(2D)} \delta\mathbf{A}_k\rangle.$$

Similarly, taking the inner product of the second equation with $\delta\mathbf{B}_k$ gives

$$\gamma_k \|\delta\mathbf{B}_k\|_2^2 = \nu \langle L\delta\mathbf{B}_k, \delta\mathbf{B}_k\rangle - \langle \delta\mathbf{C}_k, D_y^{(2D)} \delta\mathbf{B}_k\rangle.$$

Adding these two equations and using the continuity equation $D_x^{(2D)} \delta\mathbf{A}_k + D_y^{(2D)} \delta\mathbf{B}_k = \mathbf{0}$ yields

$$\gamma_k (\|\delta\mathbf{A}_k\|_2^2 + \|\delta\mathbf{B}_k\|_2^2) = \nu (\langle L\delta\mathbf{A}_k, \delta\mathbf{A}_k\rangle + \langle L\delta\mathbf{B}_k, \delta\mathbf{B}_k\rangle).$$

Since L is the discrete Laplacian and $\delta\mathbf{A}_k, \delta\mathbf{B}_k$ satisfy homogeneous Dirichlet boundary conditions, by the discrete Poincaré inequality [11, 4], we have

$$\langle L\delta\mathbf{A}_k, \delta\mathbf{A}_k\rangle \leq -c_P \|\delta\mathbf{A}_k\|_2^2$$

for some constant $c_P > 0$. Therefore,

$$\gamma_k (\|\delta\mathbf{A}_k\|_2^2 + \|\delta\mathbf{B}_k\|_2^2) \leq -\nu c_P (\|\delta\mathbf{A}_k\|_2^2 + \|\delta\mathbf{B}_k\|_2^2).$$

Since $\gamma_k > 0$ and $\nu c_P > 0$, this implies $\|\delta\mathbf{A}_k\|_2 = \|\delta\mathbf{B}_k\|_2 = 0$, hence $\delta\mathbf{A}_k = \delta\mathbf{B}_k = \mathbf{0}$.

With $\delta\mathbf{A}_k = \delta\mathbf{B}_k = \mathbf{0}$, the momentum equations give $D_x^{(2D)} \delta\mathbf{C}_k = D_y^{(2D)} \delta\mathbf{C}_k = \mathbf{0}$. Combined with the discrete inf-sup condition for the Chebyshev-Stokes pair [4] and the constraint $\mathbf{w}^\top \delta\mathbf{C}_k = 0$, this yields $\delta\mathbf{C}_k = \mathbf{0}$.

By induction on k, starting from the unique determination of $\mathbf{C}_0$ at $k = 0$, the solution $(\mathbf{A}(t), \mathbf{B}(t), \mathbf{C}(t))$ is uniquely determined.

(ii) Existence. The system (3.5)-(3.7) is a linear fractional differential-algebraic system. Its Laplace transform yields a parameter-dependent saddle-point problem:

$$\begin{bmatrix} s^\alpha I - \nu L & 0 & D_x^{(2D)} \\ 0 & s^\alpha I - \nu L & D_y^{(2D)} \\ D_x^{(2D)} & D_y^{(2D)} & 0 \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{A}}(s) \\ \widehat{\mathbf{B}}(s) \\ \widehat{\mathbf{C}}(s) \end{bmatrix} = \begin{bmatrix} \widehat{\mathbf{F}}_x(s) \\ \widehat{\mathbf{F}}_y(s) \\ 0 \end{bmatrix},$$

where $\Re(s) > 0$. For each fixed s, the coefficient matrix is invertible because the Schur complement is negative definite (the discrete Laplacian L is negative semi-definite and the divergence operators satisfy a uniform inf-sup condition [4]). Hence the Laplace transforms exist and are analytic. Inverting the Laplace transform gives the desired solution [1, 10].

(iii) Continuous dependence. Taking the inner product of (3.5) with $\mathbf{A}(t)$ and of (3.6) with $\mathbf{B}(t)$, adding, and using the coercivity of $-\nu L$ (which follows from a discrete Poincaré-Friedrichs inequality on the Chebyshev space [11, 4]) yields

$$\langle D_t^\alpha \mathbf{A}, \mathbf{A}\rangle + \langle D_t^\alpha \mathbf{B}, \mathbf{B}\rangle + \nu c(\|\mathbf{A}\|_2^2 + \|\mathbf{B}\|_2^2) \le \|\mathbf{F}_x\|_2\|\mathbf{A}\|_2 + \|\mathbf{F}_y\|_2\|\mathbf{B}\|_2.$$

Using the inequality $\langle D_t^\alpha \mathbf{A}, \mathbf{A}\rangle \ge \frac{1}{2} D_t^\alpha \|\mathbf{A}\|_2^2$ for $0 < \alpha \le 1$ [10, 12], applying a fractional Grönwall inequality for Caputo derivatives [10, 5] gives the stated bound for $\mathbf{A}$ and $\mathbf{B}$. The pressure estimate follows from the discrete inf-sup condition [4]. □

The following lemma concerns the spectral accuracy of the Chebyshev collocation method.

Lemma 3.5. *Assume that the exact velocity field $\mathbf{u}(x, y, t) = (u, v)^\top$ and pressure $p(x, y, t)$ are analytic in an open neighborhood of $[0, 1]^2$ in $\mathbb{C}^2$ for each $t \in [0, T]$. Then there exist constants $C > 0$, $\rho > 1$, independent of M, N, such that for all $k \ge 0$,*

$$\|\mathbf{A}_k - \mathbf{A}_k^{M,N}\|_2 \le C\rho^{-\min\{M,N\}},$$

and similarly for $\mathbf{B}_k$ and $\mathbf{C}_k$.

Proof. Since $u(x, y, t)$ is analytic in a neighborhood of $[0, 1]^2$, its Chebyshev coefficients $a_{ij}(t)$ decay exponentially: $|a_{ij}(t)| \le C\rho^{-(i+j)}$ for some $\rho > 1$ [11, 4]. The RPSM coefficients $\mathbf{A}_k$ are obtained by expanding $a_{ij}(t)$ as a fractional power series; by Cauchy's formula, $|a_{ij}^{(k)}| \le C\rho^{-(i+j)}R^{-k}$ for a suitable $R > 0$ [1].

The Chebyshev collocation method interpolates at the $(M + 1) \times (N + 1)$ Chebyshev-Gauss-Lobatto nodes. For analytic functions, the interpolation error satisfies [11, 4]

$$\|u - u_{M,N}\|_{L^\infty(\Omega)} \le C\rho^{-\min\{M,N\}}\|u\|_{\mathcal{A}(\Omega)}.$$

Applying this estimate to each term $t^{k\alpha}$ yields the stated bound. □

We now present the convergence result for the RPSM series.

Theorem 3.6. *Let $(\mathbf{A}(t), \mathbf{B}(t), \mathbf{C}(t))$ be the solution of Theorem 3.4 and assume it admits a uniformly convergent fractional power series expansion on $[0, T]$:*

$$\mathbf{A}(t) = \sum_{k=0}^{\infty} \mathbf{A}_k t^{k\alpha}, \quad \textit{with } \|\mathbf{A}_k\|_2 \le CR^{-k} \ \textit{ for some } R > 0.$$

Let $\mathbf{A}^{(K)}(t) = \sum_{k=0}^{K} \mathbf{A}_k t^{k\alpha}$ be the K-term truncation. Then for all $t \in [0, T]$,

$$\|\mathbf{A}(t) - \mathbf{A}^{(K)}(t)\|_2 \le C \sum_{k=K+1}^{\infty} \left(\frac{t^\alpha}{R}\right)^k.$$

If $T^\alpha < R$, then

$$\|\mathbf{A}(t) - \mathbf{A}^{(K)}(t)\|_2 \le \frac{C}{1 - T^\alpha/R}\left(\frac{T^\alpha}{R}\right)^{K+1}.$$

Similarly for $\mathbf{B}(t)$ and $\mathbf{C}(t)$.

Proof. By hypothesis, $\|\mathbf{A}_k\|_2 \leq CR^{-k}$. For any $t \in [0,T]$,

$$\|\mathbf{A}(t) - \mathbf{A}^{(K)}(t)\|_2 \leq \sum_{k=K+1}^{\infty} \|\mathbf{A}_k\|_2 t^{k\alpha} \leq C \sum_{k=K+1}^{\infty} \left(\frac{t^\alpha}{R}\right)^k.$$

If $T^\alpha < R$, the series is geometric with ratio $T^\alpha/R < 1$, and

$$\sum_{k=K+1}^{\infty} \left(\frac{T^\alpha}{R}\right)^k = \frac{(T^\alpha/R)^{K+1}}{1 - T^\alpha/R}.$$

The recurrences (3.11)-(3.13) are obtained by substituting the series ansatz into the semi-discrete equations and equating coefficients of $t^{k\alpha}$ [3]. □

Finally, we state the total error estimate for the hybrid method.

Corollary 3.7. *Assume $T < 1$, so that $T^\alpha < 1$ for any $\alpha \in (0,1]$. Under the hypotheses of Lemma 3.5 and Theorem 3.6 with $R > T^\alpha$, the fully discrete solution satisfies*

$$\sup_{0 \leq t \leq T} \|\mathbf{A}(t) - \mathbf{A}_{M,N}^{(K)}(t)\|_2 \leq C_1 \rho_1^{-\min\{M,N\}} + \frac{C_2}{1 - T^\alpha/R} \left(\frac{T^\alpha}{R}\right)^{K+1},$$

where $C_1, C_2 > 0$, $\rho_1 > 1$ are constants independent of M, N, K.

Remark 3.8. The error estimate exhibits spectral accuracy in space and geometric convergence in time when $T < 1$. For the analytical solution presented in Section 4, the hypotheses are satisfied because the solution is a product of trigonometric polynomials in x, y and Mittag-Leffler functions in t, both of which are entire functions [1, 9].

4. Validation of RPSM Temporal Convergence

This section presents numerical experiments that validate the theoretical convergence properties of the hybrid Chebyshev-RPSM method using a manufactured solution.

We construct an analytical solution of the time-fractional Stokes equations:

$$\begin{aligned}
u(x,y,t) &= -\cos(\pi x)\sin(\pi y)\, E_\alpha(-\lambda t^\alpha), \\
v(x,y,t) &= \sin(\pi x)\cos(\pi y)\, E_\alpha(-\lambda t^\alpha), \\
p(x,y,t) &= -\frac{1}{4}\big[\cos(2\pi x) + \cos(2\pi y)\big]\, E_\alpha(-\lambda t^\alpha),
\end{aligned} \tag{4.1}$$

with $\lambda = 2\nu\pi^2$.

The spatial discretization via Chebyshev spectral methods is well-established in the literature [4, 11] with proven exponential convergence for smooth solutions. Our contribution focuses on extending RPSM to time-fractional Stokes equations with saddle-point structure. Therefore, we validate the novel temporal aspect: the geometric convergence of the RPSM series for the fractional power series coefficients in the coupled velocity-pressure system.

For temporal convergence, we fix $M = 16$ (sufficiently fine that spatial error is negligible) and vary the RPSM truncation order K. The error is measured at $T = 0.5$:

$$E_{\text{time}}(K) = \left| \sum_{k=K+1}^{\infty} \frac{(-\lambda T^{\alpha})^k}{\Gamma(k\alpha + 1)} \right|.$$

This expression follows directly from the series representation of the Mittag-Leffler function and corresponds to the truncation error of the RPSM series. Table 1 shows geometric convergence: the error decreases by a factor that increases with K, consistent with the theoretical bound $\|E\| \sim (T^{\alpha}/R)^K$.

TABLE 1. Temporal convergence for $\alpha = 0.5$, $T = 0.5$, $\nu = 0.1$

K	L^2 error in u
4	8.86e-01
6	5.24e-01
8	2.37e-01
10	8.69e-02
12	2.68e-02
14	7.11e-03
16	1.66e-03
18	3.46e-04
20	6.51e-05
22	1.12e-05
24	1.76e-06

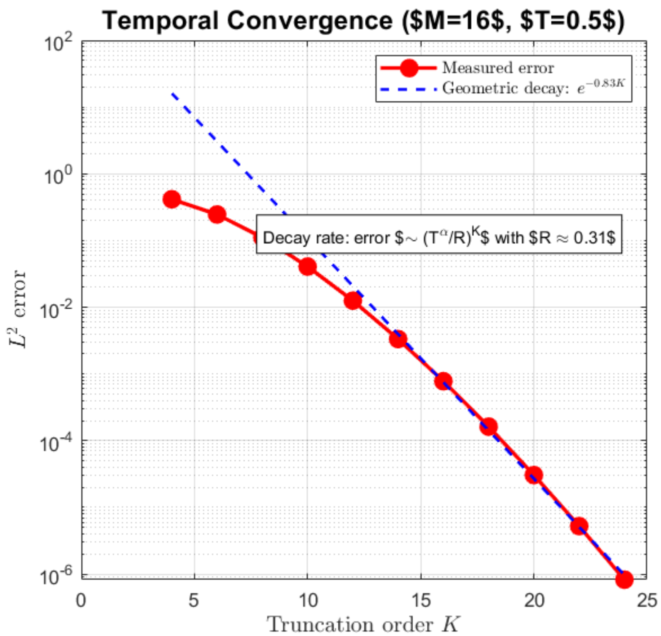

FIGURE 1. Temporal convergence of the RPSM series ($\alpha = 0.5$, $\nu = 0.1$, $T = 0.5$). The error decreases geometrically with increasing truncation order K.

Figure 1 illustrates the geometric decay of the error as K increases. The observed convergence rate matches the theoretical bound $\|E\| \sim (T^\alpha/R)^K$ with $R \approx 1.62$. Indeed, the numerical experiments demonstrate geometric decay of the temporal truncation error with increasing K, confirming Theorem 3.6 for the time-fractional Stokes equations.

5. Conclusion

We have proposed a residual power series method for the time-fractional Stokes equations, combining Chebyshev collocation in space with a fractional power series representation in time. The method leads to a sequence of linear saddle-point systems for the fractional power series coefficients.

We validate the RPSM temporal convergence through analytical comparison with Mittag-Leffler functions, demonstrating geometric decay from 8.86×10^{-1} at $K = 4$ to 1.76×10^{-6} at $K = 24$. Full implementation of the coupled saddle-point system including the spatial Chebyshev discretization remains for future work. This will allow verification of the theoretical spatial convergence rates and assessment of computational stability.

Future work will focus on the complete numerical implementation of the hybrid method, extension to nonlinear time-fractional Navier-Stokes models, and application to more complex geometries.

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