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ON THE PRIMARY SPECTRUM OF LATTICE MODULES

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Abstract. Consider a lattice module M over a $C-$lattice L. A proper element P of a lattice module M is called primary if $aA \leq P$ implies $A \leq P$ or $a^n 1_M \leq P$ for some positive integer n, for all $a \in L$ and $A \in M$. In this work, we construct a topological framework on $Spec^{pr}(M)$, the family of all primary elements of M. We examine several topological properties of this space, particularly compactness and irreducibility, and establish criteria under which $Spec^{pr}(M)$ satisfies the conditions of a spectral space.

Keywords. Lattice modules, Primary elements, Primary spectrum, Primary Zariski topology, Spectral spaces
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1. Introduction

Several extensions of the Zariski topology defined on the collection of prime submodules of an $R-$module have been explored in previous studies [5, 6, 17]. In [7], Behboodi and Noori proposed a Zariski-type topology for the classical prime spectrum of a module. Building on these ideas, Girase et al.[13] extended the Zariski topology framework to the classical prime spectrum of lattice modules, while Borkar et al.[9] developed a corresponding topology for the prime spectrum of lattice modules. Bijari et al.[8] introduced and studied a primary Zariski topology over the spectrum of primary submodules of a module. Motivated by these contributions, the present work studies primary Zariski topology on the spectrum of primary elements of lattice module M over a $C-$lattice L extending many of the results obtained in [8].

Throughout this work, let L denote a complete multiplicative lattice with least element 0_L and the greatest element 1_L. The lattice L is assumed to possess a multiplication operation "." satisfying the conditions:
(1) $x.y = y.x$, for all $x, y \in L$;
(2) $x.(y.z) = (x.y).z$, for all $x, y, z \in L$;
(3) $x.(\vee_\alpha y_\alpha) = \vee_\alpha (x.y_\alpha)$, for all $x, y_\alpha \in L$;
(4) $x.1_L = x$, for all $x \in L$.

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A compact element $x \in L$ is an element x satisfying $x \leq \vee_{\alpha \in I} y_\alpha$ implies $x \leq y_{\alpha_1} \vee y_{\alpha_2} \vee \cdots \vee y_{\alpha_n}$ for some subset $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ of I. A $C-$lattice is a multiplicative lattice generated by its compact elements under arbitrary joins, where the multiplicatively closed set of compact elements contains the identity element. For more details on $C-$lattice the reader may refer ([12],[16],[24]-[25]).

An element $x \in L$ is called *proper*, if $x < 1_L$. A proper element $q \in L$ is said to be *prime* provided that, for arbitrary $x, y \in L$, $xy \leq q$ implies $x \leq q$ or $y \leq q$. $Spec(L)$ denotes the collection of all prime elements of L.

The Zariski topology on the set $Spec(L)$ of all prime elements in multiplicative lattices is being studied in [24], by Thakare, Manjarekar and Maeda and in [23], by Thakare and Manjarekar as a generalization of the Zariski topology of a commutative ring with unity.
A proper element $m \in L$ is called *maximal* if there is no proper element lying strictly above m. Equivalently, $m < x \leq 1_L$ forces $x = 1_L$. The radical of an element $a \in L$ is defined as $\sqrt{a} = \vee\{x \in L | x$ is compact and $x^n \leq a$, for some positive integer $n\}$[3].

Let M be a complete lattice with least element 0_M and greatest element 1_M. We say that M is a lattice module over the multiplicative lattice L, or simply an $L-$module, if there exists a multiplication $L \times M \longrightarrow M,\ (x, A) \longmapsto xA$ satisfying the following conditions:

(1) $(xy)A = x(yA)$;
(2) $(\vee_\alpha x_\alpha)(\vee_\beta A_\beta) = (\vee_{\alpha\beta} x_\alpha A_\beta)$;
(3) $1_L A = A$;
(4) $0_L A = 0_M$; for all $x, y, x_\alpha \in L$, and for all $A, A_\beta \in M$.

Throughout, let M be a lattice module over a $C-$lattice L. For $A \in M, y \in L$, define the residual operation by $(A : y) = \bigvee\{K \in M \mid yK \leq A\}$. For $x, y \in L$, the corresponding residual in L is written as $(x : y) = \bigvee\{a \in L \mid ay \leq x\}$. Furthermore, if $A, B \in M$, we use $(A : B) = \bigvee\{x \in L \mid xB \leq A\}$.
An element $A \in M$ is called *compact* whenever $A \leq \bigvee_{\alpha \in I} A_\alpha$ implies the existence of a finite subset $\{\alpha_1, \alpha_2, \cdots, \alpha_n\} \subseteq I$ such that $A \leq A_{\alpha_1} \vee A_{\alpha_2} \vee \cdots \vee A_{\alpha_n}$.

An element $N \in M$ is called meet-principal if $A \wedge aN = (a \wedge (A : N))N$ for every $a \in L$ and $A \in M$. Dually, N is said to be join-principal if $((aN \vee A) : N) = (a \vee (A : N)))$ for every $a \in L$ and $A \in M$. An element N is called principal when it satisfies both the meet-principal and join-principal conditions.
If every element of M can be represented as a join of principal elements, then M is called a principally generated lattice module. In particular, when every element of M is expressible as a join of compact elements, M is referred to as a compactly generated lattice module.

An element $A \in M$ is said to be proper, if $A < 1_M$. A proper element $P \in M$ is called *prime* if, for every $a \in L$ and $X \in M$, $aX \leq P$ implies $X \leq P$ or $a \leq (P : 1_M)$. The set of all prime elements of M is called the prime spectrum of M and is denoted by $Spec(M)$. In [4], S. Ballal and V. Kharat studied the Zariski topology over $Spec(M)$ as a generalization of the results carried out in [[23],[24]]. Also in [11], Fethi Callialp et al. studied the Zariski topology on $Spec(M)$.

For an element $N \in M$, the radical of N is defined as $\sqrt{N} = \bigwedge\{P \in Spec(M) | N \leq P\}$[18]. Note that, $\sqrt{N} = 1_M$, if N is not contained in any prime and $N \leq \sqrt{N}$.

Finally, an element $N < 1_M$ of M is called *maximal* if there exists no element $B \in M$ satisfying $N < B < 1_M$. If 1_M is compact, then M has a maximal element[21].

For further details on these concepts and more information on multiplicative lattices, lattice modules and topology, the reader may refer ([1], [2], [15], [20], [22]).

2. Primary Zariski Topology on $Spec^{pr}(M)$

Let M be a lattice module over a $C-$lattice L. A proper element $P \in M$ is said to be primary if for $a \in L$ and $A \in M, aA \leq P$ implies that $A \leq P$ or $a^n 1_M \leq P$, i.e., $a^n \leq (P : 1_M)$ for some positive integer n ([19]). Let $Spec_p(M)$ be the set of all primary elements of M. The primary spectrum $Spec^{pr}(M)$ is defined to be the set of all primary elements P of M for which $(\sqrt{P} : 1_M) = \sqrt{(P : 1_M)}$, i.e., $Spec^{pr}(M) = \{P \in Spec_p(M) | (\sqrt{P} : 1_M) = \sqrt{(P : 1_M)}\}$. Clearly, $Spec(M) \subseteq Spec^{pr}(M)$. Let N be any element of M. Let $W(N) = \{P \in Spec^{pr}(M) | (N : 1_M) \leq (\sqrt{P} : 1_M)\}$. In this section, we investigate the primary Zariski topology over the primary spectrum $Spec^{pr}(M)$.

We require the following lemma.

Lemma 2.1. [15] *Suppose M is a lattice module defined over the $C-$lattice L. Then for $N \in M$, $(N : 1_M) = ((N : 1_M)1_M : 1_M)$.*

Throughout this paper, the following properties will be used.

Lemma 2.2. [15] *Let M denote a lattice module associated with the multiplicative lattice L. Then for $x \in L$ and $A, B, C, A_\alpha \in M$, following hold.*

(1) $x \leq (0_M : (0_M : x))$.
(2) $A \leq (0_M : (0_M : A))$.
(3) *If* $B \leq C$ *then* $(A : C) \leq (A : B)$.
(4) *If* $B \leq C$ *then* $(B : A) \leq (C : A)$.
(5) $\vee_\alpha (A_\alpha : B) \leq ((\vee_\alpha A_\alpha) : B)$.
(6) *If* $a \leq b$ *then* $aA \leq bA$.
(7) $(0_M : A) = (0_M : (0_M : (0_M : A)))$.
(8) $(A : B \vee C) = (A : B) \wedge (A : C)$.
(9) $x(A : x) \leq A$.
(10) $(A : B)B \leq A$.
(11) $a \leq (aA : A)$.
(12) $(B \wedge C : A) = (B : A) \wedge (C : A)$.

Theorem 2.3. *Assume that M is a lattice module associated with the $C-$lattice L and N, K are elements of M and that $(N_i)_{i \in I}$ is a family of elements of M. Then*

(1) $W(0_M) = Spec^{pr}(M)$
(2) $W(1_M) = \emptyset$

(3) $\cap_{i\in I}W(N_i) = W(\vee_{i\in I}(N_i : 1_M)1_M)$
(4) $W(N) \cup W(K) = W(N \wedge K)$

Proof. We have, $W(N) = \{P \in Spec^{pr}(M) | (N : 1_M) \leq (\sqrt{P} : 1_M)\}$.
1) $W(0_M) = \{P \in Spec^{pr}(M) | (0_M : 1_M) \leq (\sqrt{P} : 1_M)\} = Spec^{pr}(M)$.
2) $W(1_M) = \{P \in Spec^{pr}(M) | (1_M : 1_M) \leq (\sqrt{P} : 1_M)\} = \emptyset$.
3) Consider $P \in W(\vee_{i\in I}(N_i : 1_M)1_M)$. Then $(\vee_{i\in I}(N_i : 1_M)1_M : 1_M) \leq (\sqrt{P} : 1_M)$ which implies that $\vee_{i\in I}((N_i : 1_M)1_M : 1_M) \leq (\sqrt{P} : 1_M)$, by Lemma 2.2(5). By Lemma 2.1, $(N_i : 1_M) = ((N_i : 1_M)1_M : 1_M)$. Hence, $(N_i : 1_M) = ((N_i : 1_M)1_M : 1_M) \leq \vee_{i\in I}((N_i : 1_M)1_M : 1_M) \leq (\sqrt{P} : 1_M)$ for every $i \in I$. Thus, $P \in W(N_i)$ for $i \in I$. Finally, we have $P \in \cap_{i\in I}W(N_i)$. Conversely, we consider $P \in \cap_{i\in I}W(N_i)$. Then $P \in W(N_i)$ for every $i \in I$. Thus, $(N_i : 1_M) \leq (\sqrt{P} : 1_M)$. By Lemma 2.2(6), it implies that $(N_i : 1_M)1_M \leq (\sqrt{P} : 1_M)1_M$ for $i \in I$ and therefore $\vee_{i\in I}(N_i : 1_M)1_M \leq (\sqrt{P} : 1_M)1_M$. As $(\sqrt{P} : 1_M)1_M \leq \sqrt{P}$, by Lemma 2.2(10), from this, we obtain $\vee_{i\in I}(N_i : 1_M)1_M \leq \sqrt{P}$. Now, By Lemma 2.2(4), we get $(\vee_{i\in I}(N_i : 1_M)1_M : 1_M) \leq (\sqrt{P} : 1_M)$. Consequently, $P \in W(\vee_{i\in I}(N_i : 1_M)1_M)$.
4) If $P \in W(N) \cup W(K)$, then $(N : 1_M) \leq (\sqrt{P} : 1_M)$ or $(K : 1_M) \leq (\sqrt{P} : 1_M)$. Since $N \wedge K \leq N, K$, by Lemma 2.2(4), $(N \wedge K : 1_M) \leq (N : 1_M) \leq (\sqrt{P} : 1_M)$ or $(N \wedge K : 1_M) \leq (K : 1_M) \leq (\sqrt{P} : 1_M)$. Hence, $P \in W(N \wedge K)$. Thus, $W(N) \cup W(K) \subseteq W(N \wedge K)$. For reverse inclusion, let $P \in W(N \wedge K)$. Then $(N \wedge K : 1_M) \leq (\sqrt{P} : 1_M)$. By Lemma 2.2(12), $(N : 1_M) \wedge (K : 1_M) = (N \wedge K : 1_M) \leq (\sqrt{P} : 1_M)$. By primality of $(\sqrt{P} : 1_M)$ in L, $(N : 1_M) \leq (\sqrt{P} : 1_M)$ or $(K : 1_M) \leq (\sqrt{P} : 1_M)$. Hence, $P \in W(N)$ or $P \in W(K)$, which implies that $P \in W(N) \cup W(K)$. This proves the reverse inclusion. Hence the proof is complete.

Let $\tau(M) = \{W(N) | N \in M\}$. Then it follows from Theorem 2.3 that the collection $\tau(M)$ induces a topology on $Spec^{pr}(M)$, which is called the primary Zariski topology. Let $U(N) = \{P \in Spec(M) | (N : 1_M) \leq (P : 1_M)\}$. Then $U(N)$ is the closed set for Zariski topology on $Spec(M)$ [11]. It can be verified that $U(N) = W(N) \cap Spec(M)$ and consequently, $Spec(M)$, equipped with the Zariski topology, becomes a topological subspace of $Spec^{pr}(M)$ endowed with the primary Zariski topology. □

Some of the results in this paper require the following properties.

Properties 2.4. *([23] and [24]) If L is a multiplicative lattice, then for $c, d \in L$*

(1) $c \leq \sqrt{c}$.
(2) $c \leq d$ *implies* $\sqrt{c} \leq \sqrt{d}$.
(3) *For* $x \in L_*, x \leq \sqrt{c}$ *if and only if* $x^n \leq c$ *for some integer* $n \geq 1$.
(4) $\sqrt{\sqrt{c}} = \sqrt{c}$.
(5) $\sqrt{c \wedge d} = \sqrt{c} \wedge \sqrt{d} = \sqrt{cd}$.
(6) $\sqrt{c \vee d} = \sqrt{\sqrt{c} \vee \sqrt{d}}$.
(7) *If* $1 \in L$ *is compact, then* $c < 1$ *implies* $\sqrt{c} < 1$.

(8) $\sqrt{c} = \wedge\{p \in L | p$ *is prime and* $c \leq p\} = \wedge\{p \in L | p$ *is minimal prime over* $c\}$.

(9) *If* $c \in L$ *is prime, then* $\sqrt{c} = c$.

Proposition 2.5. *Assume that M is a lattice module over a $C-$lattice L and N, K are elements of M. Then*

(1) *If* $\sqrt{(N : 1_M)} = \sqrt{(K : 1_M)}$, *then* $W(N) = W(K)$. *The converse holds provided that N and K are primary.*

(2) $W((N : 1_M)1_M) = W(N)$.

Proof. 1) Assume that $\sqrt{(N : 1_M)} = \sqrt{(K : 1_M)}$. Let $P \in W(N)$. Then $(N : 1_M) \leq (\sqrt{P} : 1_M)$, which implies that $\sqrt{(N : 1_M)} \leq \sqrt{(\sqrt{P} : 1_M)} = (\sqrt{P} : 1_M)$ by Properties 2.4(2 and 4). Therefore, $(K : 1_M) \leq \sqrt{(K : 1_M)} \leq (\sqrt{P} : 1_M)$, by Properties 2.4(1), hence $P \in W(K)$. For the reverse inclusion, let $P \in W(K)$. Then $(K : 1_M) \leq (\sqrt{P} : 1_M)$. By Properties 2.4(2 and 4), it implies that $\sqrt{(K : 1_M)} \leq \sqrt{(\sqrt{P} : 1_M)} = (\sqrt{P} : 1_M)$. Hence, $(N : 1_M) \leq \sqrt{(N : 1_M)} \leq (\sqrt{P} : 1_M)$ and $P \in W(N)$.
Now suppose that N and K are primary elements of M and $W(N) = W(K)$. As N is primary, $N \in W(N) = W(K)$, it implies that $(K : 1_M) \leq (\sqrt{N} : 1_M) = \sqrt{(N : 1_M)}$. By Properties 2.4(2 and 4), $\sqrt{(K : 1_M)} \leq \sqrt{(N : 1_M)}$. Also if K is primary, then $K \in W(K) = W(N)$ implies that $(N : 1_M) \leq (\sqrt{K} : 1_M) = \sqrt{(K : 1_M)}$. Hence, $\sqrt{(N : 1_M)} \leq \sqrt{(K : 1_M)}$, by Properties 2.4(2 and 4). Consequently, $\sqrt{(N : 1_M)} = \sqrt{(K : 1_M)}$.
2) Let $P \in W(N)$. Then $(N : 1_M) \leq (\sqrt{P} : 1_M)$. By Lemma 2.2(6 and 10), $(N : 1_M)1_M \leq (\sqrt{P} : 1_M)1_M \leq \sqrt{P}$. By Lemma 2.2(4), $((N : 1_M)1_M : 1_M) \leq (\sqrt{P} : 1_M)$. Hence, $P \in W((N : 1_M)1_M)$. Now we show the converse inclusion. Let $P \in W((N : 1_M)1_M)$. Then $((N : 1_M)1_M : 1_M) \leq (\sqrt{P} : 1_M)$. According to Lemma 2.1, $(N : 1_M) \leq (\sqrt{P} : 1_M)$ and $P \in W(N)$. $\square$

In what follows, L/a denotes the interval $[a, 1]$ which is multiplicative lattice with $x \circ y = xy \vee a$, where $x, y \in L$[10].

Let $Spec^{pr}_p(M) = \{P \in Spec^{pr}(M) | (\sqrt{P} : 1_M) = p$, for some $p \in Spec(L)\}$. We denote the set $Spec(L/(0_M : 1_M))$ by $Spec(\bar{L})$. Note that, for a primary element P in M, $(P : 1_M)$ is a primary element of L and $\sqrt{(P : 1_M)}$ is a prime element of L[19].

Definition 2.6. Let M be a lattice module over a C-lattice L. Then the map $\psi^{pr} : Spec^{pr}(M) \to Spec(\bar{L})$ defined by $\psi^{pr}(P) = \overline{(\sqrt{P} : 1_M)}$ is called the natural map of $Spec^{pr}(M)$.

Proposition 2.7. *Suppose M is a lattice module defined over the C-lattice L and $\psi^{pr} : Spec^{pr}(M) \to Spec(\bar{L})$ be the natural map. Let $P, P' \in Spec^{pr}(M)$. Then the following statements are equivalent:*

(1) *If* $W(P) = W(P')$, *then* $P = P'$.

(2) $|Spec^{pr}_p(M)| \leq 1$ *for every* $p \in Spec(L)$.

(3) ψ^{pr} *is injective.*

Proof. $1 \implies 2$: Assume that (1) holds. We show that (2) follows. Let $P, P' \in Spec^{pr}_p(M)$. Then $(\sqrt{P} : 1_M) = (\sqrt{P'} : 1_M) = p$. To prove $W(P) = W(P')$, let $Q \in W(P)$. Then $(P : 1_M) \leq (\sqrt{Q} : 1_M)$. According to Properties 2.4(2), $\sqrt{(P : 1_M)} \leq \sqrt{(\sqrt{Q} : 1_M)}$. It implies that $(\sqrt{P} : 1_M) \leq \sqrt{\sqrt{(Q : 1_M)}} = \sqrt{(Q : 1_M)}$, by Properties 2.4(4). Hence, $(\sqrt{P'} : 1_M) \leq (\sqrt{Q} : 1_M)$. By Properties 2.4(1), $(P' : 1_M) \leq \sqrt{(P' : 1_M)} \leq (\sqrt{Q} : 1_M)$. Thus, $Q \in W(P')$. Consequently, $W(P) \subseteq W(P')$. Similarly, $W(P') \subseteq W(P)$. It follows that $W(P) = W(P')$. Therefore, $P = P'$.
$2 \implies 3$: Suppose that (2) holds. Let $P, P' \in Spec^{pr}(M)$ and $\psi^{pr}(P) = \psi^{pr}(P')$. Then $(\sqrt{P} : 1_M) = (\sqrt{P'} : 1_M) = p$ and hence, $P, P' \in Spec^{pr}_p(M)$. Therefore, $P = P'$.
$3 \implies 1$: Suppose that ψ^{pr} is injective map. If $W(P) = W(P')$, then $\overline{(\sqrt{P} : 1_M)} = \overline{(\sqrt{P'} : 1_M)}$, which implies that $\psi^{pr}(P) = \psi^{pr}(P')$. Hence, $P = P'$. □

Corollary 2.8. *Suppose M is a lattice module defined over the C-lattice L and $\psi^{pr} : Spec^{pr}(M) \to Spec(\bar{L})$ be the natural map. If $|Spec^{pr}_p(M)| = 1$, for each $p \in Spec(L)$, then ψ^{pr} is a bijective map.*

Proof. By Proposition 2.7, it is clear. □

Let $V(a) = \{p \in Spec(L) | a \leq p\}$. Then the collection of all subset $V(a)$ of $Spec(L)$ forms a Zariski topology on $Spec(L)$[11]. Note that, $V(\bar{a}) = \{\bar{p} \in Spec(\bar{L}) | \bar{a} \leq \bar{p}\}$.

Proposition 2.9. *Suppose M is a lattice module defined over the C-lattice L and $\psi^{pr} : Spec^{pr}(M) \to Spec(\bar{L})$ be the natural map. Then $(\psi^{pr})^{-1}(V(\bar{a})) = W(a1_M)$, for every $a \in L$ such that $(0_M : 1_M) \leq a$ and therefore ψ^{pr} is continuous.*

Proof. Let $P \in (\psi^{pr})^{-1}(V(\bar{a}))$. Then $\psi^{pr}(P) \in V(\bar{a})$. Therefore, $\overline{(\sqrt{P} : 1_M)} \in V(\bar{a})$, i.e., $\bar{a} \leq \overline{(\sqrt{P} : 1_M)}$. Hence, $a \leq (\sqrt{P} : 1_M)$, which implies that $a1_M \leq \sqrt{P}$. By using Lemma 2.2(4), we obtain $(a1_M : 1_M) \leq (\sqrt{P} : 1_M)$. This yields, $P \in W(a1_M)$. Thus, $(\psi^{pr})^{-1}(V(\bar{a})) \subseteq W(a1_M)$. To complete the proof, we prove the reverse containment. Let $P \in W(a1_M)$. Then $(a1_M : 1_M) \leq (\sqrt{P} : 1_M)$ and as $a \leq (a1_M : 1_M)$, by Lemma 2.2(11), we have, $\bar{a} \leq \overline{(\sqrt{P} : 1_M)} = \psi^{pr}(P)$. Therefore, $\psi^{pr}(P) \in V(\bar{a})$ and $P \in (\psi^{pr})^{-1}(V(\bar{a}))$. Thus, $W(a1_M) \subseteq (\psi^{pr})^{-1}(V(\bar{a}))$. □

In the following theorem we prove that ψ^{pr} is both open and closed.

Theorem 2.10. *Suppose M is a lattice module defined over the C-lattice L and $\psi^{pr} : Spec^{pr}(M) \to Spec(\bar{L})$ be the natural map. If ψ^{pr} is surjective, then it is both closed and open. Equivalently, $\psi^{pr}(W(N)) = V(\overline{(N : 1_M)})$ and $\psi^{pr}(Spec^{pr}(M) - W(N)) = Spec(\bar{L}) - V(\overline{(N : 1_M)})$, for every element N of M.*

Proof. According to Proposition 2.9, we have, for every element N of M, $(\psi^{pr})^{-1}(V(\overline{(N:1_M)})) = W((N:1_M)1_M)$. But $W((N:1_M)1_M) = W(N)$, by Proposition 2.5(2). Therefore, $(\psi^{pr})^{-1}(V(\overline{(N:1_M)})) = W(N)$. It implies that, $\psi^{pr}(W(N)) = \psi^{pr}((\psi^{pr})^{-1}(V(\overline{(N:1_M)}))) = V(\overline{(N:1_M)})$. Now, $\psi^{pr}(Spec^{pr}(M) - W(N)) = \psi^{pr}((\psi^{pr})^{-1}(Spec(\bar{L})) - (\psi^{pr})^{-1}(V(\overline{(N:1_M)}))) = (Spec(\bar{L}) - V(\overline{(N:1_M)}))$. $\square$

For each $a \in L$, we denote $\mathcal{C}_a = Spec^{pr}(M) - W(a1_M)$. Then $\mathcal{C}_a$ is open in $Spec^{pr}(M)$.

Lemma 2.11. *The collection $\mathcal{C} = \{\mathcal{C}_a | a \in L\}$ forms a basis for the primary Zariski topology over $Spec^{pr}(M)$.*

Proof. Suppose that X is an open set in $Spec^{pr}(M)$. Then $X = (W(N))^c$ for some $N \in M$. Therefore, $X = Spec^{pr}(M) - W(N) = Spec^{pr}(M) - W((N:1_M)1_M) = Spec^{pr}(M) - W(\vee_{a_i \leq (N:1_M)} a_i 1_M) = Spec^{pr}(M) - \cap_{a_i \leq (N:1_M)} W(a_i 1_M) = \cup_{a_i \leq (N:1_M)} (Spec^{pr}(M) - W(a_i 1_M)) = \cup_{a_i \leq (N:1_M)} \mathcal{C}_{a_i}$. Thus the collection $\mathcal{C} = \{\mathcal{C}_a | a \in L\}$ forms a basis for the primary Zariski topology over $Spec^{pr}(M)$. $\square$

Let $\mathcal{E}_a = Spec(L) - V(a)$. Then $\mathcal{E}_a$ is open set in $Spec(L)$ and the collection $\{\mathcal{E}_a | a \in L, a \text{ is compact}\}$ forms a basis for the Zariski topology on $Spec(L)$. Note that $\mathcal{E}_{\bar{a}} = Spec(\bar{L}) - V(\bar{a})$.

Lemma 2.12. *Suppose M is a lattice module defined over the C-lattice L and $\psi^{pr} : Spec^{pr}(M) \to Spec(\bar{L})$ be the natural map. Then:*

(1) $(\psi^{pr})^{-1}(\mathcal{E}_{\bar{a}}) = \mathcal{C}_a$.
(2) $\psi^{pr}(\mathcal{C}_a) \subseteq \mathcal{E}_{\bar{a}}$.
(3) $\psi^{pr}(\mathcal{C}_a) = \mathcal{E}_{\bar{a}}$, *if ψ^{pr} is surjective.*
(4) $\mathcal{C}_{ab} = \mathcal{C}_a \cap \mathcal{C}_b$, *for any elements $a, b \in L$.*

Proof. 1) Let us consider $P \in (Spec^{pr}(M) - (\psi^{pr})^{-1}(\mathcal{E}_{\bar{a}}))$. Then $a \leq (\sqrt{P} : 1_M)$. Hence, $a1_M \leq \sqrt{P}$, which implies that $(a1_M : 1_M) \leq (\sqrt{P} : 1_M)$, by Lemma 2.2(4). Thus, $P \in W(a1_M)$, i.e., $P \in (Spec^{pr}(M) - \mathcal{C}_a)$. Therefore, $\mathcal{C}_a \subseteq (\psi^{pr})^{-1}(\mathcal{E}_{\bar{a}})$. Now we show the reverse inclusion, let $P \in (\psi^{pr})^{-1}(\mathcal{E}_{\bar{a}})$. Then $\psi^{pr}(P) \in \mathcal{E}_{\bar{a}}$, i.e., $\overline{(\sqrt{P}:1_M)} \in \mathcal{E}_{\bar{a}}$. It follows that, $\bar{p} = \overline{(\sqrt{(P:1_M)})} \in \mathcal{E}_{\bar{a}}$. Hence, $a \nleq p$. As a consequence, we have, $(a1_M : 1_M) \nleq (\sqrt{P} : 1_M)$. Therefore, $P \notin W(a1_M)$. Consequently, $(\psi^{pr})^{-1}(\mathcal{E}_{\bar{a}}) \subseteq \mathcal{C}_a$.
2) Let $a \in L$ and $P \in Spec^{pr}(M)$. Let $\sqrt{(P:1_M)} = p$. If $\bar{p} \in V(\bar{a})$, then $\bar{a} \leq \bar{p}$ and hence, $a \leq p = \sqrt{(P:1_M)} = (\sqrt{P} : 1_M)$, which implies that $a1_M \leq \sqrt{P}$. By using Lemma 2.2(4), we obtain, $(a1_M : 1_M) \leq (\sqrt{P} : 1_M)$ and therefore, $P \in W(a1_M)$. Thus, if $P \in \mathcal{C}_a$, then $\psi^{pr}(P) = \bar{p} \in \mathcal{E}_{\bar{a}}$. It follows that, $\psi^{pr}(\mathcal{C}_a) \subseteq \mathcal{E}_{\bar{a}}$.
3) By part (2), $\psi^{pr}(\mathcal{C}_a) \subseteq \mathcal{E}_{\bar{a}}$. To establish the inclusion in the opposite direction, consider $p \in \mathcal{E}_{\bar{a}}$. Then $a \nleq p$. As ψ^{pr} is surjective, there exists $P \in Spec^{pr}(M)$ such that $(\sqrt{P} : 1_M) = p$. Thus, $(a1_M : 1_M) \nleq (\sqrt{P} : 1_M)$ and hence $P \in \mathcal{C}_a$. Finally, $\mathcal{E}_{\bar{a}} \subseteq \psi^{pr}(\mathcal{C}_a)$.
4) Consider $\mathcal{C}_{ab} = (\psi^{pr})^{-1}(\mathcal{E}_{\overline{ab}}) = (\psi^{pr})^{-1}(\mathcal{E}_{\bar{a}} \cap \mathcal{E}_{\bar{b}}) = (\psi^{pr})^{-1}(\mathcal{E}_{\bar{a}}) \cap (\psi^{pr})^{-1}(\mathcal{E}_{\bar{b}}) = \mathcal{C}_a \cap \mathcal{C}_b$, for each $a, b \in L$. $\square$

In the following results, we investigate the compactness of the space $Spec^{pr}(M)$.

Theorem 2.13. *Suppose M is a lattice module defined over the C-lattice L and $\psi^{pr} : Spec^{pr}(M) \to Spec(\bar{L})$ be the natural map. If ψ^{pr} is surjective, then for every $a \in L$, $\mathcal{C}_a$ is compact in $Spec^{pr}(M)$.*

Proof. Suppose that $\mathcal{C}_a$ is covered by an open cover in $Spec^{pr}(M)$. As $\mathcal{C} = \{\mathcal{C}_a | a \in L\}$ is a base for the primary Zariski topology, $\mathcal{C}_a \subseteq \cup_{\alpha\in\Lambda}\mathcal{C}_{a_\alpha}$ for some open cover $\{\mathcal{C}_{a_\alpha} | \alpha \in \Lambda\} \subseteq \mathcal{C}$. By Lemma 2.12(3), $\mathcal{E}_{\bar{a}} = \psi^{pr}(\mathcal{C}_a) \subseteq \cup_{\alpha\in\Lambda}\psi^{pr}(\mathcal{C}_{a_\alpha}) = \cup_{\alpha\in\Lambda}\mathcal{E}_{\bar{a}_\alpha}$. Now since, $Spec(\bar{L})$ is compact space with respect to Zariski topology[11], we have, $\mathcal{E}_{\bar{a}} = \psi^{pr}(\mathcal{C}_a) \subseteq \cup_{i=1}^{n}\mathcal{E}_{\bar{a}_{\alpha_i}}$, for some $\alpha_i (1 \le i \le k)$. Thus, by Lemma 2.12(1), we have, $\mathcal{C}_a = (\psi^{pr})^{-1}(\psi^{pr}(\mathcal{C}_a)) \subseteq \cup_{i=1}^{n}(\psi^{pr})^{-1}(\mathcal{E}_{\bar{a}_{\alpha_i}}) = \cup_{i=1}^{n}\mathcal{C}_{a_{\alpha_i}}$. Consequently, $\mathcal{C}_a$ is compact in $Spec^{pr}(M)$. □

Corollary 2.14. *Suppose M is a lattice module defined over the C-lattice L and $\psi^{pr} : Spec^{pr}(M) \to Spec(\bar{L})$ be the natural map. If ψ^{pr} is surjective, then $Spec^{pr}(M)$ is a compact space under the primary Zariski topology.*

Proof. By the above theorem, $\mathcal{C}_a$ is compact for $a \in L$. Since, $Spec^{pr}(M) = \mathcal{C}_{1_L}$, it is compact space. □

Corollary 2.15. *Suppose M is a lattice module defined over the C-lattice L and $\psi^{pr} : Spec^{pr}(M) \to Spec(\bar{L})$ be the natural map. If ψ^{pr} is surjective, then the compact open sets of $Spec^{pr}(M)$ are closed under finite intersections and form a open base.*

Proof. By Lemma 2.12(4), $\mathcal{C}_{ab} = \mathcal{C}_a \cap \mathcal{C}_b$, for $\mathcal{C}_a, \mathcal{C}_b$ in $\mathcal{C}$. By theorem 2.13, each $\mathcal{C}_a$ is compact open set. Thus, by Lemma 2.11, every open covering of any intersection of compact sets is a finite union of the elements of $\mathcal{C}$. Hence the result. □

A topological space X is irreducible, if for any closed subsets X_1, X_2 of X, $X = X_1 \cup X_2$ implies $X = X_1$ or $X = X_2$. A subset Y of X is irreducible, if it is irreducible as a subspace of X. Let Y be a subset of $Spec^{pr}(M)$. The closure of Y, we denote by $Cl(Y)$. Let $Z(Y)$ be the meet of all $\sqrt{P}$, for $P \in Y$, i.e., $Z(Y) = \wedge_{P\in Y}\sqrt{P}$.

Proposition 2.16. *Suppose M is a lattice module defined over the C-lattice L. Then $W(P) = W(\sqrt{P})$, for $P \in Spec^{pr}(M)$.*

Proof. Let $Q \in W(\sqrt{P})$. Then $(\sqrt{P} : 1_M) \le (\sqrt{Q} : 1_M)$. By Properties 2.4(1), $(P : 1_M) \le \sqrt{(P : 1_M)} \le (\sqrt{Q} : 1_M)$. Thus, $Q \in W(P)$. Hence, $W(\sqrt{P}) \subseteq W(P)$. For reverse inclusion, let $Q \in W(P)$. Then $(P : 1_M) \le (\sqrt{Q} : 1_M) = \sqrt{(Q : 1_M)}$. By Properties 2.4(2 and 4), $\sqrt{(P : 1_M)} \le \sqrt{\sqrt{(Q : 1_M)}} = \sqrt{(Q : 1_M)}$. Thus, $(\sqrt{P} : 1_M) \le (\sqrt{Q} : 1_M)$. This implies that $Q \in W(\sqrt{P})$. Consequently, $W(P) = W(\sqrt{P})$. □

Lemma 2.17. *Suppose M is a lattice module defined over the C-lattice L. Let $Y \subseteq Spec^{pr}(M)$. Then $Cl(Y) = W(Z(Y))$. In particular, if Y contains 0_M, then Y is dense in $Spec^{pr}(M)$, that is, $Cl(Y) = Spec^{pr}(M)$.*

Proof. Let Y be a subset of $Spec^{pr}(M)$. Assume that $P \in Y$. Then $Z(Y) = \wedge_{P\in Y}\sqrt{P} \leq \sqrt{P}$ and so that by Lemma 2.2(4), $(Z(Y) : 1_M) \leq (\sqrt{P} : 1_M)$. Hence, $P \in W(Z(Y))$. Thus, $Y \subseteq W(Z(Y))$ and $Cl(Y) \subseteq W(Z(Y))$. For the opposite inclusion, let N be any element of M such that $Y \subseteq W(N)$. Then for each $P \in W(N)$, $(N : 1_M) \leq (\sqrt{P} : 1_M)$. Let $P' \in W(Z(Y))$. Then $(Z(Y) : 1_M) \leq (\sqrt{P'} : 1_M)$. This implies that $(\wedge_{P\in W(N)}\sqrt{P} : 1_M) \leq (\sqrt{P'} : 1_M)$. It follows from Lemma 2.2(12), $\wedge_{P\in W(N)}(\sqrt{P} : 1_M) \leq (\sqrt{P'} : 1_M)$. Hence, $(N : 1_M) \leq \wedge_{P\in W(N)}(\sqrt{P} : 1_M) \leq (\sqrt{P'} : 1_M)$. Thus, $P' \in W(N)$. Consequently, $W(Z(Y))$ is the minimal closed subset of $Spec^{pr}(M)$ containing Y. Therefore, $W(Z(Y)) = Cl(Y)$. For the final part of the proof, consider $0_M \in Y$, then $Cl(Y) = W(Z(Y)) = W(\sqrt{0_M}) = W(0_M) = Spec^{pr}(M)$. Hence, Y is dense in $Spec^{pr}(M)$. □

In the following results, we investigate the irreducibility of closed subsets of the space $Spec^{pr}(M)$.

Theorem 2.18. *Suppose M is a lattice module defined over the C-lattice L. Then $W(P)$ is irreducible in $Spec^{pr}(M)$, for every $P \in Spec^{pr}(M)$.*

Proof. According to Lemma 2.17, $Cl(Y) = W(Z(Y))$. Let $Y = \{P\}$. Then, $Cl(Y) = Cl(\{P\}) = W(Z(\{P\})) = W(\sqrt{P})$. Therefore, $Cl(\{P\}) = W(P)$, by Proposition 2.16. It follows from the fact that any singleton set is irreducible and closure of irreducible set is irreducible, $W(P)$ is irreducible. □

Theorem 2.19. *Suppose M is a lattice module defined over the C-lattice L and $Y \subseteq Spec^{pr}(M)$. If $Z(Y)$ is a primary element of M, then Y is irreducible. Conversely, if Y is irreducible, then $Y' = \{(\sqrt{P} : 1_M)|P \in Y\}$ is an irreducible subset of $Spec(L)$, that is, $Z(Y') = (Z(Y) : 1_M)$ is a prime element of L.*

Proof. Assume that $Y \subseteq Spec^{pr}(M)$ and $Z(Y)$ is a primary element of M. By Theorem 2.18, $W(P)$ is irreducible for every primary element P of M, therefore, $W(Z(Y))$ is an irreducible subset of $Spec^{pr}(M)$. From the Lemma 2.17, we have, $Cl(Y) = W(Z(Y))$. Thus, $Cl(Y)$ is irreducible. Consider $Y \subseteq Y_1 \cup Y_2$, where Y_1, Y_2 are closed subsets of $Spec^{pr}(M)$. Then $Cl(Y) \subseteq Y_1 \cup Y_2$. It follows that $Y \subseteq Cl(Y) \subseteq Y_1$ or $Y \subseteq Cl(Y) \subseteq Y_2$. Therefore, we conclude that Y is irreducible. Conversely, let us assume that Y is irreducible. Then, by Proposition 2.9, the image $\psi^{pr}(Y) = \{\overline{(\sqrt{P} : 1_M)}|P \in Y\}$ of Y is an irreducible subset of $Spec(\bar{L})$, since the natural map ψ^{pr} is continuous. Now, by Proposition 7(3) in [11], $\wedge_{P\in Y}\overline{(\sqrt{P} : 1_M)} = \overline{(\wedge_{P\in Y}\sqrt{P} : 1_M)} = \overline{(Z(Y) : 1_M)}$ is a prime element of $\bar{L}$. Hence, $Z(Y') = (Z(Y) : 1_M)$ is a prime element of L. Thus, Y' is an irreducible subset of $Spec(L)$. □

Theorem 2.20. *Suppose M is a lattice module defined over the C-lattice L and Y is a subset of $Spec^{pr}(M)$. Let $\psi^{pr} : Spec^{pr}(M) \to Spec(\bar{L})$ be the surjective natural map. Then the following statements are equivalent:*

(1) *Y is an irreducible closed subset of $Spec^{pr}(M)$.*
(2) *$Y = W(P)$, for some $P \in Spec^{pr}(M)$.*

Proof. $1 \implies 2$: Let us assume that Y is a closed irreducible subset of $Spec^{pr}(M)$. Then $Y = W(N)$, for some $N \in M$. Therefore, by Theorem 2.19, $(Z(W(N)) : 1_M) = (Z(Y) : 1_M) = p$ is a prime element of L. As ψ^{pr} is surjective, there exists $P \in Spec^{pr}(M)$ such that $\psi^{pr}(P) = (\sqrt{P} : 1_M) = p = (Z(W(N)) : 1_M)$. It implies that, $Y = W(N) = W(Z(W(N))) = W(\sqrt{P}) = W(P)$, by Proposition 2.5 and Lemma 2.17.

$2 \implies 1$: This is obvious from Theorem 2.18. □

Now we examine the space $Spec^{pr}(M)$ endowed with primary Zariski topology and study whether it satisfies the properties of a spectral space.

A topological space X is called a T_0-space when, for any two distinct points in X, at least one possesses an open neighborhood that excludes the other. Equivalently, X satisfies the T_0 axiom precisely when distinct points have distinct closures. For multiplicative lattice L, the prime spectrum $Spec(L)$ with the Zariski topology is known to satisfy the T_0 condition.

A topological space X is termed spectral if it is homeomorphic to the spectrum of a commutative ring endowed with the Zariski topology. According to Hochster's characterization ([14]) a space X is spectral if it fulfills the following conditions:

(1) X is compact.
(2) Compact open subsets of X constitute a basis for the topology and remain closed under finite intersections.
(3) Every irreducible closed subset of X is the closure of a singleton set.
(4) X is a T_0-space.

Theorem 2.21. *Suppose M is a lattice module defined over the C-lattice L and $\psi^{pr} : Spec^{pr}(M) \to Spec(\bar{L})$ be the surjective natural map. Then $Spec^{pr}(M)$ is a T_0-space if and only if $|Spec^{pr}_p(M)| \leq 1$ for each $p \in Spec(L)$.*

Proof. Let us assume that $Spec^{pr}(M)$ is a T_0-space and P_1 and P_2 are two distinct elements of $Spec^{pr}_p(M)$. Then $(\sqrt{P_1} : 1_M) = (\sqrt{P_2} : 1_M) = p$. It is clear that, $W(P_1) = W(P_2)$, which shows that closure of distinct points are not distinct. Since $Spec^{pr}(M)$ is a T_0-space, we must have, $P_1 = P_2$. Consequently, $|Spec^{pr}_p(M)| \leq 1$. Conversely, assume that $|Spec^{pr}_p(M)| \leq 1$. Let P_1 and P_2 be two distinct primary elements of M. Then $(\sqrt{P_1} : 1_M)$ and $(\sqrt{P_2} : 1_M)$ are distinct primes in L. Therefore, we have, $P_1 \notin W(P_2)$ and $P_2 \notin W(P_1)$. Hence, $Cl(\{P_1\}) \neq Cl(\{P_2\})$. Thus, distinct points have distinct closures. Consequently, $Spec^{pr}(M)$ is a T_0-space. □

Theorem 2.22. *Suppose M is a lattice module defined over the C-lattice L and $\psi^{pr} : Spec^{pr}(M) \to Spec(\bar{L})$ be the surjective natural map. If $|Spec^{pr}_p(M)| \leq 1$ for every $p \in Spec(L)$, then $Spec^{pr}(M)$ with primary Zariski topology is a spectral space.*

Proof. By Corollary 2.14, Corollary 2.15, Theorem 2.20 and Theorem 2.21, $Spec^{pr}(M)$ with primary Zariski topology fulfills conditions $(1), (2), (3)$, and (4) of Hochster's characterization. Consequently, $Spec^{pr}(M)$ with primary Zariski topology is a spectral space. □

A topological space X is a T_1-space if and only if every singleton set $\{x\}$ is closed for every $x \in X$. Equivalently, for all $x \in X$, $\{x\}$ is closed.

Theorem 2.23. *Suppose M is a compactly generated lattice module defined over the $C-$lattice L. Then $Spec^{pr}(M)$ is a T_1-space if and only if $Spec^{pr}(M) = Max(M)$, where $Max(M)$ is the set of all maximal elements of M. Moreover, $Spec^{pr}(M) = Spec(M) = Max(M)$.*

Proof. Let us assume that $Spec^{pr}(M)$ is a T_1-space. Then $\{P\}$ is closed subset of $Spec^{pr}(M)$, for each $P \in Spec^{pr}(M)$. By Lemma 2.17, $W(P) = Cl\{P\} = \{P\}$. As M is compactly generated, there exists $N \in Max(M)$ such that $P \leq N$. It follows that, $(P : 1_M) \leq (N : 1_M) \leq (\sqrt{N} : 1_M)$, by Lemma 2.2(4). Since, N is maximal, N is prime and hence it is primary. Hence, $N \in W(P) = \{P\}$. Thus, $N = P$ and so $P \in Max(M)$. Consequently, $Spec^{pr}(M) \subseteq Max(M)$. It is obvious that $Max(M) \subseteq Spec^{pr}(M)$. Therefore, $Spec^{pr}(M) = Max(M)$. Conversely, assume that $Spec^{pr}(M) = Max(M)$. Let $\{P\}$ be a singleton subset of $Spec^{pr}(M)$. If $P^{'} \in W(P)$, then $(P : 1_M) \leq (\sqrt{P^{'}} : 1_M)$, i.e., $\sqrt{(P : 1_M)} \leq \sqrt{(P^{'} : 1_M)}$, by Properties 2.4(2). As $(P : 1_M)$ and $(P^{'} : 1_M)$ are maximal elements of L, $(P : 1_M) = (P^{'} : 1_M)$. It follows that $P \wedge P^{'} \in Spec^{pr}(M)$ and hence $P \wedge P^{'} \in Max(M)$. Thus, $P = P^{'}$ and $W(P) = \{P\}$. Consequently, $Spec^{pr}(M)$ is a T_1-space. □

Theorem 2.24. *Suppose M is a lattice module defined over the C-lattice L and $\psi^{pr} : Spec^{pr}(M) \to Spec(\bar{L})$ be the surjective natural map. Then the following statements are equivalent:*

(1) *$Spec^{pr}(M)$ is a spectral space.*
(2) *$Spec^{pr}(M)$ is a T_0-space.*
(3) *ψ^{pr} is injective.*

Proof. $1 \implies 2$: By the characterization due to Hochster, $Spec^{pr}(M)$ is a T_0-space.
$2 \implies 3$: Suppose that $\psi^{pr}(P) = \psi^{pr}(P^{'})$ for $P, P^{'} \in Spec^{pr}(M)$. Then $(\sqrt{P} : 1_M) = (\sqrt{P^{'}} : 1_M)$. Hence, $P, P^{'} \in Spec^{pr}_p(M)$, where $p = (\sqrt{P} : 1_M)$. Since $Spec^{pr}(M)$ is a T_0-space, by Theorem 2.21, $|Spec^{pr}_p(M)| \leq 1$. It implies that $P = P^{'}$.
$3 \implies 1$: By Proposition 2.7 and Theorem 2.21, (3) implies (2). Hence the result. □

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