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FOURIER REGULARIZATION FOR A TIME-FRACTIONAL SOURCE PROBLEM

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ABSTRACT. We study the inverse problem of identifying an unknown time-dependent source in a linear time-fractional diffusion equation, where the time derivative is taken in the Caputo sense of fractional order α. The problem is ill-posed in the sense of Hadamard: the solution does not depend continuously on the measured data, so a stable reconstruction requires regularization. We propose a Fourier truncation regularization method, where a cut-off frequency serves as the regularization parameter, and analyze both an a priori and an a posteriori (discrepancy-principle) choice of this parameter. For each choice we establish Hölder-type convergence estimates, confirmed by numerical experiments.

Keywords. Time-fractional diffusion equation; inverse source problem; ill-posed problem; Fourier regularization; Hölder-type error estimate.
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1. Introduction

Fractional-order operators have emerged, over the last few decades, as a natural language for describing memory-dependent and hereditary phenomena, with applications spanning porous-media flow, electrochemical processes, population dynamics, control theory and mathematical finance [6, 7, 9]. Diffusion equations built on such operators, and the inverse problems they generate—most notably the recovery of an unknown source from boundary or interior measurements—occur naturally across this range of models. As is typical of inverse problems, these are ill-posed in Hadamard's sense: even tiny errors in the data can be amplified into arbitrarily large errors in the reconstructed solution, and a stable numerical treatment therefore calls for regularization.

Many regularization strategies have been put forward for problems of this kind. Spectral and Fourier-based approaches, in particular, have shown strong performance on Cauchy problems for the time-fractional advection-dispersion equation [13] and for recovering initial data on spherically symmetric domains via

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quasi-boundary regularization [12]. In the classical (non-fractional) heat equation setting, inverse source problems typically become well-posed once an overdetermination condition is imposed, and the remaining instability is then controlled through appropriate regularization and numerical schemes [4, 8, 1, 2], with related numerical strategies, such as optimal-control and finite-element approaches for nonlinear parabolic identification problems, developed in [3]. Diffusion and heat-type equations of this kind, including non-classical variants of the Fourier law, continue to be actively studied—see, e.g., [10] for a periodic boundary-value problem for a non-Fourier heat equation. More directly relevant to our work, the Fourier truncation approach—simple to implement yet highly effective on strip-type domains—was applied in [5] to a space-dependent source problem with an a posteriori parameter rule, producing a Hölder-type estimate; [11] instead paired Fourier truncation with quasi-reversibility regularization to reconstruct a time-dependent heat source.

Building on this body of work, and on [11] especially, the present article develops the Fourier truncation regularization approach for a linear diffusion equation in which the classical first-order time derivative is replaced by a Caputo derivative of fractional order $\alpha \in (0,1)$. We recover the unknown source and derive Hölder-type convergence rates under two distinct choices of the regularization parameter, one a priori and one a posteriori.

Two features distinguish the present contribution. First, while [11] deals with a time-dependent source in the *classical* heat equation and [5] deals with a *space*-dependent source in a *space*-fractional model, here the unknown is *time*-dependent and the governing equation is *time*-fractional; the resulting Caputo derivative of order $\alpha \in (0,1)$ leads to a reconstruction kernel, and a degree of ill-posedness, that differ genuinely from both of those settings. Second, in this time-fractional framework we obtain Hölder-type convergence rates for *both* the a priori and the a posteriori (discrepancy-principle) selections of the cut-off frequency, and we test their behaviour numerically in detail. As far as we are aware, the Fourier truncation method has not been examined before in this particular setting.

More precisely, we consider the following inverse problem for a linear time-fractional diffusion equation:

$$\begin{cases} D_t^{\alpha} u\left(x,t\right) - u_{xx}\left(x,t\right) = f\left(t\right), & t>0, \quad x>0, \\ u\left(x,0\right) = 0, & x>0, \\ u\left(0,t\right)=0, & t>0, \\ u\left(x,t\right)\big|_{x\longrightarrow\infty} \text{ is bounded}, & t>0, \\ u\left(1,t\right) = g\left(t\right), & t>0, \end{cases} \tag{1.1}$$

where $f(t)$ is a heat source depending only on the time variable. The inverse problem here consists of recovering $u(x,t)$, $0 \le x < 1$, from the given supplementary condition $u(1,t) = g(t)$. The fractional derivative $D_t^{\alpha} u(x,t)$ is the

Caputo fractional derivative of order α, $(0 < \alpha \leq 1)$, defined by [7]

$$\begin{cases} D_t^{\alpha} u(x,t) = \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^t \dfrac{\partial u(x,s)}{\partial s} \dfrac{ds}{(t-s)^{\alpha}}, & \text{for } 0 < \alpha < 1, \\ D_t^{\alpha} u(x,t) = \dfrac{\partial u(x,t)}{\partial t}, & \text{for } \alpha = 1, \end{cases}$$

where $\Gamma(\cdot)$ is the Gamma function. As an inverse problem, (1.1) is ill-posed in the classical sense: its solution fails to depend continuously on the given data, so that even a minor perturbation of that data can produce a substantially different solution.

The remainder of the paper is organized as follows. Section 2 discusses the ill-posed nature of the problem and collects several auxiliary estimates. Section 3 develops the regularization scheme, deriving convergence rates first under an a priori and then under an a posteriori choice of the regularization parameter. Section 4 presents numerical experiments illustrating the accuracy and stability of the method, and Section 5 concludes the paper.

2. Ill-Posedness of the Problem and Some Auxiliary Results

The ill-posedness can be seen by solving the problem in the frequency domain. Let

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi t} f(t)\, dt$$

be the Fourier transform of the function $f(t)$.

Problem (1.1) can be formulated in frequency space as follows:

$$\begin{cases} (i\xi)^{\alpha} \widehat{u}(x,\xi) - \widehat{u}_{xx}(x,\xi) = \widehat{f}(\xi), & x > 0,\ \xi \in \mathbb{R}, \\ \widehat{u}(1,\xi) = \widehat{g}(\xi), & \xi \in \mathbb{R}, \\ \widehat{u}(0,\xi) = 0, & \xi \in \mathbb{R}, \\ \widehat{u}(x,\xi)|_{x\longrightarrow\infty} \text{ is bounded}, & \xi \in \mathbb{R}, \end{cases}$$

Let the solution of problem (1.1) be $u(x,t) \in L^2(\mathbb{R})$, and let $\widehat{u}(x,\xi)$ be its Fourier transform. By elementary calculations, we get

$$\widehat{u}(x,\xi) = \left(\frac{1 - \exp\left(- \left(\sqrt{(i\xi)^{\alpha}} \right) x \right)}{(i\xi)^{\alpha}} \right) \widehat{f}(\xi) \tag{2.1}$$

Substituting $\widehat{u}(1,\xi) = \widehat{g}(\xi)$ into (2.1) leads to

$$\widehat{f}(\xi) = \frac{(i\xi)^{\alpha}}{\left(1 - \exp\left(-\sqrt{(i\xi)^{\alpha}}\right)\right)} \widehat{g}(\xi). \tag{2.2}$$

So

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi t} \frac{(i\xi)^{\alpha}}{\left(1 - \exp\left(-\sqrt{(i\xi)^{\alpha}}\right)\right)} \widehat{g}(\xi)\, d\xi. \tag{2.3}$$

Where

$$(i\xi)^{\alpha}=\begin{cases}|\xi|^{\alpha}\left(\cos\left(\frac{\alpha\pi}{2}\right)-i\sin\left(\frac{\alpha\pi}{2}\right)\right), & \xi<0,\\ |\xi|^{\alpha}\left(\cos\left(\frac{\alpha\pi}{2}\right)+i\sin\left(\frac{\alpha\pi}{2}\right)\right), & \xi\geq 0.\end{cases}$$

The unbounded factor $\dfrac{(i\xi)^{\alpha}}{\left(1-\exp\left(-\sqrt{(i\xi)^{\alpha}}\right)\right)}$ appearing in (2.2) and (2.3) amplifies high-frequency components of $\widehat{g}(\xi)$ without bound as $\xi\longrightarrow\infty$. Consequently, working in $L^2(\mathbb{R})$ requires the exact data $\widehat{g}(\xi)$ to decay quickly at high frequencies. In practice, however, only a noisy measurement $g^{\delta}(t)\in L^2(\mathbb{R})$ of $g(t)$ is available, so recovering $f(t)$ directly would amplify the high-frequency noise beyond control, making the classical (unregularized) approach unusable for problem (1.1). This motivates the Fourier regularization scheme developed in the next section. As a first step, we impose an a priori smoothness bound on the source, i.e.,

$$\|f(\cdot)\|_{H^p}\leq E,\qquad p>0, \tag{2.4}$$

where $E>0$ is a constant and $\|f(\cdot)\|_{H^p}$ denotes the norm in the Sobolev space $H^p(\mathbb{R})$ defined by

$$\|f(\cdot)\|_{H^p}=\left(\int_{-\infty}^{\infty}\left(1+|\xi|^2\right)^p\left|\widehat{f}(\xi)\right|^2 d\xi\right)^{\frac{1}{2}}. \tag{2.5}$$

As noted above, existence and continuous dependence on the data both fail for this inverse problem in general. Since in practice g is only known through noisy measurements, we model the available data as a function $g^{\delta}\in L^2(\mathbb{R})$ satisfying

$$\left\|g(t)-g^{\delta}(t)\right\|\leq\delta, \tag{2.6}$$

where $\delta>0$ bounds the measurement error and $\|\cdot\|$ denotes the L^2 norm. Throughout, all functions are taken to vanish for $t<0$, which allows problem (1.1) to be analyzed in $L^2(\mathbb{R})$.

We collect below several auxiliary lemmas needed for the main results.

Lemma 2.1. *For any $\xi\in\mathbb{R}$ and $0<\alpha\leq 1$, we have*

$$\left|1-e^{-\sqrt{(i\xi)^{\alpha}}}\right| = \sqrt{1-2\cos\left(|\xi|^{\frac{\alpha}{2}}\sin\left(\frac{\alpha\pi}{4}\right)\right)e^{-|\xi|^{\frac{\alpha}{2}}\cos\left(\frac{\alpha\pi}{4}\right)}+e^{-2|\xi|^{\frac{\alpha}{2}}\cos\left(\frac{\alpha\pi}{4}\right)}},$$

where

$$\sqrt{(i\xi)^{\alpha}}=\begin{cases}|\xi|^{\frac{\alpha}{2}}\left(\cos\left(\frac{\alpha\pi}{4}\right)-i\sin\left(\frac{\alpha\pi}{4}\right)\right), & \xi<0,\\ |\xi|^{\frac{\alpha}{2}}\left(\cos\left(\frac{\alpha\pi}{4}\right)+i\sin\left(\frac{\alpha\pi}{4}\right)\right), & \xi\geq 0.\end{cases}$$

Proof. For $\xi\geq 0$ we have $\sqrt{(i\xi)^{\alpha}}=|\xi|^{\frac{\alpha}{2}}\left(\cos\left(\frac{\alpha\pi}{4}\right)+i\sin\left(\frac{\alpha\pi}{4}\right)\right)$, and for $\xi<0$ we have $\sqrt{(i\xi)^{\alpha}}=|\xi|^{\frac{\alpha}{2}}\left(\cos\left(\frac{\alpha\pi}{4}\right)-i\sin\left(\frac{\alpha\pi}{4}\right)\right)$. In either case, writing $z=$

$|\xi|^{\frac{\alpha}{2}}\cos\left(\frac{\alpha\pi}{4}\right)$ and $w=|\xi|^{\frac{\alpha}{2}}\sin\left(\frac{\alpha\pi}{4}\right)$, we have $\sqrt{(i\xi)^\alpha}=z\pm iw$, so that $e^{-\sqrt{(i\xi)^\alpha}}=e^{-z}\left(\cos w\mp i\sin w\right)$. Since $|1-\zeta|^2=(1-\operatorname{Re}\zeta)^2+(\operatorname{Im}\zeta)^2=1-2\operatorname{Re}\zeta+|\zeta|^2$ for any $\zeta\in\mathbb{C}$, taking $\zeta=e^{-\sqrt{(i\xi)^\alpha}}$ gives, in both cases,

$$\left|1-e^{-\sqrt{(i\xi)^\alpha}}\right| = \sqrt{1-2\cos\left(|\xi|^{\frac{\alpha}{2}}\sin\left(\frac{\alpha\pi}{4}\right)\right)e^{-|\xi|^{\frac{\alpha}{2}}\cos\left(\frac{\alpha\pi}{4}\right)}+e^{-2|\xi|^{\frac{\alpha}{2}}\cos\left(\frac{\alpha\pi}{4}\right)}}.$$

□

From Lemma 2.1 we find

$$\left|\frac{(i\xi)^\alpha}{1-e^{-\sqrt{(i\xi)^\alpha}}}\right| = \frac{|\xi|^\alpha}{\left|1-e^{-\sqrt{(i\xi)^\alpha}}\right|} = \frac{|\xi|^\alpha}{\sqrt{1-2\cos\left(|\xi|^{\frac{\alpha}{2}}\sin\left(\frac{\alpha\pi}{4}\right)\right)e^{-|\xi|^{\frac{\alpha}{2}}\cos\left(\frac{\alpha\pi}{4}\right)}+e^{-2|\xi|^{\frac{\alpha}{2}}\cos\left(\frac{\alpha\pi}{4}\right)}}},$$

which tends to ∞ as $\xi\longrightarrow\infty$, for $\alpha\in(0,1)$. So the problem is mildly ill-posed.

Lemma 2.2. *Suppose that* $0<\alpha<1$. *If* $y>1$, *then the following inequality holds:*

$$\frac{1}{1-e^{-\sqrt{\frac{y}{2}}}}<2.$$

Proof. It is known that the function $\phi(y)=\dfrac{1}{1-e^{-\sqrt{\frac{y}{2}}}}$ is monotonically decreasing for $y>1$; therefore, it follows that $\phi(y)<\phi(1)$, then

$$\frac{1}{1-e^{-\sqrt{\frac{y}{2}}}}<\frac{1}{1-e^{-\sqrt{\frac{1}{2}}}}\approx 1.968<2,$$

so $\dfrac{1}{1-e^{-\sqrt{\frac{y}{2}}}}<2.$ □

Lemma 2.3. *For any* $\xi\in\mathbb{R}$ *and* $0<\alpha<1$, *we have*

$$\frac{1}{2}\le\frac{1}{\left|1-e^{-\sqrt{(i\xi)^\alpha}}\right|}\le\frac{1}{1-e^{-|\xi|^{\frac{\alpha}{2}}\cos\left(\frac{\alpha\pi}{4}\right)}}. \tag{2.7}$$

Proof. From Lemma 2.1, for any $\xi\in\mathbb{R}$ and $0<\alpha<1$ we have the following equality:

$$\left|1-e^{-\sqrt{(i\xi)^\alpha}}\right| = \sqrt{1-2\cos\left(|\xi|^{\frac{\alpha}{2}}\sin\left(\frac{\alpha\pi}{4}\right)\right)e^{-|\xi|^{\frac{\alpha}{2}}\cos\left(\frac{\alpha\pi}{4}\right)}+e^{-2|\xi|^{\frac{\alpha}{2}}\cos\left(\frac{\alpha\pi}{4}\right)}}.$$

Then

$$1-2\cos\left(|\xi|^{\frac{\alpha}{2}}\sin\left(\frac{\alpha\pi}{4}\right)\right)e^{-|\xi|^{\frac{\alpha}{2}}\cos\left(\frac{\alpha\pi}{4}\right)}+e^{-2|\xi|^{\frac{\alpha}{2}}\cos\left(\frac{\alpha\pi}{4}\right)}\ge\left(1-e^{-|\xi|^{\frac{\alpha}{2}}\cos\left(\frac{\alpha\pi}{4}\right)}\right)^2,$$

so

$$2 \geq 1 + \left| e^{-\sqrt{(i\xi)^{\alpha}}} \right| \geq \left| 1 - e^{-\sqrt{(i\xi)^{\alpha}}} \right| \geq 1 - e^{-|\xi|^{\frac{\alpha}{2}} \cos\left(\frac{\alpha\pi}{4}\right)},$$

thus, taking reciprocals,

$$\frac{1}{2} \leq \frac{1}{\left| 1 - e^{-\sqrt{(i\xi)^{\alpha}}} \right|} \leq \frac{1}{1 - e^{-|\xi|^{\frac{\alpha}{2}} \cos\left(\frac{\alpha\pi}{4}\right)}}.$$

□

Lemma 2.4. *Suppose that $0 < \alpha < 1$. If $|\xi| \leq 1$, then the following inequality holds:*

$$\sup_{|\xi| \leq 1} \left(\frac{|\xi|^{\alpha}}{\left| 1 - e^{-|\xi|^{\frac{\alpha}{2}} \cos\left(\frac{\alpha\pi}{4}\right)} \right|} \right) \leq 2\sqrt{2}. \tag{2.8}$$

Proof. Let $z = |\xi|^{\frac{\alpha}{2}} \cos\left(\frac{\alpha\pi}{4}\right)$. For $|\xi| \leq 1$ and $0 < \alpha < 1$ we have $z \leq 1$ (because $|\xi|^{\frac{\alpha}{2}} \leq 1$ and $\cos\left(\frac{\alpha\pi}{4}\right) \leq 1$).

Using the Taylor expansion we get

$$e^{-z} = 1 - z + \frac{z^2}{2} + \cdots,$$

so that $|1 - e^{-z}| \geq |z| - |R(z)|$, where $R(z)$ is the remainder. The remainder $R(z)$ is bounded by a convergent series since $|z| \leq 1$, so

$$|R(z)| \leq \sum_{n=2}^{\infty} \frac{|z|^n}{n!} \leq \frac{|z|^2}{2}.$$

Thus

$$\left| 1 - e^{-z} \right| \geq |z| - \frac{|z|^2}{2} \geq |z| \left(1 - \frac{|z|}{2} \right).$$

Since $|z| \leq 1$, then

$$1 - \frac{|z|}{2} \geq \frac{1}{2} \Longrightarrow \left| 1 - e^{-z} \right| \geq \frac{|z|}{2}.$$

So

$$\begin{aligned} \left| 1 - e^{-|\xi|^{\frac{\alpha}{2}} \cos\left(\frac{\alpha\pi}{4}\right)} \right| &\geq \frac{\left| |\xi|^{\frac{\alpha}{2}} \cos\left(\frac{\alpha\pi}{4}\right) \right|}{2} \Longrightarrow \\ \frac{1}{\left| 1 - e^{-|\xi|^{\frac{\alpha}{2}} \cos\left(\frac{\alpha\pi}{4}\right)} \right|} &\leq \frac{2}{\left| |\xi|^{\frac{\alpha}{2}} \cos\left(\frac{\alpha\pi}{4}\right) \right|}. \end{aligned}$$

Thus

$$\frac{|\xi|^{\alpha}}{\left| 1 - e^{-|\xi|^{\frac{\alpha}{2}} \cos\left(\frac{\alpha\pi}{4}\right)} \right|} \leq \frac{2|\xi|^{\alpha}}{\left| |\xi|^{\frac{\alpha}{2}} \cos\left(\frac{\alpha\pi}{4}\right) \right|} = \frac{2|\xi|^{\frac{\alpha}{2}}}{\left| \cos\left(\frac{\alpha\pi}{4}\right) \right|}.$$

Since $0 < \alpha < 1$, we have $\frac{1}{\sqrt{2}} \leq \cos\left(\frac{\alpha\pi}{4}\right) \leq 1$, and for $|\xi| \leq 1$ we get

$$\sup_{|\xi| \leq 1} \left(\frac{|\xi|^{\alpha}}{\left| 1 - e^{-|\xi|^{\frac{\alpha}{2}} \cos\left(\frac{\alpha\pi}{4}\right)} \right|} \right) \leq 2\sqrt{2}.$$

$\square$

3. Regularization Method and Convergence Rates

In this section, we construct a Fourier truncation regularization scheme for problem (1.1) and derive Hölder-type convergence rates, first under an a priori and then under an a posteriori choice of the cut-off frequency. Given the noisy data g_δ, the regularized approximate solution is defined by

$$f_{\delta,\xi_{\max}}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi t} \frac{(i\xi)^\alpha}{1 - e^{-\sqrt{(i\xi)^\alpha}}} \widehat{g}_\delta(\xi)\, \chi_{\xi_{\max}} d\xi, \tag{3.1}$$

where $\chi_{\xi_{\max}}$ is the characteristic function of the interval $[-\xi_{\max}, \xi_{\max}]$, i.e.,

$$\chi_{\xi_{\max}} = \begin{cases} 1, & |\xi| \le \xi_{\max}, \\ 0, & |\xi| > \xi_{\max}, \end{cases} \tag{3.2}$$

and where $\xi_{\max}$ is a constant which will be selected appropriately as a regularization parameter. We also define

$$f_{\xi_{\max}}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi t} \frac{(i\xi)^\alpha}{1 - e^{-\sqrt{(i\xi)^\alpha}}} \widehat{g}(\xi)\, \chi_{\xi_{\max}} d\xi. \tag{3.3}$$

3.1. An A Priori Bound and a Conditional Stability Result.

We now give an error estimate between the regularized solution and the exact solution in the following theorem.

Theorem 3.1. *Let $f(t)$, given by (2.3), be the exact solution, and $f_{\delta,\xi_{\max}}(t)$, given by (3.1), be its regularized solution. Let assumption (2.6) and the a priori condition (2.4) hold. For $0 < \alpha < 1$ and $p > 0$, if we select*

$$\xi_{\max} = \left(\frac{E}{\delta}\right)^{\frac{1}{p+\alpha}}, \tag{3.4}$$

then the following estimate holds:

$$\|f(t) - f_{\delta,\xi_{\max}}(t)\| \le 3E^{\frac{\alpha}{p+\alpha}} \delta^{\frac{p}{p+\alpha}} + 2\sqrt{2}\delta.$$

Proof. By the Parseval formula and (2.3), (3.1), (3.3), and (3.4), we obtain

$$\begin{aligned} \|f(t) - f_{\delta,\xi_{\max}}(t)\| &= \left\| \widehat{f}(\xi) - \widehat{f}_{\delta,\xi_{\max}}(\xi) \right\| \\ &\le \left\| \widehat{f}(\xi) - \widehat{f}_{\xi_{\max}}(\xi) \right\| \\ &\quad + \left\| \widehat{f}_{\xi_{\max}}(\xi) - \widehat{f}_{\delta,\xi_{\max}}(\xi) \right\| \\ &= \left\| \frac{(i\xi)^\alpha}{1 - e^{-\sqrt{(i\xi)^\alpha}}} \widehat{g}(\xi) - \frac{(i\xi)^\alpha}{1 - e^{-\sqrt{(i\xi)^\alpha}}} \widehat{g}(\xi)\, \chi_{\xi_{\max}} \right\| \end{aligned}$$

$$
\begin{aligned}
& +\left\|\frac{(i\xi)^{\alpha}}{1-e^{-\sqrt{(i\xi)^{\alpha}}}}\widehat{g}(\xi)\chi_{\xi_{\max}}-\frac{(i\xi)^{\alpha}}{1-e^{-\sqrt{(i\xi)^{\alpha}}}}\widehat{g}_{\delta}(\xi)\chi_{\xi_{\max}}\right\| \\
= & \left\|\frac{(i\xi)^{\alpha}}{1-e^{-\sqrt{(i\xi)^{\alpha}}}}\widehat{g}(\xi)\left(1-\chi_{\xi_{\max}}\right)\right\| \\
& +\left\|\frac{(i\xi)^{\alpha}}{1-e^{-\sqrt{(i\xi)^{\alpha}}}}\left(\widehat{g}(\xi)-\widehat{g}_{\delta}(\xi)\right)\chi_{\xi_{\max}}\right\| \\
= & \left(\int\limits_{|\xi|>\xi_{\max}}\left(\frac{(i\xi)^{\alpha}}{1-e^{-\sqrt{(i\xi)^{\alpha}}}}\widehat{g}(\xi)\right)^{2}d\xi\right)^{\frac{1}{2}} \\
& +\left(\int\limits_{|\xi|\leq\xi_{\max}}\left(\frac{(i\xi)^{\alpha}}{1-e^{-\sqrt{(i\xi)^{\alpha}}}}\left(\widehat{g}(\xi)-\widehat{g}_{\delta}(\xi)\right)\right)^{2}d\xi\right)^{\frac{1}{2}} \\
= & I_1+I_2.
\end{aligned}
$$

Where

$$
\begin{aligned}
I_1 &= \left(\int\limits_{|\xi|>\xi_{\max}}\left(\frac{(i\xi)^{\alpha}}{1-e^{-\sqrt{(i\xi)^{\alpha}}}}\widehat{g}(\xi)\right)^{2}d\xi\right)^{\frac{1}{2}} \\
I_2 &= \left(\int\limits_{|\xi|\leq\xi_{\max}}\left(\frac{(i\xi)^{\alpha}}{1-e^{-\sqrt{(i\xi)^{\alpha}}}}\left(\widehat{g}(\xi)-\widehat{g}_{\delta}(\xi)\right)\right)^{2}d\xi\right)^{\frac{1}{2}}
\end{aligned}
$$

By (2.4) and (2.6) we get

$$
\begin{aligned}
I_1 &= \left(\int\limits_{|\xi|>\xi_{\max}}\left(\frac{(i\xi)^{\alpha}}{1-e^{-\sqrt{(i\xi)^{\alpha}}}}\widehat{g}(\xi)\right)^{2}d\xi\right)^{\frac{1}{2}} \\
&\leq \sup_{|\xi|>\xi_{\max}}\left(1+\xi^{2}\right)^{-\frac{p}{2}}\times\left(\int\limits_{|\xi|>\xi_{\max}}\left(\left|\widehat{f}(\xi)\right|\left(1+\xi^{2}\right)^{\frac{p}{2}}\right)^{2}d\xi\right)^{\frac{1}{2}} \\
&\leq \frac{1}{\xi_{\max}^{p}}E.
\end{aligned}
$$

So

$$
I_1\leq\frac{1}{\xi_{\max}^{p}}E. \tag{3.5}
$$

Using the inequalities

$$\begin{cases} \left|\frac{(i\xi)^\alpha}{1-e^{-\sqrt{(i\xi)^\alpha}}}\right| \le \frac{|\xi|^\alpha}{\left|1-e^{-|\xi|^{\frac{\alpha}{2}}\cos\left(\frac{\alpha\pi}{4}\right)}\right|} \le \frac{2|\xi|^{\frac{\alpha}{2}}}{\left|\cos\left(\frac{\alpha\pi}{4}\right)\right|} \le 2\sqrt{2} \quad \text{(by Lemma 2.4)}, \quad |\xi|<1 \\ \left|\frac{(i\xi)^\alpha}{1-e^{-\sqrt{(i\xi)^\alpha}}}\right| \le \frac{|\xi|^\alpha}{1-e^{-|\xi|^{\frac{\alpha}{2}}\cos\left(\frac{\alpha\pi}{4}\right)}} \le \frac{|\xi|^\alpha}{1-e^{-|\xi|^{\frac{\alpha}{2}}}} \le \frac{|\xi|^\alpha}{1-e^{-1}} \le 2\,|\xi|^\alpha, \quad |\xi|\ge 1 \end{cases}$$

Then

$$\begin{aligned} \left|\frac{(i\xi)^\alpha}{1-e^{-\sqrt{(i\xi)^\alpha}}}\right| &\le 2\left(\sqrt{2}+|\xi|^\alpha\right) \\ &\le 2\left(\sqrt{2}+\xi_{\max}^\alpha\right), \quad \text{for } |\xi|\le\xi_{\max} \end{aligned} \tag{3.6}$$

By (3.6) and (2.6) we get

$$\begin{aligned} I_2 &= \left(\int\limits_{|\xi|\le\xi_{\max}} \left(\frac{(i\xi)^\alpha}{1-e^{-\sqrt{(i\xi)^\alpha}}}\left(\widehat{g}(\xi)-\widehat{g}_\delta(\xi)\right)\right)^2 d\xi\right)^{\frac{1}{2}} \\ &\le 2\left(\sqrt{2}+\xi_{\max}^\alpha\right)\delta \end{aligned}$$

So

$$I_2 \le 2\left(\sqrt{2}+\xi_{\max}^\alpha\right)\delta \tag{3.7}$$

Combining (3.5) and (3.7) with (3.4) we have

$$\begin{aligned} \left\|f(t)-f_{\delta,\xi_{\max}}(t)\right\| &\le \frac{1}{\xi_{\max}^p}E+2\left(\sqrt{2}+\xi_{\max}^\alpha\right)\delta \\ &\le 3E^{\frac{\alpha}{p+\alpha}}\delta^{\frac{p}{p+\alpha}}+2\sqrt{2}\delta \end{aligned}$$

□

Remark 3.2. For $p>0$ and $0<\alpha\le 1$, we have

$$\left\|f(t)-f_{\delta,\xi_{\max}}(t)\right\| \le 3E^{\frac{\alpha}{p+\alpha}}\delta^{\frac{p}{p+\alpha}}+2\sqrt{2}\delta \longrightarrow 0 \text{ as } \delta\longrightarrow 0.$$

Hence $f_{\delta,\xi_{\max}}$ can be regarded as an approximation of the exact solution f.

3.2. **The Error Estimate with an A Posteriori Parameter Choice.** In this subsection, we give an a posteriori regularization-parameter choice rule which is independent of the a priori bound E. We choose the regularization parameter $\xi_{\max}^*$ as the solution of the equation

$$\left\|\left(1-\chi_{\xi_{\max}^*}(\xi)\right)\widehat{g}_\delta(\xi)\right\| = \tau\delta, \qquad \tau>1 \text{ is a constant.} \tag{3.8}$$

The following lemma and remark are required to prove the existence and uniqueness of the solution to equation (3.8).

Lemma 3.3. (See [5].) *Let* $h\left(\xi_{\max}^*\right)=\left\|\left(1-\chi_{\xi_{\max}^*}(\xi)\right)\widehat{g}_\delta(\xi)\right\|$. *Then, for* $\delta>0$, *the following hold:*

(1) $h\left(\xi_{\max}^*\right)$ *is a continuous function;*
(2) $\lim\limits_{\xi_{\max}\longmapsto\infty} h\left(\xi_{\max}^*\right)=0$;

(3) $\lim\limits_{\xi_{\max}\longmapsto 0} h\left(\xi_{\max}^{*}\right) = \|\widehat{g}_{\delta}\|$;

(4) $h\left(\xi_{\max}^{*}\right)$ *is a strictly decreasing function.*

Remark 3.4. A necessary condition for establishing the existence and uniqueness of the solution of equation (3.8) is the inequality $0 < \delta$ and $\delta < \|g_{\delta}\|$.

Lemma 3.5. *If $\xi_{\max}^{*}$ is the solution of equation (3.8), then the following inequality holds:*

$$\left|\xi_{\max}^{*}\right| \leq \left(\frac{2E}{(\tau-1)\,\delta}\right)^{\frac{1}{\alpha+p}}. \tag{3.9}$$

Proof. By (2.4), we obtain

$$\begin{aligned}
\left\|\left(1-\chi_{\xi_{\max}^{*}}(\xi)\right)\widehat{g}(\xi)\right\| &= \left(\int\limits_{|\xi|\geq\xi_{\max}^{*}} |\widehat{g}(\xi)|^{2}\,d\xi\right)^{\frac{1}{2}} \\
&= \left(\int\limits_{|\xi|\geq\xi_{\max}^{*}} \left|\frac{(i\xi)^{\alpha}}{1-e^{-\sqrt{(i\xi)^{\alpha}}}}\widehat{g}(\xi)\right|^{2}\left(1+\xi^{2}\right)^{p}\right. \\
&\quad \left.\times\left|\frac{(i\xi)^{\alpha}}{1-e^{-\sqrt{(i\xi)^{\alpha}}}}\right|^{-2}\left(1+\xi^{2}\right)^{-p}d\xi\right)^{\frac{1}{2}} \\
&\leq E\sup_{|\xi|\geq\xi_{\max}^{*}}\left|\left|\frac{(i\xi)^{\alpha}}{1-e^{-\sqrt{(i\xi)^{\alpha}}}}\right|^{-1}\left(1+\xi^{2}\right)^{-\frac{p}{2}}\right| \\
&\leq E\sup_{|\xi|\geq\xi_{\max}^{*}}\left|\left|\frac{1-e^{-\sqrt{(i\xi)^{\alpha}}}}{(i\xi)^{\alpha}}\right|\left(1+\xi^{2}\right)^{-\frac{p}{2}}\right| \\
&\leq E\frac{2}{\left|\xi_{\max}^{*}\right|^{\alpha}}\left|\xi_{\max}^{*}\right|^{-p} = \frac{2E}{\left|\xi_{\max}^{*}\right|^{\alpha+p}}.
\end{aligned}$$

So

$$\left\|\left(1-\chi_{\xi_{\max}^{*}}(\xi)\right)\widehat{g}(\xi)\right\| \leq \frac{2E}{\left|\xi_{\max}^{*}\right|^{\alpha+p}}. \tag{3.10}$$

By inequality (2.6) and (3.8), we obtain

$$\begin{aligned}
\left\|\left(1-\chi_{\xi_{\max}^{*}}(\xi)\right)\widehat{g}(\xi)\right\| &= \left\|\left(1-\chi_{\xi_{\max}^{*}}(\xi)\right)\left(\widehat{g}(\xi)-\widehat{g}_{\delta}(\xi)+\widehat{g}_{\delta}(\xi)\right)\right\| \\
&= \left\|\left(1-\chi_{\xi_{\max}^{*}}(\xi)\right)\widehat{g}_{\delta}(\xi)\right. \\
&\quad \left.+\left(1-\chi_{\xi_{\max}^{*}}(\xi)\right)\left(\widehat{g}(\xi)-\widehat{g}_{\delta}(\xi)\right)\right\| \\
&\geq \left\|\left(1-\chi_{\xi_{\max}^{*}}(\xi)\right)\widehat{g}_{\delta}(\xi)\right\| \\
&\quad -\left\|\left(1-\chi_{\xi_{\max}^{*}}(\xi)\right)\left(\widehat{g}(\xi)-\widehat{g}_{\delta}(\xi)\right)\right\| \\
&\geq \tau\delta-\delta = (\tau-1)\,\delta.
\end{aligned}$$

Then

$$\left\|\left(1-\chi_{\xi_{\max}^{*}}(\xi)\right)\widehat{g}(\xi)\right\| \geq (\tau-1)\,\delta. \tag{3.11}$$

Combining (3.7) with (3.8), we obtain

$$\frac{2E}{\left|\xi_{\max}^*\right|^{\alpha+p}} \geq \left\| \left(1-\chi_{\xi_{\max}^*}(\xi)\right) \widehat{g}(\xi) \right\| \geq (\tau-1)\,\delta. \tag{3.12}$$

So

$$\frac{2E}{\left|\xi_{\max}^*\right|^{\alpha+p}} \geq (\tau-1)\,\delta.$$

Thus

$$\left|\xi_{\max}^*\right| \leq \left(\frac{2E}{(\tau-1)\,\delta}\right)^{\frac{1}{\alpha+p}}. \tag{3.13}$$

$\square$

Lemma 3.6. *If $\xi_{\max}^*$ is the solution of equation (3.8), then the following inequality holds:*

$$\left\| \left(1-\chi_{\xi_{\max}^*}(\xi)\right) \widehat{g}(\xi) \right\| \leq (\tau+1)\,\delta. \tag{3.14}$$

Proof. By (2.6) and (3.8), we obtain

$$\begin{aligned} \left\| \left(1-\chi_{\xi_{\max}^*}(\xi)\right) \widehat{g}(\xi) \right\| &\leq \left\| \left(1-\chi_{\xi_{\max}^*}(\xi)\right) \widehat{g}_\delta(\xi) \right\| \\ &\quad + \left\| \left(1-\chi_{\xi_{\max}^*}(\xi)\right) \left(\widehat{g}(\xi)-\widehat{g}_\delta(\xi)\right) \right\| \\ &\leq \tau\delta+\delta=(\tau+1)\,\delta. \end{aligned}$$

$\square$

Theorem 3.7. *Suppose that conditions (2.6) and (2.4) hold, and take the solution $\xi_{\max}$ of equation (3.8) as the regularization parameter. Then we have the following error estimate:*

$$\begin{aligned} \left\| f(t)-f_{\delta,\xi_{\max}^*}(t) \right\| \leq & \left((2(\tau+1))^{\frac{p}{p+\alpha}} + 2\left(\frac{2}{\tau-1}\right)^{\frac{\alpha}{p+\alpha}} \right) \\ & \times E^{\frac{\alpha}{p+\alpha}} \delta^{\frac{p}{p+\alpha}} + 2\sqrt{2}\delta. \end{aligned}$$

Proof. Using the Parseval formula and the triangle inequality, we obtain

$$\begin{aligned} \left\| f(t)-f_{\delta,\xi_{\max}^*}(t) \right\| &= \left\| \widehat{f}(\xi)-\widehat{f}_{\delta,\xi_{\max}^*}(\xi) \right\| \\ &\leq \left\| \widehat{f}(\xi)-\widehat{f}_{\xi_{\max}^*}(\xi) \right\| \\ &\quad + \left\| \widehat{f}_{\xi_{\max}^*}(\xi)-\widehat{f}_{\delta,\xi_{\max}^*}(\xi) \right\| \\ &= \left\| \frac{(i\xi)^\alpha}{1-e^{-\sqrt{(i\xi)^\alpha}}}\widehat{g}(\xi) - \frac{(i\xi)^\alpha}{1-e^{-\sqrt{(i\xi)^\alpha}}}\widehat{g}(\xi)\,\chi_{\xi_{\max}^*} \right\| \\ &\quad + \left\| \frac{(i\xi)^\alpha}{1-e^{-\sqrt{(i\xi)^\alpha}}}\widehat{g}(\xi)\,\chi_{\xi_{\max}^*} - \frac{(i\xi)^\alpha}{1-e^{-\sqrt{(i\xi)^\alpha}}}\widehat{g}_\delta(\xi)\,\chi_{\xi_{\max}^*} \right\| \\ &= \left\| \frac{(i\xi)^\alpha}{1-e^{-\sqrt{(i\xi)^\alpha}}}\widehat{g}(\xi)\left(1-\chi_{\xi_{\max}^*}\right) \right\| \end{aligned}$$

$$
\begin{aligned}
& +\left\|\frac{(i\xi)^{\alpha}}{1-e^{-\sqrt{(i\xi)^{\alpha}}}}\left(\widehat{g}(\xi)-\widehat{g}_{\delta}(\xi)\right)\chi_{\xi^{*}_{\max}}\right\| \\
= & \left(\int\limits_{|\xi|>\xi^{*}_{\max}}\left|\frac{(i\xi)^{\alpha}}{1-e^{-\sqrt{(i\xi)^{\alpha}}}}\right|^{2}|\widehat{g}(\xi)|^{2}\,d\xi\right)^{\frac{1}{2}} \\
& +\left(\int\limits_{|\xi|\leq\xi^{*}_{\max}}\left(\frac{(i\xi)^{\alpha}}{1-e^{-\sqrt{(i\xi)^{\alpha}}}}\left(\widehat{g}(\xi)-\widehat{g}_{\delta}(\xi)\right)\right)^{2}d\xi\right)^{\frac{1}{2}} \\
= & \; I_1+I_2.
\end{aligned}
$$

Using the Hölder inequality and (3.14), we obtain

$$
\begin{aligned}
I_1^2 = & \int\limits_{|\xi|>\xi^{*}_{\max}}\left|\frac{(i\xi)^{\alpha}}{1-e^{-\sqrt{(i\xi)^{\alpha}}}}\right|^{2}|\widehat{g}(\xi)|^{2}\,d\xi \\
= & \int\limits_{|\xi|>\xi^{*}_{\max}}\left|\frac{(i\xi)^{\alpha}}{1-e^{-\sqrt{(i\xi)^{\alpha}}}}\right|^{2}|\widehat{g}(\xi)|^{\frac{2\alpha}{p+\alpha}}\,|\widehat{g}(\xi)|^{2\left(1-\frac{\alpha}{p+\alpha}\right)}\,d\xi \\
\leq & \left(\int\limits_{|\xi|>\xi^{*}_{\max}}\left(\left|\frac{(i\xi)^{\alpha}}{1-e^{-\sqrt{(i\xi)^{\alpha}}}}\right|^{2}|\widehat{g}(\xi)|^{\frac{2\alpha}{p+\alpha}}\right)^{\frac{p+\alpha}{\alpha}}d\xi\right)^{\frac{\alpha}{p+\alpha}} \\
& \times\left(\int\limits_{|\xi|>\xi^{*}_{\max}}\left(|\widehat{g}(\xi)|^{2\left(1-\frac{\alpha}{p+\alpha}\right)}\right)^{\frac{p+\alpha}{p}}d\xi\right)^{\frac{p}{p+\alpha}} \\
= & \left(\int\limits_{|\xi|>\xi^{*}_{\max}}\left|\frac{(i\xi)^{\alpha}}{1-e^{-\sqrt{(i\xi)^{\alpha}}}}\right|^{\frac{2(p+\alpha)}{\alpha}}|\widehat{g}(\xi)|^{2}\,d\xi\right)^{\frac{\alpha}{p+\alpha}} \\
& \times\left(\int\limits_{|\xi|>\xi^{*}_{\max}}|\widehat{g}(\xi)|^{2}\,d\xi\right)^{\frac{p}{p+\alpha}} \\
= & \left(\int\limits_{|\xi|>\xi^{*}_{\max}}\left|\frac{(i\xi)^{\alpha}}{1-e^{-\sqrt{(i\xi)^{\alpha}}}}\right|^{\frac{2p}{\alpha}+2}|\widehat{g}(\xi)|^{2}\,d\xi\right)^{\frac{\alpha}{p+\alpha}}\times\left(\int\limits_{|\xi|>\xi^{*}_{\max}}|\widehat{g}(\xi)|^{2}\,d\xi\right)^{\frac{p}{p+\alpha}} \\
= & \left(\int\limits_{|\xi|>\xi^{*}_{\max}}\left(1+\xi^{2}\right)^{-p}\left|\frac{(i\xi)^{\alpha}}{1-e^{-\sqrt{(i\xi)^{\alpha}}}}\right|^{\frac{2p}{\alpha}}\left(1+\xi^{2}\right)^{p}\left|\widehat{f}(\xi)\right|^{2}d\xi\right)^{\frac{\alpha}{p+\alpha}}
\end{aligned}
$$

$$\times\left(\int\limits_{|\xi|>\xi^*_{\max}}\left|\widehat{g}\left(\xi\right)\right|^2 d\xi\right)^{\frac{p}{p+\alpha}}.$$

Therefore

$$\begin{aligned}
I_1^2 &\le \sup_{|\xi|>\xi^*_{\max}}\left|\left(1+\xi^2\right)^{-p}\left|\frac{|\xi|^\alpha}{1-e^{-\sqrt{(i\xi)^\alpha}}}\right|^{\frac{2p}{\alpha}}\right|^{\frac{\alpha}{p+\alpha}} E^{\frac{2\alpha}{p+\alpha}}\left(\int\limits_{|\xi|>\xi^*_{\max}}\left|\widehat{g}\left(\xi\right)\right|^2 d\xi\right)^{\frac{p}{p+\alpha}}\\
&\le \sup_{|\xi|>\xi^*_{\max}}\left(|\xi|^{\frac{2\alpha p}{p+\alpha}-\frac{2\alpha p}{p+\alpha}}\left|\frac{1}{1-e^{-\sqrt{(i\xi)^\alpha}}}\right|^{\frac{2p}{p+\alpha}}\right) E^{\frac{2\alpha}{p+\alpha}}\left\|\left(1-\chi_{\xi^*_{\max}}\right)\widehat{g}\left(\xi\right)\right\|^{\frac{2p}{p+\alpha}}\\
&\le \sup_{|\xi|>\xi^*_{\max}}\left(\left|\frac{1}{1-e^{-\sqrt{(i\xi)^\alpha}}}\right|^{\frac{2p}{p+\alpha}}\right) E^{\frac{2\alpha}{p+\alpha}}\left\|\left(1-\chi_{\xi^*_{\max}}\right)\widehat{g}\left(\xi\right)\right\|^{\frac{2p}{p+\alpha}}.
\end{aligned}$$

By (3.14) and Lemma 2.2 we have

$$I_1^2 \le 2^{\frac{2p}{p+\alpha}} E^{\frac{2\alpha}{p+\alpha}}\left(\left(\tau+1\right)\delta\right)^{\frac{2p}{p+\alpha}}.$$

So

$$I_1 \le 2^{\frac{p}{p+\alpha}} E^{\frac{\alpha}{p+\alpha}}\left(\tau+1\right)^{\frac{p}{p+\alpha}}\delta^{\frac{p}{p+\alpha}}. \tag{3.15}$$

Combining (2.6) with Lemma 2.4, we obtain

$$\begin{aligned}
I_2 &= \left(\int\limits_{|\xi|\le\xi^*_{\max}}\left(\frac{(i\xi)^\alpha}{1-e^{-\sqrt{(i\xi)^\alpha}}}\left(\widehat{g}\left(\xi\right)-\widehat{g}_\delta\left(\xi\right)\right)\right)^2 d\xi\right)^{\frac{1}{2}}\\
&\le \sup_{|\xi|\le\xi^*_{\max}}\left|\frac{|\xi|^\alpha}{1-e^{-\sqrt{(i\xi)^\alpha}}}\right|\delta\\
&\le \sup_{|\xi|\le 1}\left|\frac{|\xi|^\alpha}{1-e^{-\sqrt{(i\xi)^\alpha}}}\right|\delta+\sup_{1<\xi\le\xi^*_{\max}}\left|\frac{|\xi|^\alpha}{1-e^{-\sqrt{(i\xi)^\alpha}}}\right|\delta\\
&\le \sup_{|\xi|\le 1}\left|\frac{|\xi|^\alpha}{1-e^{-\sqrt{(i\xi)^\alpha}}}\right|\delta+2\left(\xi^*_{\max}\right)^\alpha\delta.\\
&\le 2\sqrt{2}\delta+2\left(\xi^*_{\max}\right)^\alpha\delta=\left(2\sqrt{2}+2\left(\xi^*_{\max}\right)^\alpha\right)\delta.
\end{aligned}$$

Using (3.13), we obtain

$$\begin{aligned}
I_2 &\le \left(2\sqrt{2}+2\left(\xi^*_{\max}\right)^\alpha\right)\delta \le 2\left(\frac{2E}{(\tau-1)\delta}\right)^{\frac{\alpha}{p+\alpha}}\delta+2\sqrt{2}\delta\\
&= 2\left(\frac{2}{\tau-1}\right)^{\frac{\alpha}{p+\alpha}} E^{\frac{\alpha}{p+\alpha}}\delta^{1-\frac{\alpha}{p+\alpha}}+2\sqrt{2}\delta = 2\left(\frac{2}{\tau-1}\right)^{\frac{\alpha}{p+\alpha}} E^{\frac{\alpha}{p+\alpha}}\delta^{\frac{p}{p+\alpha}}+2\sqrt{2}\delta.
\end{aligned}$$

Then

$$I_2 \leq 2\left(\frac{2}{\tau-1}\right)^{\frac{\alpha}{p+\alpha}} E^{\frac{\alpha}{p+\alpha}}\delta^{\frac{p}{p+\alpha}} + 2\sqrt{2}\delta. \tag{3.16}$$

Combining (3.15) with (3.16), we obtain

$$\begin{aligned}\left\|f(t) - f_{\delta,\xi^*_{\max}}(t)\right\| &\leq 2^{\frac{p}{p+\alpha}} E^{\frac{\alpha}{p+\alpha}} (\tau+1)^{\frac{p}{p+\alpha}} \delta^{\frac{p}{p+\alpha}} \\ &\quad + 2\left(\frac{2}{\tau-1}\right)^{\frac{\alpha}{p+\alpha}} E^{\frac{\alpha}{p+\alpha}}\delta^{\frac{p}{p+\alpha}} + 2\sqrt{2}\delta \\ &= \left((2(\tau+1))^{\frac{p}{p+\alpha}} + 2\left(\frac{2}{\tau-1}\right)^{\frac{\alpha}{p+\alpha}}\right) \\ &\quad \times E^{\frac{\alpha}{p+\alpha}}\delta^{\frac{p}{p+\alpha}} + 2\sqrt{2}\delta.\end{aligned}$$

□

4. Numerical Experiment

In this section we test the proposed method on two numerical examples, with two goals: to check, in practice, the convergence rates proved above, and to assess the accuracy and stability of the reconstruction. We compare, throughout, the a posteriori parameter choice (3.8) against the a priori rule

$$\xi_{\max} = \left(\frac{E}{\delta}\right)^{\frac{1}{p+\alpha}},$$

in which $E = \|f\|_{H^p}$; in the discrepancy principle (3.8) we take $\tau = 1.1$.

All the computations are carried out in the frequency domain by means of the fast Fourier transform (FFT). Since the analysis is set on the half-line and the source is supported in $[0,1]$, the transform in time is discretized on a sufficiently large periodic window $[0,T]$ with $T = 16$ and $N = 2^{14}$ points, so that the periodization error is negligible on $[0,1]$. The limiting case $\alpha = 1$ (classical, non-fractional diffusion) is also included in the figures for reference, although the convergence analysis is established for $0 < \alpha < 1$. For a given exact source $f(t)$ supported in $[0,1]$, the corresponding exact data $g(t) = u(1,t)$ are computed from the forward relation $\widehat{g}(\xi) = \dfrac{1-e^{-\sqrt{(i\xi)^\alpha}}}{(i\xi)^\alpha}\widehat{f}(\xi)$, and the noisy data are generated by $g^\delta(t_j) = g(t_j) + \varepsilon_j$, where the ε_j are uniformly distributed random numbers rescaled so that $\left\|g^\delta - g\right\| = \delta$. The accuracy is measured by the relative L^2 error and the relative maximum (L^∞) error

$$e_r(f) = \frac{\|f - f_{\delta,\xi_{\max}}\|}{\|f\|}, \qquad e_\infty(f) = \frac{\|f - f_{\delta,\xi_{\max}}\|_{L^\infty[0,1]}}{\|f\|_{L^\infty[0,1]}}.$$

We take the exact source

$$f(t) = -\frac{t-c}{\sigma^2}\, e^{-(t-c)^2/\left(2\sigma^2\right)}, \qquad c = \tfrac{1}{2}, \quad \sigma = 0.12,$$

and, in all figures, the exact source and its regularized reconstruction are compared on the interval $[0,1]$; the reported errors are averaged over several random realizations of the noise.

4.1. **Example 1: comparison of the two rules and validation.** Figure 1 compares, for $p=1$, $\delta=0.01$ and $\alpha\in\{0.3,0.7,1\}$, the exact source with the reconstructions obtained by the a priori and the a posteriori rules; in every case both reconstructions are in excellent agreement with the exact solution. Table 1 lists the corresponding relative errors $e_r(f)$ for $p\in\left\{1,\frac{1}{2},2\right\}$ and $\alpha\in\{0.3,0.7,1\}$; the errors are small in all cases, and the a posteriori error, being independent of the source bound E, does not depend on p.

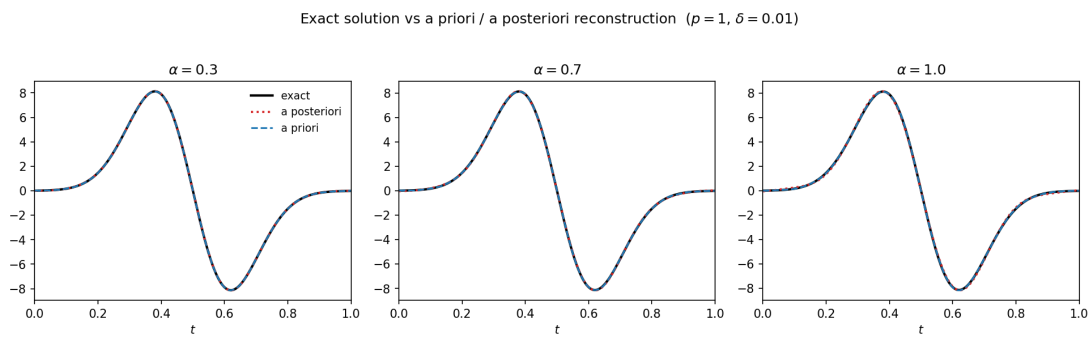

FIGURE 1. Example 1. Exact solution versus the a priori and a posteriori reconstructions for $\alpha=0.3,0.7,1$ ($p=1$, $\delta=0.01$).

TABLE 1. Example 1: relative L^2 error e_r and relative maximum error e_∞ of the a priori and a posteriori rules ($\delta=0.01$).

		a priori		a posteriori	
p	α	e_r	e_∞	e_r	e_∞
1	0.3	6.14×10^{-3}	2.61×10^{-3}	3.40×10^{-3}	2.22×10^{-3}
1	0.7	9.79×10^{-3}	3.62×10^{-3}	1.06×10^{-2}	7.22×10^{-3}
1	1	1.27×10^{-2}	4.27×10^{-3}	2.48×10^{-2}	1.74×10^{-2}
1/2	0.3	2.13×10^{-2}	7.11×10^{-3}	3.50×10^{-3}	2.26×10^{-3}
1/2	0.7	3.49×10^{-2}	1.45×10^{-2}	1.06×10^{-2}	7.07×10^{-3}
1/2	1	3.06×10^{-2}	1.03×10^{-2}	2.46×10^{-2}	1.71×10^{-2}
2	0.3	1.99×10^{-3}	7.24×10^{-4}	3.61×10^{-3}	2.30×10^{-3}
2	0.7	4.10×10^{-3}	1.50×10^{-3}	1.06×10^{-2}	7.12×10^{-3}
2	1	6.90×10^{-3}	1.87×10^{-3}	2.46×10^{-2}	1.72×10^{-2}

To validate the convergence of the method, Figure 2 and Table 2 report, for $p=1$ and $\alpha=0.7$, the relative error of both rules as the noise level δ decreases, together with the theoretical bound $3E^{\alpha/(p+\alpha)}\delta^{p/(p+\alpha)}+2\sqrt{2}\delta$ of Theorem 3.1. For every δ the error stays well below this bound and tends to zero as $\delta\to0$, which confirms the estimate numerically.

TABLE 2. Example 1: relative L^2 error e_r and relative maximum error e_∞ versus the noise level δ, and comparison with the theoretical bound ($p = 1$, $\alpha = 0.7$); the full range of δ is shown in Figure 2.

	a priori		a posteriori		
δ	e_r	e_∞	e_r	e_∞	bound
2.0×10^{-1}	3.69×10^{-2}	2.28×10^{-2}	1.65×10^{-1}	1.31×10^{-1}	6.14
2.0×10^{-2}	1.19×10^{-2}	4.11×10^{-3}	1.92×10^{-2}	1.33×10^{-2}	1.49
2.0×10^{-3}	6.14×10^{-3}	2.57×10^{-3}	2.28×10^{-3}	1.43×10^{-3}	3.77×10^{-1}
1.0×10^{-3}	5.00×10^{-3}	2.07×10^{-3}	1.17×10^{-3}	7.02×10^{-4}	2.50×10^{-1}

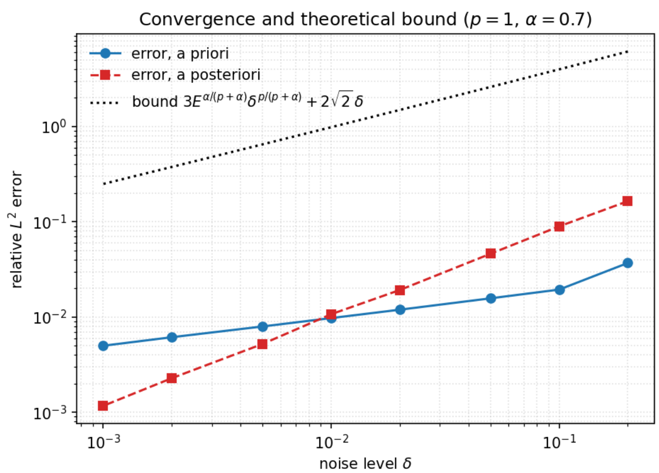

FIGURE 2. Example 1. Relative L^2 error of the a priori and a posteriori rules versus δ, compared with the theoretical bound ($p = 1$, $\alpha = 0.7$).

To complement the a priori analysis, and since the a posteriori bound of Theorem 3.7 depends explicitly on the fractional order, we now examine the a posteriori rule for several values of α. Table 3 reports, for $p = 1$ and $\delta = 0.01$, the relative errors e_r and e_∞ of the a posteriori reconstruction together with the corresponding bound of Theorem 3.7. The error increases with α—which reflects the stronger ill-posedness for larger fractional orders—yet it remains well below the theoretical bound in every case. Figure 3 confirms this behaviour: for each α the a posteriori error decreases steadily as $\delta \to 0$.

TABLE 3. A posteriori rule: relative errors e_r, e_∞ and the bound of Theorem 3.7 for several fractional orders ($p = 1$, $\delta = 0.01$).

α	e_r	e_∞	bound (Thm. 3.7)
0.3	3.45×10^{-3}	2.21×10^{-3}	5.16×10^{-1}
0.5	5.92×10^{-3}	3.90×10^{-3}	1.35
0.7	1.06×10^{-2}	7.14×10^{-3}	2.96
0.9	1.88×10^{-2}	1.30×10^{-2}	5.60

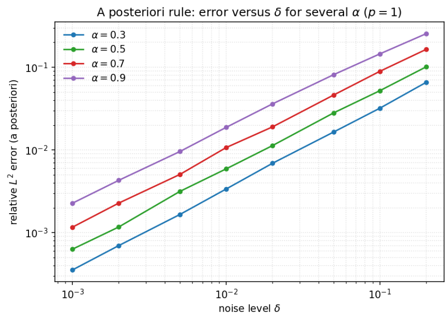

FIGURE 3. A posteriori rule: relative L^2 error versus δ for $\alpha = 0.3, 0.5, 0.7, 0.9$ $(p = 1)$.

4.2. **Example 2: a smoothed-box source.** To test the method on a qualitatively different profile, we now take a smoothed rectangular (plateau) source, obtained from the indicator function of a subinterval of $[0,1]$ regularized by raised-cosine transitions. Figure 4 compares, for $p = 1$, $\delta = 0.01$ and $\alpha \in \{0.3, 0.7, 1\}$, the exact source with the a priori and a posteriori reconstructions. The plateau and its steep transitions are well recovered; the relative L^2 errors of the a priori rule are 6.5×10^{-3}, 1.1×10^{-2} and 3.3×10^{-2} for $\alpha = 0.3, 0.7, 1$, respectively. As expected for this less regular profile, small Gibbs-type oscillations appear near the steep transitions and become more pronounced as α increases, which reflects the stronger ill-posedness of the problem for larger fractional orders.

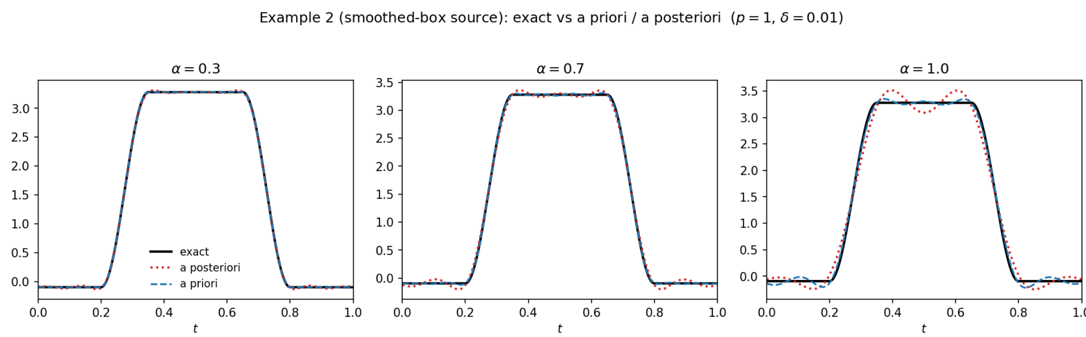

FIGURE 4. Example 2. Smoothed-box source: exact solution versus the a priori and a posteriori reconstructions for $\alpha = 0.3, 0.7, 1$ ($p = 1$, $\delta = 0.01$).

4.3. **Influence of the parameters E and δ.** For the smooth source of Example 1, Figure 5 (top panel) shows, for $p = 1$, $\delta = 0.01$ and $\alpha \in \{0.3, 0.7, 1\}$, the influence of the a priori bound E in the rule $\xi_{\max} = (E/\delta)^{1/(p+\alpha)}$: too small a value of E produces an over-smoothed reconstruction (too few frequencies are retained), whereas a sufficiently large E yields an excellent approximation of the exact solution. The bottom panel of Figure 5 shows the influence of the noise level δ for the a priori rule: the reconstruction remains stable and its agreement with the exact solution deteriorates gradually as δ increases, the effect being more pronounced for larger α, in accordance with the theoretical estimates.

4.4. **Error analysis and convergence order.** Figure 6 displays the pointwise absolute error $|f(t) - f_{\delta,\xi_{\max}}(t)|$ of the reconstruction for $p = 1$ and $\alpha = 0.7$. The error is uniformly small over $[0,1]$ for both rules and, as expected, it grows with the noise level δ (right panel).

To confirm the theoretical convergence order of Theorem 3.1, we consider a source of finite Sobolev regularity defined in the frequency domain by $\widehat{f}(\xi) = (1+\xi^2)^{-s/2}$ with $s = p + \frac{1}{2} + \varepsilon$ and $\varepsilon = 0.1$, so that f belongs to H^p but not to $H^{p+\varepsilon}$, that is, it essentially saturates the a priori condition. Table 4 reports the relative error and the estimated order of convergence,

$$\mathrm{EOC} = \frac{\ln(e_i/e_{i-1})}{\ln(\delta_i/\delta_{i-1})},$$

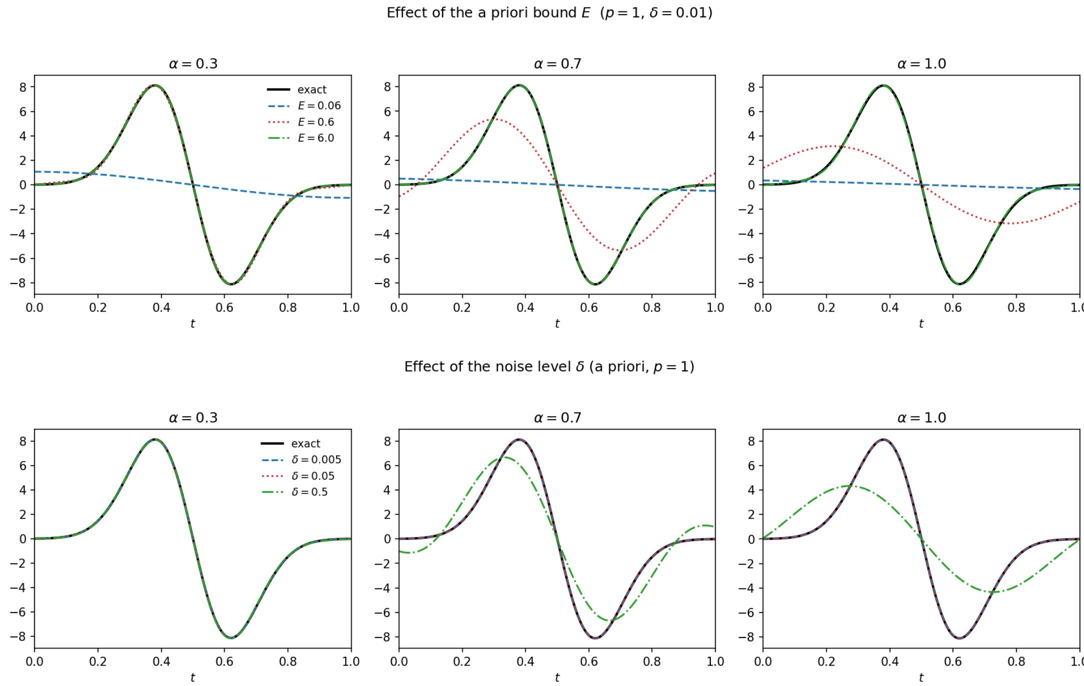

FIGURE 5. Example 1. Exact solution versus the a priori reconstruction: for three values of the a priori bound E (top; $p = 1$, $\delta = 0.01$, $\alpha = 0.3, 0.7, 1$) and for three noise levels δ (bottom; $p = 1$, $\alpha = 0.3, 0.7, 1$).

for $p = 1$ and $\alpha = 0.7$. The computed orders, together with the least-squares slopes shown in Figure 7 (0.64 for the a priori rule and 0.62 for the a posteriori rule), are in good agreement with the theoretical Hölder order $p/(p + \alpha) = 0.588$.

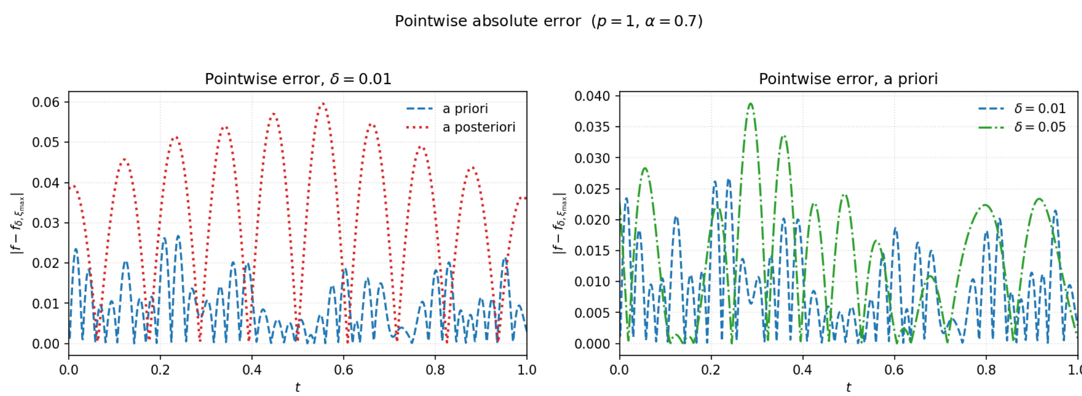

FIGURE 6. Pointwise absolute error $|f - f_{\delta,\xi_{\max}}|$ of the two rules at $\delta = 0.01$ (left) and of the a priori rule for two noise levels (right); $p = 1$, $\alpha = 0.7$.

Finally, Figure 8 illustrates the trade-off underlying the regularization. For a fixed noise level, the relative error, viewed as a function of the cut-off $\xi_{\max}$,

TABLE 4. Relative L^2 error and estimated order of convergence (EOC) for a finite-smoothness source ($p = 1$, $\alpha = 0.7$; theoretical order $p/(p+\alpha) = 0.588$); see Figure 7 for the full range of δ.

δ	e_r (a priori)	EOC	e_r (a posteriori)	EOC
2.0×10^{-1}	1.46×10^{-1}	—	2.61×10^{-1}	—
5.0×10^{-2}	6.18×10^{-2}	0.68	1.06×10^{-1}	0.60
1.0×10^{-2}	2.21×10^{-2}	0.64	3.92×10^{-2}	0.59
2.0×10^{-3}	7.89×10^{-3}	0.64	1.46×10^{-2}	0.61

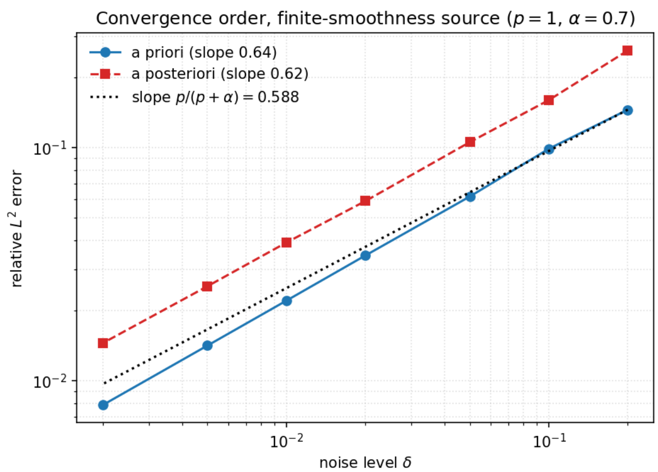

FIGURE 7. Convergence order for a finite-smoothness source: relative L^2 error versus δ in log–log scale, compared with the slope $p/(p+\alpha)$ ($p = 1$, $\alpha = 0.7$).

first decreases (the truncation error is reduced as more frequencies are retained) and then increases (the amplification of the noise by the factor $|M(\xi)| \sim |\xi|^{\alpha}$ becomes dominant), so that an optimal cut-off exists. The a priori choice $\xi_{\max} = (E/\delta)^{1/(p+\alpha)}$ (dashed lines) lies close to, and slightly to the right of, this optimum, in accordance with its order-optimality, while the a posteriori rule, being data-driven, adapts automatically to the noise level.

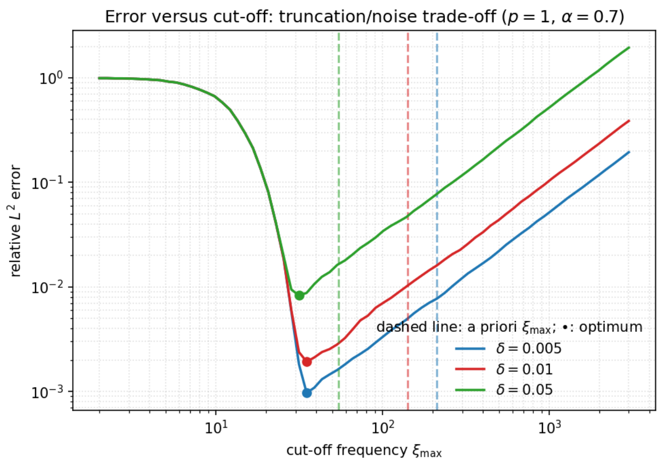

FIGURE 8. Relative L^2 error as a function of the cut-off $\xi_{\max}$ for three noise levels ($p = 1$, $\alpha = 0.7$). The dot marks the optimal cut-off and the dashed line the a priori choice.

5. Conclusion

This work addressed the recovery of a time-dependent source term in a diffusion equation governed by a Caputo derivative of fractional order $\alpha \in (0,1)$. Since the reconstruction from the supplementary measurement $u(1,t) = g(t)$ does not depend continuously on the data, the problem is ill-posed and requires regularization to produce a usable, stable approximation. We tackled this using Fourier truncation, in which the cut-off frequency $\xi_{\max}$ acts as the regularization parameter, and we examined two ways of selecting it: an a priori rule tied to the source bound E, and a data-driven a posteriori rule based on the discrepancy principle that does not require knowledge of E. Under either rule we proved Hölder-type convergence of the regularized approximation to the exact source as the noise level $\delta \to 0$. This is corroborated by the numerical experiments of Section 4: both rules reconstruct the exact source accurately, the observed errors remain below the theoretical bounds, and their dependence on E and δ matches the theory. Extending the present analysis to higher-dimensional and nonlinear time-fractional models is a natural direction for future work.

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