Gulf Journal of Mathematics
Vol 10, Issue 2 (2021) 25-30

GEOMETRIC BROWNIAN MOTION AND A NEW APPROACH TO THE SPREAD OF COVID-19 IN ITALY

NICOLA FABIANO[1*] AND STOJAN RADENOVIĆ[2]

Abstract. Starting from the problem of the asymmetry of the curve, a new model is adopted for the spread of Covid–19. The pandemic waves are modelled from a geometric Brownian motion, leading to a lognormal distribution. The infection waves are described correctly by the model and the asymmetry problem is solved. The diffusion model of the geometric Brownian motion is able to reproduce correctly the first two Covid–19 waves, and possibly show signs of a starting third wave.

1. Problem and Equation

The problem of spreading of the Covid–19 in time has been treated in [3], [4] and [5] by means of a differential equation due to [10]:

$$\frac{dx(t)}{dt} = \frac{c}{a}\left\{\frac{\exp[(b-t)/a]}{(1+\exp[(b-t)/a])^2}\right\} . \tag{1.1}$$

$x(t)$ represents the total number of cases, while its derivative $dx(t)/dt$ gives the growth of cases at the time t. The parameter a is related to the growth rapidity, c is the asymptotic total number of cases, and b indicates the peak position of the growth of cases.

The results obtained with this method so far are quite remarkable, yet as could be clearly seen in the figures of the cited papers, they share the common problem of providing a symmetric curve for the growth of cases. As already discussed there the asymmetry slowing the descent on the right handside of the curve is probably due to the fact that there is an important interaction among the infected patients spoiling a quicker recovery for the whole population.

Figure 1 shows the number of new cases per day of coronavirus in Italy with respect to the date. Data are provided by Italian ministry of health [8]. The two peaks of the different waves of the disease that happened in March and November are well visible, toghether with the curve asymmetry that makes the decay last much longer than the sharp increase.

Date: Received: Jan 10, 2021; Accepted: Apr 4, 2021.

* Corresponding author.

2010 *Mathematics Subject Classification.* Primary 60J65; Secondary 60J60, 60H10.

Key words and phrases. Coronavirus, COVID-19, Brownian motion, data fit.

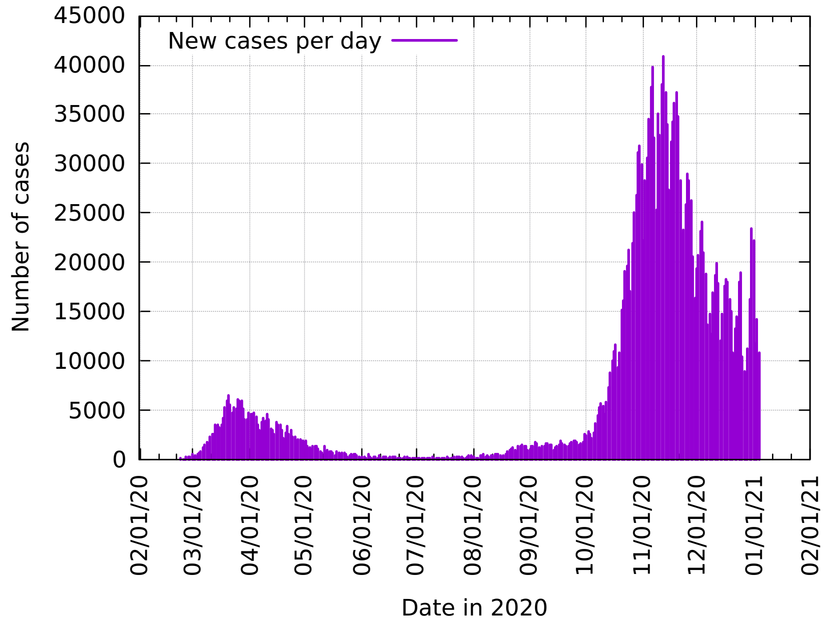

Figure 1. Number of new cases of coronavirus per day with respect to the date for the year 2020

In order to answer the question of the asymmetry we will use a different approach to the problem.

2. Brownian Motion

The problem of Brownian motion, that is the random movement of a particle in a fluid medium was first quantitatively solved by Einstein in 1905 [2]. The analogy to the spreading of a virus among the population is evident. He gave the diffusion equation for the density of particles $\rho(x,t)$

$$\frac{\partial \rho}{\partial t} = D\frac{\partial^2 \rho}{\partial x^2} \tag{2.1}$$

D being the diffusivity parameter. The solution of (2.1) for a system of N Brownian particles is given by the well–known heat kernel

$$\rho(x,t) = \frac{N}{\sqrt{4\pi D t}} \exp[-x^2/(4Dt)] \ , \tag{2.2}$$

and is readily apparent that the curve is symmetric, as each particle has an equal probability for going left or right.

In a more modern mathematical language, a Brownian motion $X(t)$ is the solution of the equation with constant drift (first term in dt) and diffusion coefficient (second term in $dW(t)$)

$$dX(t) = \mu dt + \sigma dW(t) \ , \tag{2.3}$$

where $X(0) = x_0$, $W(t)$ is a Wiener process with properties that non overlapping increments are independent, and $W(s) - W(t)$ for $0 \le t < s$ is a normal random variable with zero mean and variance $s-t$. Integrating the stochastic differential equation (2.3) one obtains the solution

$$X(t) = x_0 + \mu t + \sigma W(t) \ , \tag{2.4}$$

hence $X(t)$ is normally distributed, with mean $x_0 + \mu t$ and variance $\sigma^2 t$. Its density function is

$$f(t,x) = \frac{1}{\sigma\sqrt{2\pi t}} \exp[-(x - x_0 - \mu t)^2/(2\sigma^2 t)] \ , \tag{2.5}$$

recovering eq. (2.2) for $\mu = 0$.

3. Geometric Brownian motion

In order to circumvent the problem of a symmetric distribution, we consider a variation to the Brownian motion, namely the Geometric Brownian motion, described by a stochastic differential equation with linear drift and diffusion coefficients (compare to eq. 2.3))

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t) \ . \tag{3.1}$$

The application of Itô's lemma [6] to $\log(X)$ and the fact that $dW = O((dt)^{(1/2)})$ yelds the solution to equation (3.1)

$$X(t) = x_0 \exp[\hat{\mu} t + \sigma W(t)] \ , \tag{3.2}$$

where $\hat{\mu} = \mu - \sigma^2/2$.

As a consequence of the geometric Brownian motion, the $X(t)$ has a lognormal distribution:

$$f(t,x) = \frac{1}{\sigma x\sqrt{2\pi t}} \exp[-(\log x - \log x_0 - \hat{\mu} t)^2/(2\sigma^2 t)] \ , \tag{3.3}$$

which is not symmetric and its mean value is given by $x_0 e^{\mu t}$, always positive.

4. Lognormal fit to data

In order to use the lognormal distribution density to covid–19 data, we will use formula (3.3) with values $x_0 = 1, \ t = 1$, therefore the number of new cases at day x will be given by

$$N(x, a, \hat{\mu}, \sigma) = a\frac{1}{\sigma x\sqrt{2\pi}} \exp[-(\log x - \hat{\mu})^2/(2\sigma^2)] \ , \tag{4.1}$$

and the parameters $a, \hat{\mu}, \sigma$ will be determined by fitting the data found in [8].

The first wave of the disease, which started on 24^{th} of February 2020 and ended on the 13^{th} of July, is shown in Fig. (2) together with the fit of the lognormal function (4.1). The fit parameters obtained are presented in Table 1.

Figure (2) shows an excellent agreement with the data of the asymmetric distribution described in (4.1) for the number of new cases per day. The error on the parameters is quite small, and the two parameters $\hat{\mu}, \sigma$ are of the order of unity.

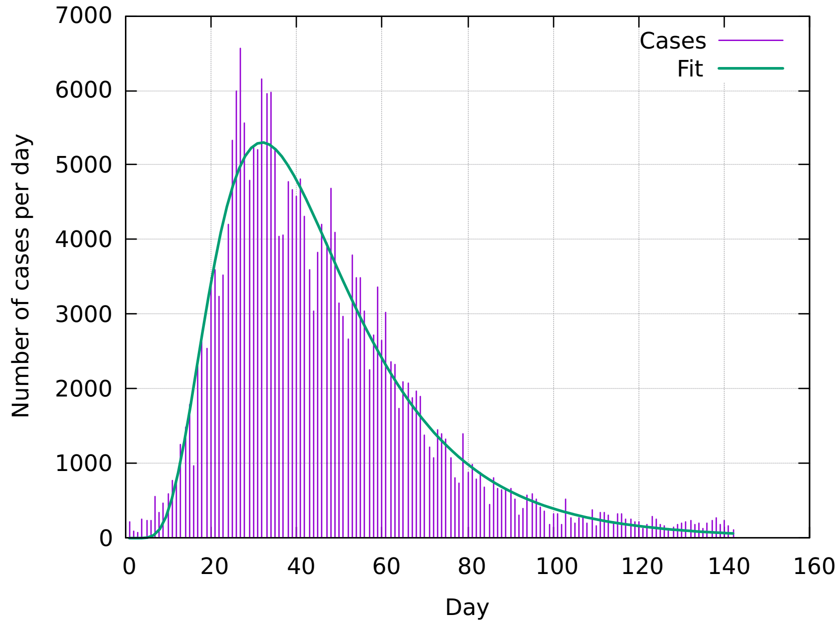

FIGURE 2. Number of new cases of coronavirus per day with respect to the numbers of days starting from 24^{th} of February 2020 the and ending on 13^{th} of July 2020

TABLE 1. Fit for the parameters $a, \hat{\mu}, \sigma$ of the first wave

Parameter	Value	Error	Error %
a	241019	4226	1.735
$\hat{\mu}$	3.71651	0.01045	0.2812
σ	0.499839	0.009351	1.871

The minimum value of new cases has been encountered on the 14^{th} of July with 114 new patients on that day. This will be the starting point of the second wave, that at the time of writing, 4^{th} of January 2021, is not over yet.

Figure (3) shows for the second wave a very good agreement of the fit with the data. Table 2 gives the fit value of parameters, that have a low error in this case as well. The two parameters $\hat{\mu}, \sigma$ are of the order of unity and are similar in both cases, while the parameter a has for the second wave a bigger value because this one is much larger than the first one.

TABLE 2. Fit for the parameters $a, \hat{\mu}, \sigma$ of the second wave

Parameter	Value	Error	Error %
a	1.9623×10^6	4.38×10^4	2.232
$\hat{\mu}$	4.86056	0.004919	0.1012
σ	0.191687	0.00505	2.634

Other useful informations on geometric Brownian motion could be found for instance in [7] and [1].

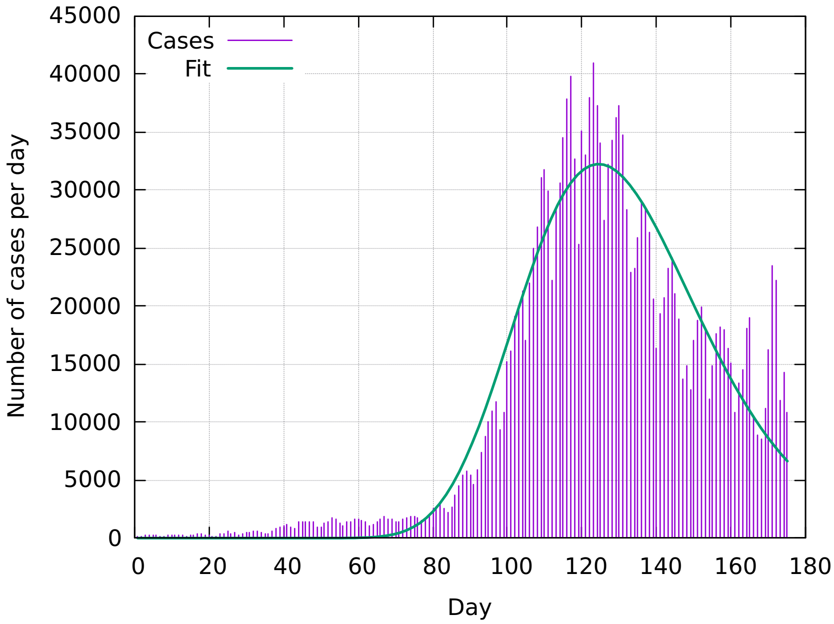

FIGURE 3. Number of new cases of coronavirus per day with respect to the numbers of days starting from the 14^{th} of July 2020 up to the 4^{th} of January 2021

5. Conclusion

The approach of the geometric Brownian motion to the problem of calculating the number of new cases per day of Covid–19 has proven to be very effective and the obtained lognormal distribution solves the problem of the asymmetry of the density of number of new cases. Compared to the existing data the two virus waves are described in an excellent manner. For the second wave the results prove that the end of it is yet to come, and the apparent discrepancy with the right limit of the data could be the sign that there is actually a third wave on the raise, maybe due to the holidays period travelling or to the new mutations of Covid–19, known as the lineage B.1.1.7, whose preliminar data are discussed in [9].

References

1. S. Altinkaya, S. Yalcin, *On the Chebyshev polynomial coefficient problem of some subclasses of bi–univalent functions*, Gulf Journal of Mathematics, **5(3)** (2017).
2. A. Einstein, A. 1905. *Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen [Investigations on the theory of Brownian Movement]*, Annalen der Physik, **322 (8)**: 549–560 (1905).
3. N. Fabiano, S. Radenović, *On COVID-19 diffusion in Italy: data analysis and possible outcome*, Vojnotehnički glasnik/Military Technical Courier, **68(1)**, pp.216-224 (2020). Available at: https://doi.org/10.5937/vojtehg68-25948.
4. N. Fabiano, S. Radenović, *What could happen after the first wave of COVID-19 diffusion in Italy: learning from the 1918 influenza pandemic*, Vojnotehnički glasnik/Military Technical Courier, **68(3)**, pp.413-423 (2020). Available at: https://doi.org/10.5937/vojtehg68-26500.

5. N. Fabiano, S. Radenović, *The Second Covid–19 wave of 2020 in Italy: a brief analysis*, Vojnotehnički glasnik/Military Technical Courier, **69(1)** (2021). Available at: https://doi.org/10.5937/vojtehg68-26511.
6. K. Itô, *Stochastic Integral*, Proc. Imperial Acad. Tokyo **20**, 519–524 (1944).
7. E.H. Lakhel, S. Hajji, *Neutral stochastic functional differential equation driven by fractional Brownian motion and Poisson point processes*. Gulf Journal of Mathematics, **4(3)** (2016).
8. Ministero della Salute, *Data of COVID-19 disease situation in Italy 2020* [online] (2020). Available at: https://raw.githubusercontent.com/pcm-dpc/COVID-19/master/dati-andamento-nazionale/dpc-covid19-ita-andamento-nazionale.csv (in Italian) [Accessed: 4 January 2021].
9. A. Rambaut et al., *Preliminary genomic characterisation of an emergent SARS-CoV-2 lineage in the UK defined by a novel set of spike mutations*, report of COVID-19 Genomics Consortium UK (2020), https://virological.org/t/preliminary-genomic-characterisation-of-an-emergent-sars-cov-2-lineage-in-the-uk-defined-by-a-novel-set-of-spike-mutations/563
10. PF. Verhulst, *Notice sur la loi que la population suit dans son accroissement. Correspondance mathématique et physique*, **10**, pp.113-121 (1838) (in French).

[1] Independent researcher, Rome, Italy
Email address: nicola.fabiano@gmail.com

[2] Faculty of Mechanical Engineering, University of Belgrade, Kraljice Marije 16, 11120 Beograd 35, Serbia
Email address: radens@beotel.net, sradenovic@mas.bg.ac.rs