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PROPERTIES OF A CLASS OF MEROMORPHIC UNIVALENT FUNCTIONS

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Abstract. In this paper, we study a subclass of meromorphic functions defined in the unit disk having a simple pole in it. A sufficient condition that guarantees the univalence of functions belonging to this subclass is derived. Inspired by this condition, we define a new subclass of meromorphic univalent functions in the unit disk characterized by a specific differential inequality. We derive a formula to represent functions in this subclass. Furthermore, conditions which are necessary and sufficient for membership of functions in terms of its Taylor's coefficient are obtained adjoined with a sharp estimate for the associated Fekete–Szegö functional.

Keywords. Meromorphic functions, Univalent functions, Taylor's coefficient.
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1. Introduction and preliminaries

Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denote the extended complex plane, where $\mathbb{C}$ represents the complex plane. The open unit disk is defined by $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{H}$ be the class of all holomorphic functions in $\mathbb{D}$, and define $\mathcal{A} = \{f \in \mathcal{H} : f(0) = 0,\ f'(0) = 1\}$. Moreover, let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of functions that are univalent in $\mathbb{D}$. Comprehensive discussions on these function classes are available in [8, 11]. Every function $f \in \mathcal{S}$ admits the Taylor series representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{1.1}$$

Over the last century, geometric function theory has witnessed the development of many profound and significant results. Introduced by Ludwig Bieberbach in 1916, the Bieberbach conjecture soon emerged as a fundamental problem in geometric function theory.The conjecture states that if a function $f \in \mathcal{S}$ is represented by the expansion given in (1.1), then its coefficients satisfy the estimate $|a_n| \leq n$ for every $n \geq 2$. This long-standing conjecture was finally established by de Branges in 1985 (see [7]). Prior to this achievement, considerable efforts were devoted to proving the conjecture for various geometric subclasses of $\mathcal{S}$. In particular, it was

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verified for the classes of convex, starlike, and close-to-convex functions (see [8]). In more recent years, the class $\mathcal{U}(\lambda)$, which forms a subclass of $\mathcal{S}$, has been the subject of extensive investigation in geometric function theory. For a parameter λ satisfying $0 < \lambda \leq 1$, let $\mathcal{U}(\lambda)$ stand for the family of functions in $\mathcal{A}$ defined by

$$\mathcal{U}(\lambda) = \{f \in \mathcal{A} : |U_f(z)| < \lambda,\ z \in \mathbb{D}\},$$

where the operator U_f is given by

$$U_f(z) = \left(\frac{z}{f(z)}\right)^2 f'(z) - 1.$$

It is a well-established fact that the inclusion $\mathcal{U}(\lambda) \subseteq \mathcal{S}$ holds (see [1, 13]. The geometric properties and various aspects of this class have been investigated extensively in [10],[12]-[14].

Motivated by these developments, Sivasubramanian [15] introduced the generalized subclass $\mathcal{U}(\alpha, \mu, \lambda)$ of analytic functions. Let $\lambda > 0$, $0 < \alpha \leq 1$, and $0 \leq \mu < \alpha$. A function $f \in \mathcal{A}$ is said to be a member of the class $\mathcal{U}(\alpha, \mu, \lambda)$ if it satisfies

$$\left|(1-\alpha)\left(\frac{z}{f(z)}\right)^{\mu} + \alpha\left(\frac{z}{f(z)}\right)^{\mu+1} f'(z) - 1\right| < \lambda, \quad z \in \mathbb{D}.$$

Each function f in the class $\mathcal{U}(\alpha, \mu, \lambda)$ has a representation formula

$$\left(\frac{z}{f(z)}\right)^{\mu} = 1 - \frac{\lambda\mu}{\alpha}\int_0^1 \frac{w(sz)}{s^{\frac{\mu}{\alpha}+1}} ds$$

where w is analytic in $\mathbb{D}$, satisfies $|w(z)| < 1$, and fulfills the conditions $w(0) = 0$. It follows from the Schwarz lemma that $|w(z)| < |z|$. Let $\mathcal{B}$ denote the family of all functions w that are analytic in $\mathbb{D}$ and satisfy $|w(z)| < 1$ for every $z \in \mathbb{D}$. Inspired by the investigations on the class $\mathcal{U}(\lambda)$, the authors in [3, 4] introduced its meromorphic counterpart, denoted by $\mathcal{V}_p(\lambda)$, for $0 < \lambda \leq 1$. We briefly recall the definition of this class. Let $\mathcal{A}(p)$ denote the family of functions that are analytic in $\mathbb{D} \setminus \{p\}$, where $p \in (0,1)$, and possess a simple pole at $z = p$ with residue $m \neq 0$. These functions are normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Furthermore, let $\Sigma(p)$ represent the subclass of $\mathcal{A}(p)$ consisting of functions that are univalent in $\mathbb{D}$. Several geometric properties and further developments concerning this family of meromorphic functions can be found in [4, 5, 6]. Motivated by the work of Bhowmik *et al.* [4], we introduce in this paper the class $\mathcal{U}_p(\alpha, \lambda)$, which serves as a meromorphic counterpart of the analytic class $\mathcal{U}(\alpha, \mu, \lambda)$. In Section 2, we derive a condition which is sufficient for functions in the class $\mathcal{A}(p)$ to be univalent.This result naturally leads to the definition of the subclass $\mathcal{U}_p(\alpha, \lambda)$ of $\Sigma(p)$. In Section 3, we establish a representation theorem for functions in the class $\mathcal{U}_p(\alpha, \lambda)$ having the expansion

$$\frac{z}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots, \quad z \in \mathbb{D}.$$

Furthermore, we derive sharp estimates for the coefficients $|b_n|$ for every $n \geq 1$. In the final section, we obtain the optimal upper estimate for the Fekete–Szegö functional corresponding to functions in the class $\mathcal{U}_p(\alpha, \lambda)$.

2. A Univalence Criterion for functions in $\mathcal{A}(p)$

Lemma 2.1. *For each $f \in \mathcal{A}(p)$ and an arbitrary but fixed point x of $\mathbb{D} \setminus \{p\}$ with $f'(x) \neq 0$, if $|h'(z,x)| \leq |m|/(1+p)^2$ for $z \in \mathbb{D}$ where $'$ the derivative taken with respect to z where*

$$h(z,x) = -\frac{m}{z-p} + \frac{\alpha m(1-p^2)^{-1}(1-|x|^2)f'(x)}{f\left(\dfrac{\frac{z-p}{1-pz}+x}{1+\bar{x}(\frac{z-p}{1-pz})}\right)} + m(1-\alpha)\int_0^z \frac{1}{(\zeta-p)(1-p\zeta)f\left(\dfrac{z-p}{1-pz}\right)}d\zeta,$$

then then f is univalent in $\mathbb{D}$.

Proof. Define

$$g(z) = \frac{\alpha m(1-p^2)^{-1}(1-|x|^2)f'(x)}{f\left(\dfrac{\frac{z-p}{1-pz}+x}{1+\bar{x}(\frac{z-p}{1-pz})}\right)} + m(1-\alpha)\int_0^z \frac{1}{(\zeta-p)(1-p\zeta)f\left(\frac{z-p}{1-pz}\right)}d\zeta \tag{2.1}$$

where $f \in \mathcal{A}(p)$ and $x \in \mathbb{D} \setminus \{p\}$. Since $g(p) = \infty$ and h' is bounded,the function g, defined by (2.1), has no poles in $\mathbb{D}$ except at the point $z = p$. Moreover, it follows from (2.1) that the function f is univalent in $\mathbb{D}$ if and only if g is univalent in $\mathbb{D}$. A straightforward computation shows that the residue of g at its pole $z = p$ is equal to m and we can express the function g as:

$$g(z) = \frac{m}{z-p} + h(z,x)\cdot \tag{2.2}$$

We prove that f to be univalent in $\mathbb{D}$ by establishing the same for g in $\mathbb{D}$. Evidently $g(z) - m/(z-p) = h(z,x)$ is analytic in $\mathbb{D}$. Now for $z_1, z_2 \in \mathbb{D}\setminus\{p\}$, $z_1 \neq z_2$, we have

$$\begin{aligned}\left(g(z_2) - \frac{m}{z_1-p}\right) - \left(g(z_1) - \frac{m}{z_1-p}\right) &= h(z_2,x) - h(z_1,x)\\ &= \int_{z_1}^{z_2} h'(z,x)dz.\end{aligned}$$

where the integration is carried out along the line segment connecting the points z_1 and z_2. By introducing the parametrization $z = z_1 + t(z_2 - z_1)$, where $0 \leq t \leq 1$, the above integral can be expressed as

$$\int_0^1 (z_2 - z_1)h'(z,x)dt.$$

Now multiplying $\left|-\dfrac{(z_1-p)(z_2-p)}{m(z_2-z_1)}\right|$ on both sides we have,

$$\begin{aligned}\left|\frac{(g(z_1)-g(z_2))(z_1-p)(z_2-p)}{m(z_2-z_1)}\right| &= \left|\frac{(z_1-p)(z_2-p)}{m}\int_0^1 h'(z,x)dt\right| \\ &\leq \frac{|(z_1-p)||(z_2-p)|}{|m|}\int_0^1 |h'(z,x)|dt \\ &< \frac{(1+p)^2}{|m|}\frac{|m|}{(1+p)^2}\int_0^1 dt \\ &= 1.\end{aligned}$$

This shows that $g(z_1)-g(z_2)\neq 0$ for $z_1\neq z_2$ *i.e.*, g is univalent in $\mathbb{D}$, and consequently so is f. □

Theorem 2.2. *Let $f\in\mathcal{A}(p)$ and $\alpha>1$.If* $\left|(\alpha-1)\left(\frac{z}{f(z)}\right)+\alpha\left(\frac{z}{f(z)}\right)^2 f'(z)-1\right|\leq\left(\frac{1-p}{1+p}\right)^2$ *for $z\in\mathbb{D}$, then f is univalent in $\mathbb{D}$.*

Proof. It follows from Lemma 2.1 that, f is univalent in $\mathbb{D}$ whenever $|h'(z,x)|\leq |m|/(1+p)^2$ for $z\in\mathbb{D}$. The fact that $|z^2h'(z,0)|\leq|m|/(1+p)^2$ for $z\in\mathbb{D}$ leads to $|h'(z,0)|\leq|m|/(1+p)^2$ for $z\in\mathbb{D}$ by an application of the Maximum Modulus Principle for holomorphic functions. Using (2.1) and (2.2), we get

$$h(z,x)=-\frac{m}{z-p}+\frac{\alpha m(1-p^2)^{-1}(1-|x|^2)f'(x)}{f\left(\dfrac{\frac{z-p}{1-pz}+x}{1+\bar{x}(\frac{z-p}{1-pz})}\right)}-m(\alpha-1)\int_0^z\frac{1}{(\zeta-p)(1-p\zeta)f\left(\frac{z-p}{1-pz}\right)}d\zeta$$

where $h(z,x)$ is holomorphic in $\mathbb{D}$.Differentiating $h(z,x)$ with respect to z and evaluating at $x=0$ gives

$$h'(z,0)=\frac{m}{(z-p)^2}\left[1-\frac{\alpha\left(\frac{z-p}{1-pz}\right)^2 f'\left(\frac{z-p}{1-pz}\right)}{f^2\left(\frac{z-p}{1-pz}\right)}-\frac{\left(\frac{z-p}{1-pz}\right)(\alpha-1)}{f\left(\frac{z-p}{1-pz}\right)}\right].$$

A change in variable by $\xi=\dfrac{z-p}{1-pz}$ i.e. $z=\dfrac{\xi+p}{1+p\xi}$, gives $|\xi|<1$ if and only if $|z|<1$. From the above equality, we get

$$\xi^2h'\left(\frac{\xi+p}{1+p\xi},0\right)=\frac{m(1+p\xi)^2}{(1-p^2)^2}\left[1-\frac{\alpha\xi^2f'(\xi)}{f^2(\xi)}-\frac{(\alpha-1)\xi}{f(\xi)}\right].$$

Therefore, from the given condition, we get

$$\left|\xi^2h'\left(\frac{\xi+p}{1+p\xi},0\right)\right|\leq\frac{|m|}{(1+p)^2},$$

which will force f to be univalent through lemma(2.1). □

Definition 2.3. Let $\mathcal{U}_p(\alpha,\lambda)$ denote the class of functions $f \in \mathcal{A}(p)$ satisfying $\left|(1-\alpha)\left(\frac{z}{f(z)}\right)+\alpha\left(\frac{z}{f(z)}\right)^2 f'(z)-1\right| < \lambda\mu$ for $\alpha > 1$,
$\mu = \left(\frac{1-p}{1+p}\right)^2$ and $\lambda > 0$.

3. Fundamental properties for functions in $\mathcal{U}_p(\alpha,\lambda)$

Theorem 3.1. *The family $\mathcal{U}_p(\alpha,\lambda)$ is invariant under conjugation.*

Proof. Since $f \in \mathcal{U}_p(\alpha,\lambda)$, we have $f(0) = 0 = f'(0) = 1$, $f(p) = \infty$ and $\left|(1-\alpha)\left(\frac{z}{f(z)}\right)+\alpha\left(\frac{z}{f(z)}\right)^2 f'(z)-1\right| < \lambda\mu$ for $z \in \mathbb{D}$. Define $h(z) = \overline{f(\overline{z})}$. Note that h is analytic in $\mathbb{D}\setminus\{p\}$ with $h(p) = \infty$. Also, $h'(z) = \overline{f'(\overline{z})}$ and $h(0) = 0 = h'(0) - 1$. A computation gives,

$$\begin{aligned}
|\mathcal{U}_h(z)| &= \left|(1-\alpha)\left(\frac{z}{h(z)}\right)+\alpha h'(z)\left(\frac{z}{h(z)}\right)^2-1\right| \\
&= \left|(1-\alpha)\left(\frac{z}{\overline{f(\overline{z})}}\right)+\alpha\overline{f'(\overline{z})}\left(\frac{z}{\overline{f(\overline{z})}}\right)^2-1\right| \\
&= \left|(1-\alpha)\left(\frac{\overline{z}}{f(\overline{z})}\right)+\alpha f'(\overline{z})\left(\frac{\overline{z}}{f(\overline{z})}\right)^2-1\right| \\
&< \lambda\mu.
\end{aligned}$$

This shows that $h \in \mathcal{U}_p(\alpha,\lambda)$. □

Theorem 3.2. *Each $f \in \mathcal{U}_p(\alpha,\lambda)$ can be represented as*

$$\frac{z}{f(z)} = 1 - \frac{z}{p}\left(1-\frac{\lambda\mu p^{1/\alpha}}{\alpha}\int_0^p \frac{w(\zeta)}{\zeta^{1+1/\alpha}}d\zeta\right) - \frac{\lambda\mu z^{1/\alpha}}{\alpha}\int_0^z \frac{w(\zeta)}{\zeta^{1+1/\alpha}}d\zeta,$$

where $w \in \mathcal{B}$.

Proof. Let $f \in \mathcal{U}_p(\alpha,\lambda)$. Then

$$|\mathcal{U}_{f_\alpha}(z)| = \left|(1-\alpha)\left(\frac{z}{f(z)}\right)+\alpha f'(z)\left(\frac{z}{f(z)}\right)^2-1\right| < \lambda\mu.$$

Introduce

$$h(z) = \frac{cf(z)}{c+f(z)} \text{ where } -c \notin f(\mathbb{D}). \tag{3.1}$$

Note that h is analytic and univalent in $\mathbb{D}$, and satisfies $h(0) = 0$ and $h'(0) = 1$. Therefore, $h \in \mathcal{S}$ and

$$(1-\alpha)\left(\frac{z}{h(z)}\right)+\alpha h'(z)\left(\frac{z}{h(z)}\right)^2-1 = (1-\alpha)\left(\frac{z}{f(z)}\right)+\alpha f'(z)\left(\frac{z}{f(z)}\right)^2-1,$$

that is, $\mathcal{U}_{h_\alpha}(z) = \mathcal{U}_{f_\alpha}(z)$. Thus $h \in \mathcal{S}$ with $|\mathcal{U}_h(z)| < \lambda\mu$. Consequently $h \in \mathcal{U}(\alpha, \lambda)$ and it can be represented in the following form (see [13]):

$$\frac{z}{h(z)} = 1 - \frac{\lambda\mu z^{1/\alpha}}{\alpha}\int_0^z \frac{w(\zeta)}{\zeta^{1+1/\alpha}}d\zeta, \tag{3.2}$$

where $w \in \mathcal{B}$. Next, from (3.1) we obtain

$$\frac{z}{h(z)} = \frac{z}{f(z)} + \frac{z}{c}\cdot$$

Substituting the above expression for $z/h(z)$ into equation (3.2), we obtain

$$\frac{z}{f(z)} = 1 - \frac{z}{c} - \frac{\lambda\mu z^{1/\alpha}}{\alpha}\int_0^z \frac{w(\zeta)}{\zeta^{1+1/\alpha}}d\zeta.$$

$$\frac{1}{c} = \frac{1}{p}\left(1 - \frac{\lambda\mu p^{1/\alpha}}{\alpha}\int_0^p \frac{w(\zeta)}{\zeta^{1+1/\alpha}}d\zeta\right).$$

hence, for every function $f \in \mathcal{U}_p(\alpha, \lambda)$ admits the representation

$$\frac{z}{f(z)} = 1 - \frac{z}{p}\left(1 - \frac{\lambda\mu p^{1/\alpha}}{\alpha}\int_0^p \frac{w(\zeta)}{\zeta^{1+1/\alpha}}d\zeta\right) - \frac{\lambda\mu z^{1/\alpha}}{\alpha}\int_0^z \frac{w(\zeta)}{\zeta^{1+1/\alpha}}d\zeta\cdot$$

□

Corollary 3.3. *Let $f \in \mathcal{U}_p(\alpha, \lambda)$ for some $\lambda > 0, \alpha > 1$ and $0 < p < 1$. Then*

$$\left|\frac{z}{f(z)} - 1\right| \leq \frac{|z|}{p}\left(1 + \frac{\lambda\mu p}{(\alpha-1)}\right) + \frac{\lambda\mu|z|}{(\alpha-1)}\cdot$$

Proof. Since for each $f \in \mathcal{U}_p(\alpha, \lambda)$ can be expressed as

$$\frac{z}{f(z)} = 1 - \frac{z}{p}\left(1 - \frac{\lambda\mu p^{1/\alpha}}{\alpha}\int_0^p \frac{w(\zeta)}{\zeta^{1+1/\alpha}}d\zeta\right) - \frac{\lambda\mu z^{1/\alpha}}{\alpha}\int_0^z \frac{w(\zeta)}{\zeta^{1+1/\alpha}}d\zeta, \zeta \in \mathbb{D} \text{ and } \omega \in \mathcal{B}.$$

$$\begin{aligned}\left|\frac{z}{f(z)} - 1\right| &= \left| - \frac{z}{p}\left(1 - \frac{\lambda\mu p^{1/\alpha}}{\alpha}\int_0^p \frac{w(\zeta)}{\zeta^{1+1/\alpha}}d\zeta\right) - \frac{\lambda\mu z^{1/\alpha}}{\alpha}\int_0^z \frac{w(\zeta)}{\zeta^{1+1/\alpha}}d\zeta\right| \\ &\leq \frac{|z|}{p}\left(1 + \frac{\lambda\mu p^{1/\alpha}}{\alpha}\int_0^p \frac{|w(\zeta)|}{|\zeta^{1+1/\alpha}|}d\zeta\right) + \frac{\lambda\mu |z|^{1/\alpha}}{\alpha}\int_0^z \frac{|w(\zeta)|}{|\zeta|^{1+1/\alpha}}d\zeta \\ &\leq \frac{|z|}{p}\left(1 + \frac{\lambda\mu p}{(\alpha-1)}\right) + \frac{\lambda\mu|z|}{(\alpha-1)}.\end{aligned}$$

□

Corollary 3.4. *Let $f \in \mathcal{U}_p(\alpha, \lambda)$ for some $\lambda > 0, \alpha > 1$ and $0 < p < 1$.Then $|U_{f_\alpha}(z)| \leq (\alpha - 1)|a_2||z| + \lambda\mu|z|$.*

Proof. Using the representation of $f \in \mathcal{U}_p(\alpha, \lambda)$

$$
\begin{aligned}
U_{f_\alpha}(z) &= -\alpha z\left(\frac{z}{f(z)}\right)' + \frac{z}{f(z)} - 1 \\
&= -\alpha z\left(1 - a_2 z - \frac{\lambda\mu z^{1/\alpha}}{\alpha}\int_0^z \frac{w(\zeta)}{\zeta^{1+1/\alpha}}d\zeta\right)' + 1 - a_2 z - \frac{\lambda\mu z^{1/\alpha}}{\alpha}\int_0^z \frac{w(\zeta)}{\zeta^{1+1/\alpha}}d\zeta - 1 \\
&= (\alpha - 1)a_2 z + \lambda\mu z w(z)
\end{aligned}
$$

where $w \in \mathcal{B}$. Applying the Schwarz lemma, we obtain $|w(z)| < |z|$, and hence the desired inequality follows.Next, we consider $f_\theta, \theta \in [0, 2\pi)$, one can calculate $U_{f_\theta}(z) = (\alpha - 1)a_2 e^{i\theta} z + \lambda\mu e^{i\theta} z$,which shows that equality holds for the given function. □

4. Second coefficient estimates for functions in $\mathcal{U}_p(\alpha, \lambda)$

Theorem 4.1. *For each $f \in \mathcal{U}_p(\alpha, \lambda), \alpha > 1$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is its Taylor series expansion, in the disc $\{z : |z| < p\}$.Then the region of variability of second Taylor coefficient a_2 is the disc given by*

$$
|a_2 - 1/p| < \frac{\lambda\mu}{\alpha - 1}
$$

.

Proof. For each $f \in \mathcal{U}_p(\alpha, \lambda)$ has the representation formula of the form

$$
\frac{z}{f(z)} = 1 - a_2 z - \frac{\lambda\mu z^{1/\alpha}}{\alpha}\int_0^z \frac{w(\zeta)}{\zeta^{1+1/\alpha}}d\zeta,
$$

where $w \in \mathcal{B}$. Now at $z = p$, the above equality implies that

$$
a_2 = \frac{1}{p}\left(1 - \frac{\lambda\mu p^{1/\alpha}}{\alpha}\int_0^p \frac{w(\zeta)}{\zeta^{1+1/\alpha}}d\zeta\right).
$$

Consequently,we obtain the estimate

$$
|a_2 - 1/p| = \left| - \frac{\lambda\mu p^{1/\alpha - 1}}{\alpha}\int_0^p \frac{w(\zeta)}{\zeta^{1+1/\alpha}}d\zeta\right|
$$

Since $|w(\zeta)| \leq |\zeta|$

$$
\begin{aligned}
|a_2 - 1/p| &\leq \frac{\lambda\mu p^{1/\alpha - 1}}{\alpha}\int_0^p |\zeta|^{-1/\alpha}|d\zeta| \\
&\leq \frac{\lambda\mu}{\alpha - 1}.
\end{aligned}
$$

□

5. Necessary and Sufficient condition for functions in $\mathcal{U}_p(\alpha, \lambda)$

Throughout this section for each $f \in \mathcal{A}(p)$, the Taylor expansion of the function z/f is considered to be :

$$\frac{z}{f} = 1 + b_1 z + b_2 z^2 + \cdots, z \in \mathbb{D} \tag{5.1}$$

Theorem 5.1. *Let $f \in \mathcal{U}_p(\alpha, \lambda), \alpha > 1$ have an expansion of the form (5.1). Then $g \in \mathcal{A}(p)$ defined by*

$$\left(\frac{z}{g(z)}\right) = 1 + \sum_{k=1}^{\infty} (Reb_k) z^k, with \frac{z}{g(z)} \neq 0$$

in $\mathbb{D} \setminus \{p\}$, is also a member of the class $\mathcal{U}_p(\alpha, \lambda)$.

Proof. For $f \in \mathcal{U}_p(\alpha, \lambda)$, call $h(z) = \overline{f(\overline{z})}$. Since $\mathcal{U}_p(\alpha, \lambda)$ is invariant under conjugation, $h \in \mathcal{U}_p(\alpha, \lambda)$. It maybe observed that

$$\begin{aligned}
\frac{z}{g(z)} &= 1 + \Sigma_{k=1}^{\infty} (Reb_k) z^k \\
&= \frac{1}{2}\Big[\big(1 + \Sigma_{k=1}^{\infty} b_k z^k\big) + \big(\overline{1 + \Sigma_{k=1}^{\infty} b_k \overline{z}^k}\big)\Big] \\
&= \frac{1}{2}\Big[\frac{z}{f(z)} + \Big(\overline{\frac{\overline{z}}{f(\overline{z})}}\Big)\Big] \\
&= \frac{1}{2}\Big[\frac{z}{f(z)} + \frac{z}{h(z)}\Big].
\end{aligned}$$

Further,

$$\begin{aligned}
\mathcal{U}_{g_\alpha}(z) &= \frac{z}{g(z)} - \alpha z \left(\frac{z}{g(z)}\right)' - 1. \\
&= \frac{1}{2}\big[\mathcal{U}_{f_\alpha}(z) + \mathcal{U}_{h_\alpha}(z)\big]
\end{aligned}$$

Hence $|U_{g_\alpha}(z)| < \lambda\mu$ which implies $g \in \mathcal{U}_p(\alpha, \lambda)$. $\square$

Theorem 5.2. *If each $f \in \mathcal{A}(p)$ of the form (5.1) satisfies the condition $\sum_{n=1}^{\infty} (n\alpha - 1)|b_n| \leq \lambda\mu$, then $f \in \mathcal{U}_p(\alpha, \lambda)$.*

Proof. For each f of the form (5.1)

$$\begin{aligned}\left|\mathcal{U}_{f_\alpha}(z)\right| &= \left|(1-\alpha)\left(\frac{z}{f(z)}\right)+\alpha\left(\frac{z}{f(z)}\right)f'(z)-1\right| \\ &= \left|\left(\frac{z}{f(z)}\right)-\alpha z\left(\frac{z}{f(z)}\right)'-1\right| \\ &= \left|\sum_{n=1}^{\infty}(1-n\alpha)b_n z^n\right| \\ &= \left|-\sum_{n=1}^{\infty}(n\alpha-1)b_n z^n\right| \\ &< \sum_{n=1}^{\infty}(n\alpha-1)|b_n| \\ &\leq \lambda\mu,\end{aligned}$$

which implies $f \in \mathcal{U}_p(\alpha,\lambda)$. $\square$

Theorem 5.3. *Let $f \in \mathcal{U}_p(\alpha,\lambda)$ be of the form (5.1). Then*

$$\sum_{n=1}^{\infty}(n\alpha-1)^2|b_n|^2 \leq \lambda^2\mu^2$$

and for each $n \geq 1$, $|b_n| \leq \frac{\lambda\mu}{(n\alpha-1)}$. The result is sharp for the function

$$f(z) = \frac{z}{1-\left(\frac{1}{p}+\frac{\lambda\mu p^{n-1}}{(n\alpha-1)}\right)z+\frac{\lambda\mu z^n}{(n\alpha-1)}}, \; z \in \mathbb{D}. \tag{5.2}$$

Proof. From (5.1) we find that

$$|\mathcal{U}_{f_\alpha}(z)| = \left|\sum_{n=1}^{\infty}(n\alpha-1)b_n z^n\right| < \lambda\mu, z \in \mathbb{D}.$$

Let $z = re^{i\theta}$ where $0 < r < 1$ and $0 \leq \theta \leq 2\pi$. Then

$$\sum_{n=1}^{\infty}(n\alpha-1)^2|b_n|^2 r^{2n} = \frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{n=1}^{\infty}(n\alpha-1)b_n r^n e^{in\theta}\right|^2 d\theta < \lambda^2\mu^2.$$

By letting $r \to 1$,, we deduce the desired inequality. For establishing the coefficient bound, for $n \geq 1$, it follows

$$(n\alpha-1)^2|b_n|^2 \leq \sum_{n=1}^{\infty}(n\alpha-1)^2|b_n|^2 \leq \lambda^2\mu^2,$$

which implies

$$|b_n| \leq \frac{\lambda\mu}{(n\alpha-1)}, \; \forall n \geq 1.$$

To verify the sharpness, consider the function f defined by(5.2), the coefficient $b_n = \frac{\lambda\mu}{(n\alpha-1)}$. We now show that this function belongs to $\mathcal{U}_p(\alpha,\lambda)$. A direct computation shows that the function satisfies $f(p)=\infty$, $f(0)=0$, $f'(0)=1$ and

$$\begin{aligned}
|\mathcal{U}_{f_\alpha}(z)| &= \left| -\alpha z\left(\frac{z}{f(z)}\right)' + \frac{z}{f(z)} - 1\right| \\
&= \left| -(1-\alpha)\frac{z}{p} - \frac{(1-\alpha)\lambda\mu z p^{n-1}}{n\alpha-1} - \lambda\mu z^n\right| \\
&< |\lambda\mu z^n| < \lambda\mu,
\end{aligned}$$

for $z\in\mathbb{D}$. We establish that $z=p$ is the unique pole of the function in the unit disk $\mathbb{D}$. To prove this, observe that

$$\begin{aligned}
\frac{z}{f(z)} &= 1 - \left(\frac{1}{p} + \frac{\lambda\mu p^{n-1}}{(n\alpha-1)}\right)z + \frac{\lambda z^n}{(n\alpha-1)} \\
&= (z-p)\Big[\frac{\lambda\mu p^{n-1}}{(n\alpha-1)}\sum_{k=1}^{n-1}\frac{z^k}{p^k} - \frac{1}{p}\Big]. \qquad (5.3)
\end{aligned}$$

Now, for $z\in\mathbb{D}$,

$$\left|\frac{\lambda\mu p^{n-1}}{(n\alpha-1)}\sum_{k=1}^{n-1}\frac{z^k}{p^k} - \frac{1}{p}\right| \geq \left|\frac{1}{p} - \left|\frac{\lambda\mu p^{n-1}}{(n\alpha-1)}\sum_{k=1}^{n-1}\frac{z^k}{p^k}\right|\right|,$$

and it is easy to see that

$$\left|\frac{\lambda\mu p^{n-1}}{(n\alpha-1)}\sum_{k=1}^{n-1}\frac{z^k}{p^k}\right| \leq \frac{\lambda\mu p^{n-1}}{n\alpha-1} < \lambda\mu < 1.$$

It follows that rightmost bracketed entity in (5.3) is nonzero throughout the unit disk, since $1/p>1$. Consequently, z/f is nonzero at every $z\in\mathbb{D}$ except at $z=p$. Therefore, the function f defined in (5.2) belongs to the class $\mathcal{U}_p(\alpha,\lambda)$. □

6. Fekete-Szegö estimate for $\mathcal{U}_p(\alpha,\lambda)$

Theorem 6.1. *For some $0<\lambda\leq 1$, $\alpha>1$ if $f\in\mathcal{U}_p(\alpha,\lambda)$ has the Taylor's series expansion of the form $f(z)=z+\sum_{n=2}^{\infty}a_n z^n$ in the disc $\{z:|z|<p\}$ then*

$$|a_3 - a_2^2| \leq \frac{\lambda\mu}{(2\alpha-1)}.$$

Proof. Let $f\in\mathcal{U}_p(\alpha,\lambda)$. Then the Taylor expansion given by (5.1) implies

$$z = (1+b_1 z + b_2 z^2 + b_3 z^3 + \cdots)(z + a_2 z^2 + a_3 z^3 + \cdots)$$

from which it follows that $b_1 + a_2 = 0$ and $b_2 + b_1 a_2 + a_3 = 0$.
Therefore, $a_3 - a_2^2 = -b_2$. The result follows from Theorem (5.2). □

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