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LOCAL–NONLOCAL EQUATION WITH LOGARITHMIC AND SINGULAR NONLINEARITIES

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Abstract. This paper investigates a class of local–nonlocal elliptic equations driven by the classical and fractional p-Laplacian operators and involving logarithmic and singular nonlinearities. Using variational methods and critical point theory, we establish the existence of at least two distinct positive weak solutions under suitable assumptions on the parameters.

Keywords. Local–nonlocal equation, Singular nonlinearity, Positive solution, Logarithmic nonlinearity.
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1. Introduction and Main Result

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$. In this paper, we investigate the following mixed local–nonlocal elliptic problem involving logarithmic and singular nonlinearities:

$$\begin{cases} -\Delta_p u + (-\Delta)_p^s u = |u|^{q-2}u \ln |u|^2 + \dfrac{\lambda}{u^\gamma}, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where $0 < s < 1 < p < N$, $0 < \gamma < 1$, $2p < q < q + 2 < p^* = \frac{Np}{N-p}$, and $\lambda > 0$ is a real parameter. Here, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ denotes the classical p-Laplacian, whereas the fractional p-Laplacian is given by

$$(-\Delta)_p^s u(x) := \text{P. V.} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}}\, dy,$$

where P. V. stands for the principal value. We refer the reader to [15] for a comprehensive account of the fractional p-Laplacian and its fundamental properties.

Problems driven by the combination of local and nonlocal diffusion operators have attracted considerable attention during the past decade. Besides their mathematical interest, such models naturally arise in several applications, including

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anomalous diffusion, continuum mechanics, biological processes, and phase transition phenomena; see, for instance, [16, 10, 17]. Their study has generated a substantial body of work devoted to existence, multiplicity, regularity, and qualitative properties of weak solutions; see, among others, [13, 5, 7, 2, 8, 11, 18]. The presence of singular nonlinearities considerably increases the analytical complexity of these problems since it affects both the regularity of solutions and the variational structure of the associated energy functional.

A representative contribution in this direction was obtained by Das et al. [5], who considered the mixed local–nonlocal equation

$$\begin{cases} -\Delta_p u + (-\Delta)_p^s u = \dfrac{\lambda}{u^\gamma} + u^r, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\lambda > 0$, $\gamma \in (0,1)$, $r \in (p-1, p_s^* - 1)$, $s \in (0,1)$, and $N > p > 1$. Using variational techniques together with suitable a priori estimates, they proved the existence of at least two positive weak solutions for sufficiently small values of the parameter λ.

Subsequently, Arora et al. [2] and Garain et al. [20] investigated singular equations governed by mixed local–nonlocal operators of the form

$$\begin{cases} -\Delta_p u + (-\Delta)_p^s u = \dfrac{f}{u^\gamma}, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

obtaining several existence, uniqueness, and regularity results. We also refer the interested reader to [9, 6, 25, 4] for further developments concerning related singular mixed local–nonlocal problems.

On the other hand, logarithmic nonlinearities have received increasing attention because of their occurrence in various mathematical models and their delicate analytical features. Numerous existence and multiplicity results have been established for elliptic equations involving logarithmic terms; see, for example, [3, 23, 14, 29, 22, 28]. In particular, Zhang et al. [28] analyzed a mixed local–nonlocal elliptic system with logarithmic nonlinearities and symmetric coupling terms on bounded Lipschitz domains. By combining the Nehari manifold approach with fibering map techniques, they obtained two sign-changing solutions and further established the existence of a ground state solution through a minimization argument.

Inspired by the above works, we investigate problem (1.1), where logarithmic and singular nonlinearities appear simultaneously in the presence of a mixed local–nonlocal diffusion operator. To the best of our knowledge, this combination has not been addressed in the existing literature. The coexistence of these nonlinear effects gives rise to substantial analytical challenges, including the singular behavior near the origin, the logarithmic growth of the reaction term, and the nonsmooth character of the associated energy functional. These difficulties prevent the direct application of standard variational arguments and require a more careful analysis of the corresponding Nehari manifold. By developing an appropriate variational framework together with refined compactness arguments, we establish the multiplicity of positive weak solutions under suitable assumptions on the parameter λ.

Our main result is stated as follows.

Theorem 1.1. *Assume that $0 < \gamma < 1$. Then there exists $\lambda_0 > 0$ such that, for every $\lambda \in (0, \lambda_0)$, problem (1.1) admits at least two positive weak solutions.*

The paper is organized as follows. Section 2 introduces the functional framework together with the auxiliary results required throughout the paper. The proof of Theorem 1.1 is presented in Section 3.

2. Preliminaries

In this section, we introduce the functional framework and the main analytical tools that will be used throughout the paper.

For the standard properties of Lebesgue, Sobolev, and fractional Sobolev spaces, including the classical embedding theorems, we refer the reader to [15, 21].

To investigate problem (1.1), we introduce the Banach space

$$\mathbb{X}_0(\Omega) = \left\{ u \in W^{1,p}(\mathbb{R}^N) : u|_\Omega \in W_0^{1,p}(\Omega),\ u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \right\},$$

endowed with the norm

$$\|u\|_{\mathbb{X}_0} = \left(\int_\Omega |\nabla u|^p \, dx + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \, dx \, dy \right)^{\frac{1}{p}}.$$

According to the standard theory of mixed local–nonlocal Sobolev spaces (see, for example, [7, 15]), the space $\mathbb{X}_0(\Omega)$ is a separable and reflexive Banach space. Furthermore, the embedding $\mathbb{X}_0(\Omega) \hookrightarrow L^r(\Omega)$, is continuous for every $1 \le r \le p^*$ and compact whenever $1 \le r < p^*$.

Throughout the paper, we use the notation

$$S_\rho = \{ u \in \mathbb{X}_0(\Omega) : \|u\|_{\mathbb{X}_0} = \rho \}, \qquad B_\rho = \{ u \in \mathbb{X}_0(\Omega) : \|u\|_{\mathbb{X}_0} \le \rho \},$$

for the sphere and the closed ball of radius ρ in $\mathbb{X}_0(\Omega)$, respectively. Moreover, we set

$$\mathcal{A}(u(x,y)) = |u(x) - u(y)|^{p-2}(u(x) - u(y)), \qquad d\nu = |x - y|^{-N-ps} \, dx \, dy,$$

and denote by $u^+(x) = \max\{u(x), 0\}$, the positive part of u.

The energy functional associated with problem (1.1) is defined on $\mathbb{X}_0(\Omega)$ by

$$\begin{aligned} J_\lambda(u) =& \frac{1}{p} \int_\Omega |\nabla u|^p \, dx + \frac{1}{p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \, dx \, dy \\ &+ \frac{2}{q^2} \int_\Omega |u|^q \, dx - \frac{1}{q} \int_\Omega |u|^q \ln |u|^2 \, dx - \frac{\lambda}{1-\gamma} \int_\Omega |u|^{1-\gamma} \, dx, \end{aligned}$$

for every $u \in \mathbb{X}_0(\Omega)$.

We next recall the basic notions of nonsmooth critical point theory that will be used in the sequel. For further details, we refer the reader to [12, 24].

Definition 2.1. Let $(\mathcal{U}, d)$ be a complete metric space, $f : \mathcal{U} \to \mathbb{R}$ be a continuous functional in $\mathcal{U}$. Denote by $|Df|(u)$ the supremum of c in $[0, \infty)$ such that there exist $\delta > 0$, a neighborhood V of $u \in \mathcal{U}$, and a continuous map $\sigma : V \times [0, \delta] \to \mathcal{U}$ satisfying

$$\begin{cases} f(\sigma(v,t)) \leq f(v) - ct, & (v,t) \in V \times [0, \delta], \\ d(\sigma(v,t), v) \leq t, & (v,t) \in V \times [0, \delta]. \end{cases} \tag{2.1}$$

The extended real number $|Df|(u)$ is called the weak slope of f at u, we say that $u \in \mathcal{U}$ is a critical point of f if $|Df|(u) = 0$, we say that $k \in \mathbb{R}$ is a critical value of f if there exists a critical point $u \in \mathcal{U}$ of f with $f(u) = k$.

Definition 2.2. A sequence $\{u_n\}$ of $\mathcal{U}$ is called a Palais–Smale (PS) sequence of the functional f, if $|Df|(u_n) \to 0$ as $n \to \infty$ and $f(u_n)$ is bounded. Since $u \mapsto |Df|(u)$ is lower semicontinuous, any accumulation point of a (PS) sequence is clearly a critical point of f.

We consider the energy functional J_λ on the closed positive cone

$$\mathcal{U}^+ = \{u \in \mathbb{X}_0(\Omega), u(x) \geq 0, \text{ a.e. } x \in \Omega\}.$$

$\mathcal{U}^+$ is a complete metric space and J_λ is a continuous functional on $\mathcal{U}^+$. Therefore, we obtain the following lemma.

Lemma 2.3. *Assume that $|DJ_\lambda|(u) < +\infty$ holds, then for all $v \in \mathcal{U}^+$ we have*

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla(v-u) dx + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \mathcal{A}(u(x,y))[(v-u)(x) - (v-u)(y)] d\nu \\ - \int_\Omega |u|^{q-2} u(v-u) \ln |u|^2 dx + |DJ_\lambda|(u) \|v-u\|_{\mathbb{X}_0} \geq \lambda \int_\Omega \frac{v-u}{u^\gamma} dx. \tag{2.2}$$

Proof. Let $|DJ_\lambda|(u) < \mu, \delta < \frac{1}{2}\|v-u\|_{\mathbb{X}_0}, v \in \mathcal{U}^+$ and $v \neq u$. Define the mapping $\sigma : V \times [0, \delta] \to \mathcal{U}^+$ by

$$\sigma(z,t) = z + t \frac{v-z}{\|v-z\|_{\mathbb{X}_0}},$$

where V is a neighborhood of u. Then, we get $\|\sigma(z,t) - z\|_{\mathbb{X}_0} = t$, by (2.1), there exists a pair $(z,t) \in V \times [0, \delta]$ such that $J_\lambda(\sigma(z,t)) > J_\lambda(z) - \mu t$. Therefore, we suppose that there exist sequences $\{u_n\} \subset \mathcal{U}^+$ and $\{t_n\} \subset [0, \infty)$, such that $u_n \to u, t_n \to 0^+$, and

$$J_\lambda\left(u_n + t_n \frac{v-u_n}{\|v-u_n\|_{\mathbb{X}_0}}\right) \geq J_\lambda(u_n) - \mu t_n,$$

which implies

$$J_\lambda(u_n + s_n(v-u_n)) \geq J_\lambda(u_n) - \mu s_n \|v-u_n\|_{\mathbb{X}_0}, \tag{2.3}$$

where $s_n = \frac{t_n}{\|v-u_n\|_{\mathbb{X}_0}} \to 0^+$as $n \to \infty$. Dividing by s_n in (2.3), we obtain

$$\frac{1}{p}\frac{\|u_n + s_n(v-u_n)\|_{\mathbb{X}_0}^p - \|u_n\|_{\mathbb{X}_0}^p}{s_n} + \frac{T(u_n + s_n(v-u_n)) - T(u_n)}{s_n} + \mu\|v-u_n\|_{\mathbb{X}_0}$$
$$\geq \frac{\lambda}{1-\gamma}\int_\Omega \frac{|u_n + s_n(v-u_n)|^{1-\gamma} - |u_n|^{1-\gamma}}{s_n}dx,$$

with

$$T(u_n) = \frac{2}{q^2}\int_\Omega |u_n|^q\,dx - \frac{1}{q}\int_\Omega |u_n|^q \ln|u_n|^2\,dx.$$

Note that

$$\begin{aligned}
&\lim_{n\to\infty}\frac{T(u_n + s_n(v-u_n)) - T(u_n)}{s_n} = \lim_{n\to\infty}\frac{2}{q^2}\int_\Omega \frac{|u_n + s_n(v-u_n)|^q - |u_n|^q}{s_n}dx\\
&-\lim_{n\to\infty}\frac{1}{q}\int_\Omega \frac{(|u_n + s_n(v-u_n)|^q - |u_n|^q)\ln|u_n + s_n(v-u_n)|^2}{s_n}dx\\
&-\lim_{n\to\infty}\frac{1}{q}\int_\Omega \frac{|u_n|^q\left(\ln|u_n + s_n(v-u_n)|^2 - \ln|u_n|^2\right)}{s_n}dx\\
=&\frac{2}{q}\int_\Omega |u|^{q-2}u(v-u)dx - \int_\Omega |u|^{q-2}u(v-u)\ln|u|^2dx - \frac{2}{q}\int_\Omega |u|^{q-2}u(v-u)dx\\
=&-\int_\Omega |u|^{q-2}u(v-u)\ln|u|^2dx.
\end{aligned}$$

Using [23], for all $\theta \in (q, p^*)$ and $2p < q < p^*$, we have that

$$\lim_{t\to 0}\frac{|t|^{q-1}\ln|t|^2}{|t|^{p-1}} = 0, \qquad \text{and} \qquad \lim_{t\to\infty}\frac{|t|^{q-1}\ln|t|^2}{|t|^{\theta-1}} = 0.$$

Hence, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|t|^{q-1}\ln|t|^2 \leq \varepsilon|t|^{p-1} + C_\varepsilon|t|^{\theta-1}. \tag{2.4}$$

Since $u_n(x) \to u(x)$ a.e in Ω and the map $t \longmapsto |t|^q\ln|t|^2$ is continuous, we conclude that $|u_n(x)|^q\ln|u_n(x)|^2 \to |u(x)|^q\ln|u(x)|^2$, a.e. in Ω. Then, from (2.4) and the Lebesgue dominated convergence theorem, we have

$$\int_\Omega |u_n|^q\ln|u_n|^2\,dx \to \int_\Omega |u|^q\ln|u|^2dx, \qquad \text{as } n\to\infty.$$

We pose

$$J_{1,n} = \int_\Omega \frac{|u_n + s_n(v-u_n)|^{1-\gamma} - |(1-s_n)u_n|^{1-\gamma}}{s_n(1-\gamma)}dx,$$

and

$$J_{2,n} = \frac{(1-s_n)^{1-\gamma} - 1}{s_n(1-\gamma)}\int_\Omega |u_n|^{1-\gamma}\,dx.$$

Thus we have

$$J_{1,n} = \int_\Omega \frac{\xi_n^{-\gamma}s_n v}{s_n}dx = \int_\Omega \xi_n^{-\gamma}v dx,$$

where $\xi_n \in (u_n - s_n u_n, u_n + s_n (v - u_n))$, then $\xi_n \to u\,(u_n \to u)$ as $s_n \to 0^+$. Hence $J_{1,n} \geq 0$ for all n, by the Fatou lemma, we get

$$\liminf_{n\to\infty} J_{1,n} \geq \int_\Omega \frac{v}{u^\gamma} dx,$$

for all $v \in \mathcal{U}^+$. For $J_{2,n}$, by the Lebesgue dominated convergence theorem, we obtain

$$\lim_{n\to\infty} J_{2,n} = -\int_\Omega u^{1-\gamma} dx.$$

It follows from the above that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla(v-u)dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathcal{A}(u(x,y))[(v-u)(x)-(v-u)(y)]d\nu$$
$$-\int_\Omega |u|^{q-2}u(v-u)\ln|u|^2 dx + \mu\|v-u\|_{\mathbb{X}_0} \geq \liminf_{n\to\infty}(J_{1,n}+J_{2,n}) \geq \lambda\int_\Omega \frac{v-u}{u^\gamma}dx,$$

for every $v \in \mathcal{U}^+$. Since $|DJ_\lambda|(u) < \mu$ is arbitrary. The proof is complete. □

Lemma 2.4. *The functional J_λ satisfies the (PS) condition.*

Proof. Let $\{u_n\} \subset \mathcal{U}^+$ be a (PS) sequence for J_λ at the level c, that is

$$J_\lambda(u_n) \to c, \text{ and } \quad |DJ_\lambda|(u_n) \to 0 \text{ as } n \to \infty. \tag{2.5}$$

We deduce from (2.2) and (2.5) that

$$\begin{aligned}\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla(v-u_n)dx &+ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathcal{A}(u_n(x,y))[(v-u_n)(x)-(v-u_n)(y)]d\nu \\ &- \int_\Omega |u_n|^{q-2}u_n(v-u_n)\ln|u_n|^2 dx + |DJ_\lambda|(u_n)\|v-u_n\|_{\mathbb{X}_0} \\ &\geq \lambda\int_\Omega \frac{v-u_n}{u_n^\gamma}dx.\end{aligned} \tag{2.6}$$

We take $v = 2u_n \in \mathcal{U}^+$ in (2.6), we have

$$\|u_n\|^p_{\mathbb{X}_0} - \int_\Omega |u_n|^q \ln|u_n|^2 dx + |DJ_\lambda|(u_n)\|u_n\|_{\mathbb{X}_0} \geq \lambda\int_\Omega u_n^{1-\gamma}dx. \tag{2.7}$$

Since (2.5), (2.7) and the Hölder inequality, there exists a constant $C > 0$, such that

$$\begin{aligned} c + 1 + o\left(\|u_n\|_{\mathbb{X}_0}\right) &\geq J_\lambda(u_n) + \frac{1}{q}|DJ_\lambda|(u_n)\|u_n\|_{\mathbb{X}_0} \\ &\geq \left(\frac{1}{p}-\frac{1}{q}\right)\|u_n\|^p_{\mathbb{X}_0} + \frac{2}{q^2}\int_\Omega |u_n|^q dx - \lambda\left(\frac{1}{1-\gamma}-\frac{1}{q}\right)\int_\Omega |u_n|^{1-\gamma}dx \\ &\geq \left(\frac{1}{p}-\frac{1}{q}\right)\|u_n\|^p_{\mathbb{X}_0} - \lambda\left(\frac{1}{1-\gamma}-\frac{1}{q}\right)C\|u_n\|^{1-\gamma}_{\mathbb{X}_0}. \end{aligned}$$

From $1-\gamma < 1 < p$, we conclude that $\{u_n\}$ is bounded in $\mathbb{X}_0(\Omega)$. Then, we may assume up to a subsequence, still denoted by $\{u_n\}$, there exists $u \in \mathbb{X}_0(\Omega)$ such that

$$\begin{cases} u_n \rightharpoonup u, & \text{weakly in } \mathbb{X}_0(\Omega), \\ u_n \to u, & \text{strongly in } L^r(\Omega)\,(1 \le r < p^*)\,, \\ u_n(x) \to u(x), & \text{a.e. in } \Omega, \end{cases} \tag{2.8}$$

as $n \to \infty$. Choosing $v = u_m$ in (2.6), we obtain that

$$\begin{aligned} &\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla(u_m - u_n)dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathcal{A}(u_n(x,y))[(u_m-u_n)(x) - (u_m-u_n)(y)]d\nu \\ &\qquad - \int_\Omega |u_n|^{q-2}\, u_n\,(u_m - u_n) \ln |u_n|^2\, dx + o(1)\, \|u_m - u_n\|_{\mathbb{X}_0} \\ &\qquad \ge \lambda \int_\Omega \frac{u_m - u_n}{u_n^\gamma} dx. \end{aligned} \tag{2.9}$$

If we change the role of u_m and u_n in (2.9), we obtain a similar inequality. By adding the two inequalities, we have

$$\begin{aligned} &\int_\Omega \left(|\nabla u_n|^{p-2}\,\nabla u_n - |\nabla u_m|^{p-2}\,\nabla u_m\right) \nabla\left(u_n - u_m\right) dx \\ &+ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} [\mathcal{A}(u_n(x,y)) - \mathcal{A}(u_m(x,y))]\,[(u_n - u_m)\,(x) - (u_n - u_m)\,(y)]\,d\nu \\ &\le \int_\Omega \left(|u_n|^{q-2}\, u_n \ln |u_n|^2 - |u_m|^{q-2}\, u_m \ln |u_m|^2\right) (u_n - u_m)\, dx \\ &\quad + \lambda \int_\Omega \left(u_n^{-\gamma} - u_m^{-\gamma}\right) (u_n - u_m)\, dx + o(1)\, \|u_m - u_n\|_{\mathbb{X}_0} \\ &\le \int_\Omega \left(|u_n|^{q-2}\, u_n \ln |u_n|^2 - |u_m|^{q-2}\, u_m \ln |u_m|^2\right) (u_n - u_m)\, dx + o(1)\, \|u_m - u_n\|_{\mathbb{X}_0}\,. \end{aligned} \tag{2.10}$$

By using (2.4), (2.8) and $\{u_n\}$ is bounded in $\mathbb{X}_0(\Omega)$, for all $r \in (q, p^*)$, we get

$$\begin{aligned} &\left|\int_\Omega |u_n|^{q-2}\, u_n \ln |u_n|^2\, (u_n - u_m)\, dx\right| \\ &\quad \le C\varepsilon \int_\Omega |u_n|^{p-1}\, |u_n - u_m|\, dx + C_\varepsilon \int_\Omega |u_n|^{r-1}\, |u_n - u_m|\, dx \\ &\quad \le C\varepsilon \left(\int_\Omega |u_n|^p\, dx\right)^{\frac{p-1}{p}} \left(\int_\Omega |u_n - u_m|^p\, dx\right)^{\frac{1}{p}} \\ &\quad + C_\varepsilon \left(\int_\Omega |u_n|^r\, dx\right)^{\frac{r-1}{r}} \left(\int_\Omega |u_n - u_m|^r\, dx\right)^{\frac{1}{r}} \\ &\le C\varepsilon\, \|u_n - u_m\|_p + C_\varepsilon\, \|u_n - u_m\|_r \to 0, \end{aligned} \tag{2.11}$$

as $n \to \infty$. From a similar calculation in (2.11), we have

$$\left|\int_\Omega |u_m|^{q-2}\, u_m \ln |u_m|^2\, (u_n - u_m)\, dx\right| \le C\varepsilon\, \|u_n - u_m\|_p + C_\varepsilon\, \|u_n - u_m\|_r \to 0, \tag{2.12}$$

as $n \to \infty$. Combining (2.11) and (2.12), we get that

$$\lim_{n\to\infty} \int_{\Omega} \left(|u_n|^{q-2} u_n \ln |u_n|^2 - |u_m|^{q-2} u_m \ln |u_m|^2\right) (u_n - u_m)\, dx = 0. \tag{2.13}$$

We recall the well-known Simon inequalities [27], for all $\xi, \zeta \in \mathbb{R}$ such that

$$|\xi - \zeta|^p \leq \begin{cases} c_p \left(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta\right)(\xi - \zeta), & \text{for } p \geq 2 \\ C_p \left[\left(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta\right)(\xi - \zeta)\right]^{\frac{p}{2}} \left(|\xi|^p + |\zeta|^p\right)^{\frac{2-p}{2}}, & \text{for } 1 < p < 2, \end{cases} \tag{2.14}$$

where $c_p, C_p > 0$ depending only on p.

Then, by (2.10),(2.13) and (2.14), we obtain

$$\lim_{n\to\infty} \int_{\Omega} \left(|\nabla u_n|^{p-2} \nabla u_n - |\nabla u_m|^{p-2} \nabla u_m\right) \nabla (u_n - u_m)\, dx$$
$$+ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} [\mathcal{A}(u_n(x,y)) - \mathcal{A}(u_m(x,y))] \left[(u_n - u_m)(x) - (u_n - u_m)(y)\right] d\nu = 0. \tag{2.15}$$

Therefore, we distinguish two cases.

Case 1: $p \geq 2$**.** It follows from (2.15) and Simon's inequality (2.14) that

$$\begin{aligned}\|u_n - u_m\|^p_{\mathbb{X}_0} &= \int_{\Omega} |\nabla u_n - \nabla u_m|^p\, dx \\ &\quad + \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|(u_n - u_m)(x) - (u_n - u_m)(y)|^p}{|x-y|^{N+ps}}\, dx\, dy \\ &\leq c_p \Bigg(\int_{\Omega} \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m\right) \cdot \nabla(u_n - u_m)\, dx \\ &\qquad + \iint_{\mathbb{R}^N\times\mathbb{R}^N} \big(\mathcal{A}(u_n) - \mathcal{A}(u_m)\big)\big((u_n - u_m)(x) - (u_n - u_m)(y)\big)\, d\nu \Bigg).\end{aligned}$$

Hence, by (2.15), $\|u_n - u_m\|_{\mathbb{X}_0} \longrightarrow 0$. Consequently $u_n \longrightarrow u$ in $\mathbb{X}_0(\Omega)$ as $n \to \infty$.

Case 2: $1 < p < 2$**.** Since $\{u_n\}$ is bounded in $\mathbb{X}_0(\Omega)$, the factors appearing in Simon's inequality remain uniformly bounded. Combining the subadditivity inequality

$$(\xi + \zeta)^{\frac{2-p}{2}} \leq \xi^{\frac{2-p}{2}} + \zeta^{\frac{2-p}{2}}, \qquad \xi, \zeta \geq 0,$$

with (2.14), we obtain

$$\begin{aligned}\|u_n - u_m\|^p_{\mathbb{X}_0} \leq C \Bigg(&\int_{\Omega} \left[\left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m\right) \cdot \nabla(u_n - u_m)\right]^{\frac{p}{2}} dx \\ &+ \iint_{\mathbb{R}^N\times\mathbb{R}^N} \left[\big(\mathcal{A}(u_n) - \mathcal{A}(u_m)\big)\big((u_n - u_m)(x) - (u_n - u_m)(y)\big)\right]^{\frac{p}{2}} d\nu \Bigg),\end{aligned}$$

where $C > 0$ is independent of n and m. Using (2.15), the right-hand side tends to zero as $n, m \to \infty$. Consequently, $\|u_n - u_m\|_{\mathbb{X}_0} \longrightarrow 0$. Hence, $u_n \longrightarrow u$ in $\mathbb{X}_0(\Omega)$ as $n \to \infty$. $\square$

Lemma 2.5. *If $|DJ_\lambda|(u) = 0$, then u is a weak solution of (1.1). That is, $u^{-\gamma}\phi \in L^1(\Omega)$ for all $\phi \in \mathbb{X}_0(\Omega)$ such that*

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\phi \, dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathcal{A}(u(x,y))[\phi(x)-\phi(y)] \, d\nu = \int_\Omega |u|^{q-2}u\phi \ln|u|^2 \, dx + \lambda\int_\Omega \frac{\phi}{u^\gamma} \, dx.$$

Proof. Assume that $|DJ_\lambda|(u) = 0$. By Lemma 2.3, for every $v \in \mathcal{U}^+$,

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla(v-u) \, dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathcal{A}(u(x,y))[(v-u)(x)-(v-u)(y)] \, d\nu - \int_\Omega |u|^{q-2}u(v-u)\ln|u|^2 \, dx \geq \lambda\int_\Omega \frac{v-u}{u^\gamma} \, dx. \tag{2.16}$$

Let $t \in \mathbb{R}$, we take $v = (u+t\phi)^+ \in \mathcal{U}^+$ in (2.16), for any $\phi \in \mathbb{X}_0(\Omega)$, we have

$$\begin{aligned} 0 \leq & t\left[\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\phi dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathcal{A}(u(x,y))[\phi(x)-\phi(y)]d\nu \right. \\ & \left. - \int_\Omega |u|^{q-2}u\phi\ln|u|^2 dx - \lambda\int_\Omega \frac{\phi}{u^\gamma}dx\right] \\ & - t\left[\int_{u+t\phi<0} |\nabla u|^{p-2}\nabla u\nabla\phi dx + \int_{u+t\phi<0}\int_{u+t\phi<0} \mathcal{A}(u(x,y))[\phi(x)-\phi(y)]d\nu\right] \\ & + \int_{u+t\phi<0} |u|^{q-2}u(u+t\phi)\ln|u|^2 dx. \end{aligned}$$

From $\nabla u(x) = 0$ for a.e. $x \in \Omega$ with $u(x) = 0$, and

$$\operatorname{meas}\{x \in \Omega \mid u(x)+t\phi(x) < 0, u(x) > 0\} \to 0, \qquad \text{as } t \to 0.$$

We obtain that

$$\begin{aligned} 0 \leq t & \left[\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\phi dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathcal{A}(u(x,y))[\phi(x)-\phi(y)]d\nu \right. \\ & \left. - \int_\Omega |u|^{q-2}u\phi\ln|u|^2 dx - \lambda\int_\Omega \frac{\phi}{u^\gamma}dx\right] + o(1). \end{aligned}$$

Therefore, we have

$$\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\phi dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathcal{A}(u(x,y))[\phi(x)-\phi(y)]d\nu - \int_\Omega |u|^{q-2}u\phi\ln|u|^2 dx - \lambda\int_\Omega \frac{\phi}{u^\gamma}dx \geq 0.$$

Replacing ϕ by $-\phi$ yields the reverse inequality. Hence,

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\phi \, dx + \iint_{\mathbb{R}^N\times\mathbb{R}^N} \mathcal{A}(u(x,y))[\phi(x)-\phi(y)] \, d\nu = \int_\Omega |u|^{q-2}u\phi\ln|u|^2 \, dx + \lambda\int_\Omega \frac{\phi}{u^\gamma} \, dx,$$

for every $\phi \in \mathbb{X}_0(\Omega)$. $\square$

3. Proof of theorem (1.1)

This section is devoted to the proof of Theorem (1.1). We first present several key lemmas that will play a crucial role in the proof of the main theorem.

Lemma 3.1. *There exist constants $\alpha, \rho, \lambda_0 > 0$, for all $\lambda \in (0, \lambda_0)$. Then the functional J_λ satisfies the following conditions:*

(1) $J_\lambda|_{u\in S_\rho} \geq \alpha > 0; \inf_{u\in B_\rho} J_\lambda(u) < 0$.
(2) *There exists $e \in \mathbb{X}_0(\Omega)$ with $\|e\|_{\mathbb{X}_0} > \rho$ such that $J_\lambda(e) < 0$.*

Proof. (1) Using the Hölder and Sobolev inequalities, and $\ln|u|^2 \leq |u|^2$, we get

$$\begin{aligned}
J_\lambda(u) &= \frac{1}{p}\|u\|^p_{\mathbb{X}_0} + \frac{2}{q^2}\int_\Omega |u|^q dx - \frac{1}{q}\int_\Omega |u|^q \ln|u|^2 dx - \frac{\lambda}{1-\gamma}\int_\Omega |u|^{1-\gamma}dx \\
&\geq \frac{1}{p}\|u\|^p_{\mathbb{X}_0} - \frac{1}{q}\int_\Omega |u|^{q+2}dx - \frac{\lambda}{1-\gamma}\int_\Omega |u|^{1-\gamma}dx \\
&\geq \frac{1}{p}\|u\|^p_{\mathbb{X}_0} - \frac{1}{q}|\Omega|^{\frac{p^*-q-2}{p^*}}\left(\int_\Omega |u|^{p^*}dx\right)^{\frac{q+2}{p^*}} - \frac{\lambda}{1-\gamma}|\Omega|^{\frac{p^*-1+\gamma}{p^*}}\left(\int_\Omega |u|^{p^*}dx\right)^{\frac{1-\gamma}{p^*}} \\
&\geq \frac{1}{p}\|u\|^p_{\mathbb{X}_0} - \frac{1}{q}C_1\|u\|^{q+2}_{\mathbb{X}_0} - \frac{\lambda}{1-\gamma}C_2\|u\|^{1-\gamma}_{\mathbb{X}_0} \\
&= \|u\|^{1-\gamma}_{\mathbb{X}_0}\left(\frac{1}{p}\|u\|^{p-1+\gamma}_{\mathbb{X}_0} - \frac{1}{q}C_1\|u\|^{q+1+\gamma}_{\mathbb{X}_0} - \frac{\lambda}{1-\gamma}C_2\right),
\end{aligned}$$

where C_1 and C_2 are a positive constants.
Set

$$g(t) = \frac{1}{p}t^{p-1+\gamma} - \frac{1}{q}C_1 t^{q+1+\gamma},$$

for $t > 0$, so, there exists a constant

$$\rho = \left[\frac{q(p-1+\gamma)}{p(q+1+\gamma)C_1}\right]^{\frac{1}{q+2-p}} > 0,$$

such that $\max_{t>0} g(t) = g(\rho) > 0$. Let

$$\lambda_0 = \frac{g(\rho)(1-\gamma)}{2C_2},$$

then, for every $0 < \lambda < \lambda_0$ and $u \in S_\rho$ we have

$$J_\lambda(u) \geq \rho^{1-\gamma}\left(g(\rho) - \frac{\lambda}{1-\gamma}C_2\right) \geq \frac{\rho^{1-\lambda}g(\rho)}{2} = \alpha > 0.$$

Furthermore, for $u \in \mathbb{X}_0(\Omega)\backslash\{0\}$, we have

$$\lim_{t\to 0^+}\frac{J_\lambda(tu)}{t^{1-\gamma}} = -\frac{\lambda}{1-\gamma}\int_\Omega |u|^{1-\gamma}dx < 0.$$

Thus, we get that $J_\lambda(tu) < 0$ for t small enough. Therefore, for $\|u\|_{\mathbb{X}_0}$ small enough, we have

$$d = \inf_{u \in B_\rho} J_\lambda(u) < 0. \tag{3.1}$$

(2) For all $u \in \mathbb{X}_0(\Omega)\backslash\{0\}$ and $t > 0$, we have

$$\begin{aligned} J_\lambda(tu) =& \frac{t^p}{p}\|u\|_{\mathbb{X}_0}^p + \frac{2t^q}{q^2}\int_\Omega |u|^q dx - \frac{t^q}{q}\int_\Omega |u|^q \ln |tu|^2 dx - \frac{\lambda t^{1-\gamma}}{1-\gamma}\int_\Omega |u|^{1-\gamma} dx \\ =& \frac{t^p}{p}\|u\|_{\mathbb{X}_0}^p + \frac{2t^q}{q^2}\int_\Omega |u|^q dx - \frac{2t^q}{q}\int_\Omega |u|^q \ln |u| dx \\ & - \frac{2t^q}{q}\int_\Omega |u|^q \ln t dx - \frac{\lambda t^{1-\gamma}}{1-\gamma}\int_\Omega |u|^{1-\gamma} dx \to -\infty, \end{aligned}$$

as $t \to +\infty$,therefore $J_\lambda(tu) < 0$ for $t > 0$ large enough. Then, we can find $e \in \mathbb{X}_0(\Omega)$ with $\|e\|_{\mathbb{X}_0} > \rho$ such that $J_\lambda(e) < 0$.

□

Theorem 3.2. *Assume that $0 < \gamma < 1$. Then there exists a parameter $\lambda_0 > 0$ such that for every $\lambda \in (0, \lambda_0)$, problem (1.1) has a positive solution u_* satisfying $J_\lambda(u_*) < 0$.*

Proof. From the definition of d in (3.1) and Lemma (3.1), there exists a minimizing sequence $\{u_n\} \subset B_\rho \cap \mathcal{U}^+$ such that $\lim_{n\to\infty} J_\lambda(u_n) = d < 0$. It is clearly, $\{u_n\}$ is bounded in B_ρ, so it admits a subsequence, still denoted by $\{u_n\}$, thus there exists $u_* \in \mathbb{X}_0(\Omega)$ such that

$$\begin{cases} u_n \rightharpoonup u_*, & \text{weakly in } \mathbb{X}_0(\Omega), \\ u_n \to u_*, & \text{strongly in } L^r(\Omega), 1 \le r < p^*, \\ u_n(x) \to u_*(x), & \text{a.e. in } \Omega, \end{cases}$$

as $n \to \infty$.
We then prove that $u_n \to u_*$ as $n \to \infty$ in $\mathbb{X}_0(\Omega)$. Let $t_n = u_n - u_*$, by the Brézis-Lieb Lemma see [19], we obtain

$$\|u_n\|_{\mathbb{X}_0}^p = \|t_n\|_{\mathbb{X}_0}^p + \|u_*\|_{\mathbb{X}_0}^p + o(1).$$

Consequently, by Lemma (2.4), we get

$$\begin{aligned} d &= \lim_{n\to\infty} J_\lambda(u_n) \\ &= J_\lambda(u_*) + \lim_{n\to\infty} \frac{1}{p}\|t_n\|_{\mathbb{X}_0}^p \\ &\ge J_\lambda(u_*) \ge d, \end{aligned}$$

from which it follows that $\|t_n\|_{\mathbb{X}_0} \to 0$ as $n \to \infty$. Since B_ρ is closed and convex, then $u_* \in B_\rho$. Therefore, we can conclude that $J_\lambda(u_*) = d < 0$, which implies that u_* is a local minimizer of J_λ and $u \not\equiv 0$ in Ω. For $v \in \mathcal{U}^+$ and $t > 0$ small enough such that $u_* + t(v - u_*) \in B_\rho$, similar to the proof of Lemma (2.5), we have

$$\int_\Omega |\nabla u_*|^{p-2}\nabla u_* \nabla(v-u_*)dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathcal{A}(u_*(x,y))[(v-u_*)(x)-(v-u_*)(y)]d\nu$$
$$- \int_\Omega |u_*|^{q-2} u_* (v-u_*) \ln |u_*|^2 dx$$
$$\geq \lambda \int_\Omega \frac{v-u_*}{u_*^\gamma} dx.$$

Thus, u_* is a critical point of J_λ, by Lemma (2.5), we get $u_* \in \mathcal{U}^+$ is a solution of (1.1) with $J_\lambda(u_*) = d < 0$, which implies that $u_* \geq 0$ and $u_* \not\equiv 0$. We consider the function

$$h(t) = 2\ln t + \frac{\lambda}{t^{q-1+\gamma}}.$$

We obtain

$$\lim_{t\to 0^+} h(t) = +\infty, \quad \text{and} \quad \lim_{t\to+\infty} h(t) = +\infty.$$

Then, h achieves its minimum at

$$t_* = \left[\frac{\lambda(q-1+\gamma)}{2}\right]^{\frac{1}{q-1+\gamma}}.$$

Hence

$$\min_{t>0} h(t) = h(t_*) = \frac{2}{q-1+\gamma}\ln\frac{\lambda(q-1+\gamma)}{2} + \frac{2}{q-1+\gamma} = C.$$

Therefore, we have that

$$-\Delta_p u_* + (-\Delta)_p^s u_* = u_*^{q-1}\ln u_*^2 + \frac{\lambda}{u_*^\gamma} \geq C u_*^{q-1} \geq 0.$$

By using the strong maximum principle in [26], we deduce that $u_* \in \mathcal{U}^+$ is a positive solution of (1.1). $\square$

Theorem 3.3. *Assume that $0<\gamma<1$. Then there exists a parameter $\lambda_0 > 0$ such that for every $\lambda \in (0,\lambda_0)$, problem (1.1) has a positive solution v_* satisfying $J_\lambda(v_*) > 0$.*

Proof. Using Lemma (3.1) and the mountain pass Lemma in [1], there exists a sequence $\{u_n\} \subset \mathbb{X}_0(\Omega)$ such that

$$J_\lambda(u_n) \to c, \quad \text{and} \quad |DJ_\lambda|(u_n) \to 0, \quad \text{as } n\to\infty,$$

where

$$c = \inf_{\eta\in\Gamma}\max_{t\in[0,1]} J_\lambda(\eta(t)),$$

and

$$\Gamma = \{\eta \in C([0,1],\mathbb{X}_0(\Omega)) : \eta(0)=0, \eta(1)=e\}.$$

By Lemma (2.4), we know that $\{u_n\} \subset \mathbb{X}_0(\Omega)$ has a convergent subsequence, still denoted by $\{u_n\}$, we may assume that $u_n \to v_*$ in $\mathbb{X}_0(\Omega)$ as $n\to\infty$, we get

$$J_\lambda(v_*) = \lim_{n\to\infty} J_\lambda(u_n) \geq \alpha > 0,$$

which implies that $v_* \not\equiv 0$. It is similar to Step 1 that $v_* > 0$, we obtain that v_* is a positive solution of equation (1.1) such that $J_\lambda(v_*) > 0$. $\square$

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