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ON LOCALLY ATTRACTIVE SOLUTIONS FOR QUADRATIC FRACTIONAL INTEGRAL EQUATIONS

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Abstract. In this paper, we investigate the existence and attractivity of solutions for ψ-fractional-order quadratic functional integral equations in a Banach algebra over $\mathbb{R}_+ = [0, \infty)$. Using Dhage's hybrid fixed point theorem and appropriate analytical techniques, we establish existence results ensuring the well-posedness of the considered equations under suitable assumptions. The main novelty lies in deriving existence and local attractivity results for a coupled quadratic functional integral structure governed by ψ-fractional operators within a Banach algebra framework, extending beyond classical fractional and non-quadratic models. Two illustrative examples validate the applicability and effectiveness of the proposed approach, providing further insight into the qualitative analysis of fractional integral equations.

Keywords. Quadratic functional fractional integral equation; locally attractive solution; fixed point theorem; existence result; Banach algebra.
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1. Introduction

Quadratic functional integral equations (QFIEs) have gained a lot of attention in recent years. These equations constitute a rich branch of nonlinear analysis, offering a powerful mathematical environment to describe a wide range of physical and engineering phenomena. Parallel to the growth of QFIEs, fractional calculus also has emerged in recent decades as a highly effective tool in most fields of science and engineering [8, 22, 30, 31, 39], particularly due to its ability to model memory effects and hereditary properties that cannot be adequately captured by classical integer-order operators.

QFIEs have been studied in various branches of science, including radioactive transfer theory, kinetic gas theory, neutron transport theory, and traffic flow theory [1, 4, 17]. In these applications, quadratic integral terms naturally arise from nonlinear interaction mechanisms involving multiple dependent variables, and hence become precious tools for mathematical physics and engineering.

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There is extensive literature on the examination of quadratic integral problems of arbitrary order. Furthermore, the existence and uniqueness theorems, some fundamental theoretical issues, and practical implications for such equations have been investigated by monographs and research articles [5, 9, 15, 16, 20, 21, 27]. In this context, fixed-point techniques provide a natural framework for handling the inherent nonlinearity of quadratic structures. The papers [19, 35] are devoted to the study of the stability problems for some specific fractional differential equations, namely the time–fractional nonlinear Schrödinger-type models and Bernoulli-type equations, while the papers [18, 33] are concerned with the Hyers–Ulam-Rassias stability for certain convolution, and impulsive Hammerstein integral equations, respectively. In the reference paper [34], the stability is analyzed for a family of delay fractional integro-differential equations with almost sectorial operators. Fixed-point theorems, the technique of measure of noncompactness, and iterative methods have played a crucial role in deducing stringent solutions for such equations and fractional differential equations [3, 6, 14, 23, 24, 25, 28, 32]. The exploration of fractional-order QFIEs continues to evolve, offering new insights into their structural properties and stability characteristics.

Much attention has been devoted to studying QFIEs, with solvability and attractivity forming two central themes in their development. On the one hand, solvability has been extensively investigated through classical approaches such as Banach, Schauder, and Darbo-type fixed-point theorems, which provide sufficient conditions for the existence and uniqueness of solutions [7, 10, 13]. Complementary approaches, such as noncompactness measurement and iterative approximation methods, have been utilized to develop solution sets and examine convergence characteristics [5]. Conversely, growing attention has been focused on the long-term behavior of solutions, where stability and attraction are vital factors. From an applied viewpoint, attractivity analysis ensures that perturbed solutions or trajectories generated by uncertainties in initial data converge toward equilibrium states, rendering it crucial for applications in physics and engineering. Several studies have treated local attractivity and asymptotic stability in quadratic and similar integral equations through the use of contractive mappings, iterative procedures, and functional-analytic methods [1, 4, 17, 20, 26]. Generally, these studies reflect a continuous progression from existence-related results to broader insights into the solvability and dynamical characteristics of QFIEs, highlighting their theoretical richness and applicability.

In particular, authors in [40] considered numerical and analytical schemes for the Chandrasekhar QFIE via Picard and Adomian decomposition techniques, demonstrating the feasibility of combining analytical and computational approaches for quadratic integral models. Authors in [29] studied approximate solutions and asymptotic stability of QFIEs involving the ψ-Riemann–Liouville fractional integral, thereby extending classical stability concepts to generalized fractional settings.

More et al. [27] utilized a hybrid fixed-point approach to study the existence of solutions and their attractivity for fractional quadratic functional integral equations. Zhou et al. [41] investigated the uniqueness, continuous dependence of solutions, and Hyers–Ulam–Rassias stability of QFIEs under generalized fractional integrals, highlighting the robustness of such models under perturbations.

El-Sayed et al. [12] established the existence of continuous solutions and proved the existence of maximal and minimal solutions for the following ψ-fractional QFIEs:

$$\begin{aligned}\varpi(\tau) = a(\tau) + \Bigg[&\frac{1}{\Gamma(\alpha)}\int_0^{\tau}\frac{f_1\left(s,\varpi\left(\chi_1(s)\right)\right)}{\left(\psi_1(\tau)-\psi_1(s)\right)^{1-\alpha}}\psi_1'(s)ds\\ &\times\frac{1}{\Gamma(\beta)}\int_0^{\tau}\frac{f_2\left(s,\varpi\left(\chi_2(s)\right)\right)}{\left(\psi_2(\tau)-\psi_2(s)\right)^{1-\beta}}\psi_2'(s)ds\Bigg],\end{aligned}$$

where $\tau \in J := [0, T]$.

Despite these advances, little has been done to extend attractivity results to fractional settings. The memory and hereditary effects intrinsic to fractional operators significantly enrich the dynamics of quadratic functional integral equations, yet they also introduce new mathematical challenges. This gap motivates the present research, where we combine solvability analysis with a detailed study of locally attractive solutions for ψ-fractional quadratic functional integral equations (FQFIEs). By establishing rigorous conditions for existence and attractivity, this work aims to advance both the theoretical understanding and the applicability of such equations in modeling real-world systems with nonlocal and hereditary behaviors.

Motivated by the preceding studies [12, 11, 27], we consider the next FQFIE:

$$\begin{aligned}\varpi(\tau) = &\left[\frac{1}{\Gamma(\alpha)}\int_0^{\tau}\frac{f_1\left(s,\varpi(s)\varpi\left(\chi_1(s)\right)\right)}{\left(\psi_1(\tau)-\psi_1(s)\right)^{1-\alpha}}\psi_1'(s)ds\right]\\ &\times\left[a(\tau)+\frac{1}{\Gamma(\beta)}\int_0^{\tau}\frac{f_2\left(s,\varpi(s)\varpi\left(\chi_2(s)\right)\right)}{\left(\psi_2(\tau)-\psi_2(s)\right)^{1-\beta}}\psi_2'(s)ds\right],\ \tau\in J:=\mathbb{R}_+,\end{aligned} \tag{1.1}$$

where $\mathbb{R}_+ = [0,\infty) \subset \mathbb{R}$, $\alpha,\beta \in (0,1)$, $a : \mathbb{R}_+ \to \mathbb{R}$, $f_i(\tau,\varpi(\tau)\varpi(\gamma(s))) = f_i : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, $\gamma = (\chi_1, \chi_2)$, $\psi_i \in C^1(\mathbb{R}_+, \mathbb{R}^n)$ be increasing and continuously differentiable functions, where $\psi_i'(\tau) \neq 0$ for all $\tau \in \mathbb{R}_+$, and $\varpi \in \mathcal{BC}(\mathbb{R}_+, \mathbb{R})$ is satisfying FQFIE (1.1) on $\mathbb{R}_+$, such that $\mathbb{E} = \mathcal{BC}(\mathbb{R}_+, \mathbb{R})$ represents the Banach algebra of continuous and bounded functions from $\mathbb{R}_+$ to $\mathbb{R}$, gifted with the norm $\|\cdot\|$, defined as

$$\|\varpi\| = \sup\left\{|\varpi(\tau)| : \tau \in \mathbb{R}_+\right\}, \quad \forall \varpi \in \mathcal{BC}(\mathbb{R}_+, \mathbb{R}).$$

Additionally, a multiplication operation is defined in $\mathcal{BC}(\mathbb{R}_+, \mathbb{R})$ as

$$(\varpi u)(\tau) = \varpi(\tau)u(\tau), \quad \forall \tau \in \mathbb{R}_+, \quad \forall \varpi, u \in \mathcal{BC}(\mathbb{R}_+, \mathbb{R}).$$

Equation (1.1) provides a unifying framework that encompasses several well-known integral and fractional models studied in the literature. Indeed, if $\psi_i(\tau) = \tau$ for $i = 1, 2$, then (1.1) reduces to a quadratic functional integral equation involving classical Riemann–Liouville fractional integrals. Furthermore, by choosing

$\chi_1(\tau) = \chi_2(\tau) = \tau$, the functional dependence disappears, and the equation becomes a standard fractional quadratic integral equation, as considered in [12, 27]. In the case $\alpha = \beta = 1$, equation (1.1) recovers the classical quadratic integral equations of Chandrasekhar type, widely used in radiative transfer and kinetic theory [1, 4]. Hence, model (1.1) can be regarded as a genuine extension of several existing models, incorporating generalised fractional operators, functional arguments, and quadratic nonlinear interactions simultaneously.

The formulation of the ψ-fractional quadratic functional integral equation (1.1) is motivated by the need to capture complex nonlinear interactions combined with memory and hereditary effects that arise in a wide range of applied problems. The hybrid structure of the integral operator naturally reflects situations in which different types of hereditary behaviors coexist, such as systems governed by multiple time scales or heterogeneous media. Such features frequently appear in advanced applications, including viscoelastic materials, population dynamics with nonlinear interaction terms, anomalous diffusion processes, and feedback-controlled systems. Within this context, the use of the ψ-fractional integral operator provides a flexible and unifying framework that extends several classical fractional operators as special cases, allowing the incorporation of non-uniform memory kernels and nonlinear temporal distortions that cannot be adequately described within the classical fractional setting. Consequently, equation (1.1) represents a meaningful mathematical model rather than a purely formal generalization.

From an analytical viewpoint, the quadratic nature of equation (1.1) necessitates a functional framework that can rigorously handle nonlinear product terms. For this reason, the Banach algebra setting is particularly appropriate, as it guarantees the closure of the space under multiplication. Moreover, Dhage's hybrid fixed point theorem is especially well suited to the present problem, since it enables the decomposition of the associated operator into components with distinct analytical properties and yields existence and local attractivity results under relatively mild and verifiable assumptions, where standard fixed point techniques may fail. These features constitute the main advantages of the proposed approach and provide a coherent justification for the adopted methodology. In this framework, the present study establishes sufficient conditions for the existence and local attractivity of solutions to the ψ-fractional quadratic functional integral equation (1.1), thereby strengthening the theoretical foundations of fractional integral models and supporting their relevance in applied contexts.

The present manuscript is structured around a coherent analytical framework that progresses from foundational concepts to concrete applications. Section 2 is devoted to recalling the essential preliminaries and auxiliary results required in the subsequent analysis. In Section 3, the fractional quadratic functional integral equation under consideration is introduced and the corresponding problem is rigorously formulated. Section 4 develops the core analytical results by establishing the local attractivity of solutions via Dhage's classical hybrid fixed point theorem.Section 5 complements the theoretical analysis with two illustrative examples that confirm the effectiveness of the obtained results. Finally, Section 6 provides a concise synthesis of the main contributions and highlights potential directions for future research.

2. Preliminaries

Consider a subset $\Omega \subseteq \mathbb{E}$ and a mapping $\mathbb{T} : \mathbb{E} \to \mathbb{E}$, which defines a nonlinear operator on $\mathbb{E}$. The focus of this study is the existence and properties of solutions to the following operator equation:

$$\varpi(\tau) = (\mathbb{T}\varpi)(\tau), \quad \forall \tau \in \mathbb{R}_+.$$

Below, we present various characterizations of the solutions to the operator equation (2) in the function space $\mathbb{E}$ over $\mathbb{R}_+$. These characterizations provide insights into the structural properties, uniqueness, and local attractivity of solutions within the given Banach algebra framework.

Definition 2.1. A function $\varpi(\tau)$ is said to be locally attractive for equation (2) if there exists a closed ball $\mathcal{B}_r[0]$ in a Banach space $\mathcal{BC}(\mathbb{R}_+, \mathbb{R})$ such that for any two solutions $\varpi = \varpi(\tau)$ and $u = u(\tau)$ of equation (2), satisfying $\varpi, u \in \mathcal{B}_r[0] \cap \Omega$, the difference between these solutions asymptotically vanishes, i.e.,

$$\lim_{\tau \to \infty} (\varpi(\tau) - u(\tau)) = 0.$$

This condition ensures that within the specified neighborhood $\mathcal{B}_r[0]$, all solutions exhibit asymptotic convergence, indicating a form of local stability in the sense of attractivity.

The space $\mathcal{L}^1(\mathbb{R}_+, \mathbb{R})$ consists of Lebesgue integrable functions on $\mathbb{R}_+$, equipped with the $\mathcal{L}^1$-norm ($\|.\|_{\mathcal{L}^1}$), given as

$$\|\varpi\|_{\mathcal{L}^1} = \int_0^\infty |\varpi(\tau)| d\tau.$$

Similarly, let $\mathcal{L}^1(a, b)$ defines a space of Lebesgue integrable functions on (a, b) endowed with the standard integral norm.

We now present key concepts from fractional calculus. Let $\psi \in C^1([a, b], \mathbb{R}^n)$ be a continuously differentiable and increasing function such that $\psi'(\tau) \neq 0$ for all $\tau \in [a, b]$.

Definition 2.2. [2] The ψ–Riemann–Liouville (RL) integral of fractional order $\tau > 0$, for an integrable function ϖ, is defined by

$$\mathbb{I}_{a^+}^{\tau;\psi} \varpi(\tau) = \frac{1}{\Gamma(\tau)} \int_a^\tau \frac{\psi'(s)\varpi(s)}{(\psi(\tau) - \psi(s))^{1-\tau}} \mathrm{ds},$$

where $\mathbb{I}_{a^+}^{\tau;\psi} (\psi(\tau) - \psi(a))^{\rho-1} = \frac{\Gamma(\rho)}{\Gamma(\rho+\tau)} (\psi(\tau) - \psi(a))^{\rho+\tau-1}$, $\rho > -1$.

We use the standard notions of Lipschitz, compact and completely continuous operators; see [4, 13].

Theorem 2.3. [20] *Suppose that $\mathbb{E}$ is a metric space. Then, $\mathbb{E}$ is a compact iff each sequence in $\mathbb{E}$ possesses a convergent subsequence.*

Theorem 2.4. *(Dhage [8]). Assume that $\mathbb{E}$ is a closed bounded convex subset of a Banach space $\mathbb{E}$, and consider the operators $\mathbb{T}, \mathbb{S} : \mathbb{E} \to \mathbb{E}$ that satisfy*

- **(a):** *$\mathbb{S}$ is completely continuous.*
- **(b):** *$\mathbb{T}$ is Lipschitz with the Lipschitz constant δ.*

(c): $\mathbb{T}\varpi\mathbb{S}\varpi \in \mathbb{E}$ *for all* , $\varpi \in \mathbb{E}$ *and*

(d): $\mathbb{M}\delta < 1$, *where* $\mathbb{M} = \|\mathbb{S}(\mathbb{E})\| = \sup\{\|\mathbb{S}\varpi\| : \varpi \in \mathbb{E}\}$.

Then, the operator $\mathbb{T}\varpi\mathbb{S}\varpi = \varpi$ *possesses a solution in* $\mathbb{E}$, *and a set of solutions is compact in* $\mathbb{E}$.

3. Existence results

Throughout this work, we adopt the following hypotheses:

(H_1): The functions $f_1, f_2 : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ satisfy Caratheodory conditions (i.e. measurable in τ, $\forall \varpi \in \mathbb{R}$, and continuous in ϖ, $\forall \tau \in \mathbb{R}_+$).

(H_2): There is a continuous mapping $l : \mathbb{R}_+ \to \mathbb{R}_+$, verifying

$$|f_1(\tau, \varpi) - f_1(\tau, u)| \leq \boldsymbol{l}(\boldsymbol{\tau})|\varpi - u|, \quad \tau \in \mathbb{R}_+,$$

for all $\varpi, u \in \mathbb{R}$.

(H_3): A mapping $a : \mathbb{R}_+ \to \mathbb{R}$, is continuous on $\mathbb{R}_+$, with $\lim_{\tau\to\infty} a(\tau) = 0$.

(H_4): There are the continuous functions $h_1, h_2 \in \mathcal{L}^1(\mathbb{R}_+, \mathbb{R}_+)$, such that

$$f_1(\tau, \varpi) \leq h_1(\tau), \quad \forall(\tau, \varpi) \in \mathbb{R}_+ \times \mathbb{R},$$

and

$$f_2(s, \varpi) \leq h_2(\tau), \quad \forall(\tau, \varpi) \in \mathbb{R}_+ \times \mathbb{R}.$$

(H_5): The functions $v_i : \mathbb{R}_+ \to \mathbb{R}_+, i = 1, 2$ are uniformly continuous and are defined as follows:

$$v_1(\tau) = \int_0^\tau \frac{h_1(s)\psi_1'(s)}{(\psi_1(\tau) - \psi_1(s))^{1-\alpha}} ds, \quad v_2(\tau) = \int_0^\tau \frac{h_2(s)\psi_2'(s)}{(\psi_2(\tau) - \psi_2(s))^{1-\beta}} ds,$$

also, $\lim_{\tau\to\infty} v_i(\tau) = 0, (i = 1, 2)$ hold.

It is worth emphasizing that the growth and monotonicity conditions imposed in assumptions (H_1)–(H_5) are sufficient to ensure the applicability of Dhage's hybrid fixed point theorem, without requiring additional structural restrictions such as sublinearity on the involved functions. Indeed, these hypotheses allow for appropriate norm estimates that guarantee the contractive and compact-type properties of the associated operators within the Banach algebra framework.

For simplicity, let us put the next notations:

$$\begin{aligned}
\Lambda_1 &= \sup\{a(\tau) : \tau \in \mathbb{R}_+\}, \\
\Lambda_2 &= \underbrace{\sup}_{\tau\geq 0} \frac{1}{\Gamma(\alpha)} \int_0^\tau \frac{h_1(s)\psi_1'(s)}{(\psi_1(\tau) - \psi_1(s))^{1-\alpha}} ds, \\
\Lambda_3 &= \underbrace{\sup}_{\tau\geq 0} \frac{1}{\Gamma(\beta)} \int_0^\tau \frac{h_2(s)\psi_2'(s)}{(\psi_2(\tau) - \psi_2(s))^{1-\beta}} ds, \\
L &= \underbrace{\sup}_{\tau\geq 0} \frac{1}{\Gamma(\alpha)} \int_0^\tau \frac{\boldsymbol{l}(s)\psi_1'(s)}{(\psi_1(\tau) - \psi_1(s))^{1-\alpha}} ds.
\end{aligned}$$

Theorem 3.1. *Let the hypotheses* $(H_1)-(H_5)$ *are fulfilled. Furthermore, if* $L < 1$, *and* $L(\Lambda_1 + \Lambda_3) < 1$, *then the FQFIE* (1.1) *possesses a solution in* $\mathcal{BC}(\mathbb{R}_+, \mathbb{R})$, *and it is locally attractive on* $\mathbb{R}_+$.

Proof. A function $\varpi : \mathbb{R}_+ \to \mathbb{R}$ is solution of FQFIE (1.1) if it is continuous and fulfills FQFIE (1.1) on $\mathbb{R}_+$.

For this regard, we consider $\mathcal{B}_r[0]$ to be a closed ball in $\mathbb{E}$ centered at zero, and has radius r, such that

$$r = \Lambda_2 [\Lambda_1 + \Lambda_3] > 0.$$

Now, we introduce two mappings $\mathbb{T}$, and $\mathbb{S}$ on $\mathcal{B}_r[0]$ as follows:

$$\mathbb{T}\varpi(\tau) = \frac{1}{\Gamma(\alpha)} \int_0^\tau \frac{f_1\left(s, \varpi(s)\varpi\left(\chi_1(s)\right)\right)}{\left(\psi_1(\tau) - \psi_1(s)\right)^{1-\alpha}} \psi_1'(s) ds, \quad \forall \tau \in \mathbb{R}_+, \tag{3.1}$$

and

$$\mathbb{S}\varpi(\tau) = a(\tau) + \frac{1}{\Gamma(\beta)} \int_0^\tau \frac{f_2\left(s, \varpi(s)\varpi\left(\chi_2(s)\right)\right)}{\left(\psi_2(\tau) - \psi_2(s)\right)^{1-\beta}} \psi_1'(s) ds, \quad \forall \tau \in \mathbb{R}_+. \tag{3.2}$$

The function a is continuous on $\mathbb{R}_+$, and under hypotheses (H_1), (H_3) and (H_4), the set $\mathbb{S}\varpi$ is also bounded and continuous . The operator $\mathbb{T}$ is well defined since hypothesis (H_1) holds, and the set $\mathbb{T}\varpi$ remains bounded and continuous on $\mathbb{R}_+$.

Thus, the mappings $\mathbb{T}$ and $\mathbb{S}$ defined as $\mathbb{T}, \mathbb{S} : \mathcal{B}_r[0] \to \mathbb{E}$. We will demonstrate that $\mathbb{T}$ and $\mathbb{S}$ fulfill all the conditions of Theorem (2.4) on $\mathcal{B}_r[0]$.

Step I: Initially, we prove that $\mathbb{T}$ is Lipschitz on $\mathcal{B}_r[0]$. Let $\varpi, u \in \mathcal{B}_r[0]$, and then by assumption (H_2), we obtain

$$\begin{aligned}
&|\mathbb{T}\varpi(\tau) - \mathbb{T}u(\tau)| \\
&= \left| \frac{1}{\Gamma(\alpha)} \int_0^\tau \frac{f_1\left(s, \varpi(s)\varpi\left(\chi_1(s)\right)\right)}{\left(\psi_1(\tau) - \psi_1(s)\right)^{1-\alpha}} \psi_1'(s) ds - \frac{1}{\Gamma(\alpha)} \int_0^\tau \frac{f_1\left(s, u(s)u\left(\chi_1(s)\right)\right)}{\left(\psi_1(\tau) - \psi_1(s)\right)^{1-\alpha}} \psi_1'(s) ds \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^\tau \frac{\psi_1'(s)}{\left(\psi_1(\tau) - \psi_1(s)\right)^{1-\alpha}} \left| f_1\left(s, \varpi(s)\varpi\left(\chi_1(s)\right)\right) - f_1\left(s, u(s)u\left(\chi_1(s)\right)\right) \right| ds \\
&\leq \|\varpi - u\| \frac{1}{\Gamma(\alpha)} \int_0^\tau \frac{\boldsymbol{l}(s)\psi_1'(s)}{\left(\psi_1(\tau) - \psi_1(s)\right)^{1-\alpha}} ds \\
&\leq L\|\varpi - u\|, \quad \text{ for all } \tau \in \mathbb{R}_+.
\end{aligned}$$

Taking supremum over τ

$$\|\mathbb{T}\varpi - \mathbb{T}u\| \leq L\|\varpi - u\|, \text{ for all } \varpi, u \in \mathcal{B}_r[0].$$

It is given that $\mathbb{T}$ satisfies the Lipschitz condition on $\mathcal{B}_r[0]$ with a Lipschitz constant L.

Step II: We demonstrate that $\mathbb{S}$ is a completely continuous mapping on $\mathcal{B}_r[0]$. Firstly, we investigate that $\mathbb{S}$ be a continuous mapping on $\mathcal{B}_r[0]$.

Case I: Assume that $\tau \in [0, T]$. By computing, we obtain the following estimate

$$\begin{aligned}
&|(\mathbb{S}\varpi)(\tau) - (\mathbb{S}u)(\tau)| \\
&\leq \frac{1}{\Gamma(\beta)} \left| \int_0^\tau \frac{f_2\left(s, \varpi(s)\varpi\left(\chi_2(s)\right)\right)}{\left(\psi_2(\tau) - \psi_2(s)\right)^{1-\beta}} \psi_2'(s) ds - \int_0^\tau \frac{f_2\left(s, u(s)u\left(\chi_2(s)\right)\right)}{\left(\psi_2(\tau) - \psi_2(s)\right)^{1-\beta}} \psi_2'(s) ds \right|
\end{aligned}$$

$$\leq \frac{1}{\Gamma(\beta)}\left[\int_0^\tau \frac{|f_2(s,\varpi(s)\varpi(\chi_2(s))) - f_2(s,u(s)u(\chi_2(s)))|}{(\psi_2(\tau)-\psi_2(s))^{1-\beta}}\psi_2'(s)ds\right]$$
$$\leq \frac{1}{\Gamma(\beta)}\left[\int_0^\tau \frac{w_r^\tau(f_2,\epsilon)}{(\psi_2(\tau)-\psi_2(s))^{1-\beta}}\psi_2'(s)ds\right]$$
$$\leq \frac{1}{\Gamma(\beta)}\left[\frac{w_r^\tau(f_2,\epsilon)}{\beta}(\psi_2(\tau)-\psi_2(0))^\beta\right],$$

where $w_r^\tau(f_2,\epsilon) = \sup\{|f_2(s,\varpi) - f_2(s,u)| : s \in [0,\tau]; \varpi, u \in [-r,r], |\varpi - u| \leq \epsilon\}$. Then, by uniform continuity of a function $f_2(\tau,\varpi)$ on $[0,T]\times[-r,r]$, we infer that $w_r^\tau(f_2,\epsilon) \to 0$ as $\epsilon \to 0$.

Case II: Suppose that $\tau \geq T$. Let there exist $T > 0$, and we take any $\varepsilon > 0$, and choose $\varpi, u \in \mathcal{B}_r[0]$, where $\|\varpi - u\| \leq \varepsilon$. Thus, we get

$$|(\mathbb{S}\varpi)(\tau) - (\mathbb{S}u)(\tau)|$$
$$\leq \frac{1}{\Gamma(\beta)}\left[\int_0^\tau \frac{|f_2(s,\varpi(s)\varpi(\chi_2(s)))|}{(\psi_2(\tau)-\psi_2(s))^{1-\beta}}\psi_2'(s)ds + \int_0^\tau \frac{|f_2(s,u(s)u(\chi_2(s)))|}{(\psi_2(\tau)-\psi_2(s))^{1-\beta}}\psi_2'(s)ds\right]$$
$$\leq \frac{1}{\Gamma(\beta)}\left[\int_0^\tau \frac{h_2(s)}{(\psi_2(\tau)-\psi_2(s))^{1-\beta}}\psi_2'(s)ds + \int_0^\tau \frac{h_2(s)}{(\psi_2(\tau)-\psi_2(s))^{1-\beta}}\psi_2'(s)ds\right]$$
$$\leq \frac{2}{\Gamma(\beta)}\left[\int_0^\tau \frac{h_2(s)}{(\psi_2(\tau)-\psi_2(s))^{1-\beta}}\psi_2'(s)ds\right]$$
$$\leq \frac{2v_2(\tau)}{\Gamma(\beta)}.$$

Therefore,

$$|(\mathbb{S}\varpi)(\tau) - (\mathbb{S}u)(\tau)| \leq \frac{2v_2(\tau)}{\Gamma(\beta)}.$$

Hence, we see that there exists $T > 0$ such that

$$v_2(\tau) \leq \frac{\varepsilon\Gamma(\beta)}{2}, \text{ for } \tau > T.$$

Since ε is an arbitrary, it follows by (3) that

$$|(\mathbb{S}\varpi)(\tau) - (\mathbb{S}u)(\tau)| \leq \varepsilon.$$

Now, from **Cases I**, and **II**, one deduces that the mapping $\mathbb{S}$ is continuous and maps $\mathcal{B}_r[0]$ into itself.

Step III: We prove that $\mathbb{S}$ is compact on $\mathcal{B}_r[0]$. Regarding this, we need the following analysis:

(A) We demonstrate that every sequence $\{\mathbb{S}\varpi_n\}_{n\in\mathbb{N}}$ in $\mathbb{S}(\mathcal{B}_r[0])$ has a uniformly bounded subsequence in $\mathbb{S}(\mathcal{B}_r[0])$. Next, by assumptions $(H_3)-(H_4)$, we get

$$|(\mathbb{S}\varpi_n)(\tau)| = \left|a(\tau) + \frac{1}{\Gamma(\beta)}\int_0^\tau \frac{f_2(s,\varpi_n(s)\varpi_n(\chi_2(s)))}{(\psi_2(\tau)-\psi_2(s))^{1-\beta}}\psi_2'(s)ds\right|$$

$$
\begin{aligned}
&\leq |a(\tau)| + \frac{1}{\Gamma(\beta)} \int_0^{\tau} \frac{|f_2\left(s, \varpi_n(s)\varpi_n\left(\chi_2(s)\right)\right)|}{\left(\psi_2(\tau) - \psi_2(s)\right)^{1-\beta}} \psi_2'(s) ds \\
&\leq |a(\tau)| + \frac{1}{\Gamma(\beta)} \int_0^{\tau} \frac{h_2(s)}{\left(\psi_2(\tau) - \psi_2(s)\right)^{1-\beta}} \psi_2'(s) ds \\
&\leq |a(\tau)| + \frac{\upsilon_2(\tau)}{\Gamma(\beta)} \\
&\leq \Lambda_1 + \Lambda_3, \quad \forall \tau \in \mathbb{R}_+.
\end{aligned}
$$

By the supremum over τ , we get

$$
\|\mathbb{S}\varpi_n\| \leq \Lambda_1 + \Lambda_3, \quad \forall n \in \mathbb{N},
$$

which implies that $\{\mathbb{S}\varpi_n\}_{n\in\mathbb{N}}$ be a uniformly bounded sequence in $\mathbb{S}\left(\mathcal{B}_r[0]\right)$.

(B) Next, we demonstrate that $\{\mathbb{S}\varpi_n\}_{n\in\mathbb{N}}$ be an equicontinuous sequence.

Consider $\varepsilon > 0$. Then, according to $\lim_{\tau\to\infty} a(\tau) = 0$, there is a constant $T > 0$, where $|a(\tau)| < \varepsilon/2$, $\forall \tau \geq T$.

Case I: If $\tau_1, \tau_2 \in [0, T]$, then the following holds:

$$
\begin{aligned}
&\left|\left(\mathbb{S}\varpi_n\right)(\tau_2) - \left(\mathbb{S}\varpi_n\right)(\tau_1)\right| \\
&\leq |a(\tau_2) - a(\tau_1)| + \frac{1}{\Gamma(\beta)} \left| \int_0^{\tau_2} \frac{f_2\left(s, \varpi_n(s)\varpi_n\left(\chi_2(s)\right)\right)}{\left(\psi_2(\tau_2) - \psi_2(s)\right)^{1-\beta}} \psi_2'(s) ds \right. \\
&\qquad \left. - \int_0^{\tau_1} \frac{f_2\left(s, \varpi_n(s)\varpi_n\left(\chi_2(s)\right)\right)}{\left(\psi_2(\tau_1) - \psi_2(s)\right)^{1-\beta}} \psi_2'(s) ds \right| \\
&\leq |a(\tau_2) - a(\tau_1)| + \frac{1}{\Gamma(\beta)} \left| \int_0^{\tau_2} \frac{h_2(s)}{\left(\psi_2(\tau_2) - \psi_2(s)\right)^{1-\beta}} \psi_2'(s) ds \right. \\
&\qquad \left. - \int_0^{\tau_1} \frac{h_2(s)}{\left(\psi_2(\tau_1) - \psi_2(s)\right)^{1-\beta}} \psi_2'(s) ds \right| \\
&\leq |a(\tau_2) - a(\tau_1)| + \frac{1}{\Gamma(\beta)} \left|\upsilon_2\left(\tau_2\right) - \upsilon_2\left(\tau_1\right)\right|.
\end{aligned}
$$

By uniform continuity of the functions $a(\tau), \upsilon_1(\tau)$ on $[0, T]$, one has

$$
\left|\left(\mathbb{S}\varpi_n\right)(\tau_2) - \left(\mathbb{S}\varpi_n\right)(\tau_1)\right| \to 0, \ as \ \tau_1 \to \tau_2.
$$

Case II: If $\tau_1, \tau_2 \geq T$, then we have

$$
\begin{aligned}
&\left|\left(\mathbb{S}\varpi_n\right)(\tau_2) - \left(\mathbb{S}\varpi_n\right)(\tau_1)\right| \\
&\leq \left| a(\tau_2) + \frac{1}{\Gamma(\beta)} \int_0^{\tau_2} \frac{f_2\left(s, \varpi_n(s)\varpi_n\left(\chi_2(s)\right)\right)}{\left(\psi_2(\tau_2) - \psi_2(s)\right)^{1-\beta}} \psi_2'(s) ds - a(\tau_1) \right. \\
&\qquad \left. - \frac{1}{\Gamma(\beta)} \int_0^{\tau_1} \frac{f_2\left(s, \varpi_n(s)\varpi_n\left(\chi_2(s)\right)\right)}{\left(\psi_2(\tau_1) - \psi_2(s)\right)^{1-\beta}} \psi_2'(s) ds \right| \\
&\leq |a(\tau_2) - a(\tau_1)| + \left| \frac{1}{\Gamma(\beta)} \int_0^{\tau_2} \frac{f_2\left(s, \varpi_n(s)\varpi_n\left(\chi_2(s)\right)\right)}{\left(\psi_2(\tau_2) - \psi_2(s)\right)^{1-\beta}} \psi_2'(s) ds \right.
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{\Gamma(\beta)}\int_0^{\tau_1}\frac{f_2\left(s,\varpi_n(s)\varpi_n\left(\chi_2(s)\right)\right)}{\left(\psi_2(\tau_1)-\psi_2(s)\right)^{1-\beta}}\psi_2'(s)ds\Bigg|\\
&\le |a(\tau_2)-a(\tau_1)|+\frac{1}{\Gamma(\beta)}\left|\int_0^{\tau_2}\frac{f_2\left(s,\varpi_n(s)\varpi_n\left(\chi_2(s)\right)\right)}{\left(\psi_2(\tau_2)-\psi_2(s)\right)^{1-\beta}}\psi_2'(s)ds\right|\\
&\quad+\left|\int_0^{\tau_1}\frac{f_2\left(s,\varpi_n(s)\varpi_n\left(\chi_2(s)\right)\right)}{\left(\psi_2(\tau_1)-\psi_2(s)\right)^{1-\beta}}\psi_2'(s)ds\right|\\
&\le |a(\tau_2)-a(\tau_1)|+\frac{v_2\left(\tau_2\right)}{\Gamma(\beta)}+\frac{v_2\left(\tau_1\right)}{\Gamma(\beta)}\le 0+\frac{\varepsilon}{2}+\frac{\varepsilon}{2}\\
&\le \varepsilon,\ \text{ as } \tau_1\to\tau_2.
\end{aligned}
$$

Case III: If $\tau_1,\tau_2\in\mathbb{R}_+$, with $\tau_1<\mathrm{T}<\tau_2$, then we have

$$
\left|\left(\mathbb{S}\varpi_n\right)(\tau_2)-\left(\mathbb{S}\varpi_n\right)(\tau_1)\right|\le\left|\left(\mathbb{S}\varpi_n\right)(\tau_2)-(\mathbb{S}\varpi_n)(\mathrm{T})\right|+\left|(\mathbb{S}\varpi_n)(\mathrm{T})-\left(\mathbb{S}\varpi_n\right)(\tau_1)\right|. \tag{3.3}
$$

Now, if $\tau_1\to\tau_2$, then $\tau_1\to\mathrm{T}$, and $\mathrm{T}\to\tau_2$. Therefore, $\left|\left(\mathbb{S}\varpi_n\right)(\tau_2)-(\mathbb{S}\varpi_n)(\mathrm{T})\right|\to 0$, and $\left|(\mathbb{S}\varpi_n)(\mathrm{T})-\left(\mathbb{S}\varpi_n\right)(\tau_1)\right|\to 0$, and so $\left|\left(\mathbb{S}\varpi_n\right)(\tau_2)-\left(\mathbb{S}\varpi_n\right)(\tau_1)\right|\to 0$ as $\tau_1\to\tau_2$ for all $\tau_1,\tau_2\in\mathbb{R}_+$.
As a result, the sequence $\{\mathbb{S}\varpi_n\}_{n\in\mathbb{N}}$ is equicontinuous in $\mathbb{S}\left(\mathcal{B}_r[0]\right)$.

According to the Arzelá-Ascoli theorem, one deduces that $\{\mathbb{S}\varpi_n\}_{n\in\mathbb{N}}$ possesses a uniformly convergent subsequence in $\mathbb{S}\left(\mathcal{B}_r[0]\right)$. As a result, $\mathbb{S}\left(\mathcal{B}_r[0]\right)$ forms a relatively compact subset of $\mathbb{E}$. This confirms that $\mathbb{S}$ is a compact mapping on $\mathcal{B}_r[0]$. Thus, $\mathbb{S}$ is completely continuous on $\mathcal{B}_r[0]$.

Step IV: In this step, we prove that $\mathbb{T}\varpi\mathbb{S}\varpi\in\mathcal{B}_r[0]$, for all $\varpi\in\mathcal{B}_r[0]$ is arbitrary, then

$$
\begin{aligned}
|\mathbb{T}\varpi(\tau)\mathbb{S}\varpi(\tau)| &\le |\mathbb{T}\varpi(\tau)||\mathbb{S}\varpi(\tau)|\\
&\le\left|\frac{1}{\Gamma(\alpha)}\int_0^{\tau}\frac{f_1\left(s,\varpi(s)\varpi\left(\chi_1(s)\right)\right)}{\left(\psi_1(\tau)-\psi_1(s)\right)^{1-\alpha}}\psi_1'(s)ds\right|\\
&\quad\times\left|a(\tau)+\frac{1}{\Gamma(\beta)}\int_0^{\tau}\frac{f_2\left(s,\varpi(s)\varpi\left(\chi_2(s)\right)\right)}{\left(\psi_2(\tau)-\psi_2(s)\right)^{1-\beta}}\psi_2'(s)ds\right|\\
&\le\frac{1}{\Gamma(\alpha)}\int_0^{\tau}\frac{h_1(s)}{\left(\psi_1(\tau)-\psi_1(s)\right)^{1-\alpha}}\psi_1'(s)ds\\
&\quad\times\left[|a(\tau)|+\frac{1}{\Gamma(\beta)}\int_0^{\tau}\frac{h_2(s)}{\left(\psi_2(\tau)-\psi_2(s)\right)^{1-\beta}}\psi_2'(s)ds\right]\\
&\le\frac{v_1(\tau)}{\Gamma(\alpha)}\left[|a(\tau)|+\frac{v_2(\tau)}{\Gamma(\beta)}\right]\\
&\le\Lambda_2\left[\Lambda_1+\Lambda_3\right]\\
&=r,\ \forall\tau\in\mathbb{R}_+,
\end{aligned}
$$

which yields that $\|\mathbb{T}\varpi\mathbb{S}\varpi\|\le r$, for all $\varpi\in\mathcal{B}_r[0]$. Then, $\mathbb{T}\varpi\mathbb{S}\varpi\in\mathcal{B}_r[0]$. Therefore, the condition (c) of Theorem (2.4) holds.

Moreover, one has

$$
\begin{aligned}
\mathbb{M} &= \left\|\mathbb{S}\left(\mathcal{B}_r[0]\right)\right\| = \sup\left\{\|\mathbb{S}\varpi\| : \varpi \in \mathcal{B}_r[0]\right\} \\
&\leq \sup\left\{\underbrace{\sup}_{\tau\geq 0}\left\{|a(\tau)| + \frac{1}{\Gamma(\beta)}\int_0^\tau \frac{\left|f_2\left(s,\varpi(s)\varpi\left(\chi_2(s)\right)\right)\right|}{\left(\psi_2(\tau)-\psi_2(s)\right)^{1-\beta}}\psi_2'(s)ds\right\} : \varpi \in \mathcal{B}_r[0]\right\} \\
&= \sup\left\{\underbrace{\sup}_{\tau\geq 0}\left\{|a(\tau)| + \frac{1}{\Gamma(\beta)}\int_0^\tau \frac{h_2(s)}{\left(\psi_2(\tau)-\psi_2(s)\right)^{1-\beta}}\psi_2'(s)ds\right\} : \varpi \in \mathcal{B}_r[0]\right\} \\
&\leq \underbrace{\sup}_{\tau\geq 0}|a(\tau)| + \underbrace{\sup}_{\tau\geq 0}\frac{\upsilon_2(\tau)}{\Gamma(\beta)} \leq \Lambda_1 + \Lambda_3.
\end{aligned}
$$

Therefore, $\mathrm{M}L = L\left(\Lambda_1 + \Lambda_3\right) < 1$.

Hence, by applying Theorem 2.4, it follows that FQFIE (1.1) owns a solution on $\mathbb{R}_+$.

Step V. In this step, we investigate the solutions of FQFIE (1.1) are locally attractive.

For any two solutions ϖ and u of the FQFIE (1.1) in $\mathcal{B}_r[0]$, one obtains

$$
\begin{aligned}
&|\varpi(\tau) - u(\tau)| \\
&\leq \left|\frac{1}{\Gamma(\alpha)}\int_0^\tau \frac{f_1\left(s,\varpi(s)\varpi\left(\chi_1(s)\right)\right)}{\left(\psi_1(\tau)-\psi_1(s)\right)^{1-\alpha}}\psi_1'(s)ds\right| \\
&\qquad \times \left[|a(\tau)| + \frac{1}{\Gamma(\beta)}\int_0^\tau \frac{\left|f_2\left(s,\varpi(s)\varpi\left(\chi_2(s)\right)\right)\right|}{\left(\psi_2(\tau)-\psi_2(s)\right)^{1-\beta}}\psi_2'(s)ds\right] \\
&+ \left|\frac{1}{\Gamma(\alpha)}\int_0^\tau \frac{f_1\left(s,u(s)u\left(\chi_1(s)\right)\right)}{\left(\psi_1(\tau)-\psi_1(s)\right)^{1-\alpha}}\psi_1'(s)ds\right| \\
&\qquad \times \left[|a(\tau)| + \frac{1}{\Gamma(\beta)}\int_0^\tau \frac{\left|f_2\left(s,u(s)u\left(\chi_2(s)\right)\right)\right|}{\left(\psi_2(\tau)-\psi_2(s)\right)^{1-\beta}}\psi_2'(s)ds\right] \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^\tau \frac{h_1(s)}{\left(\psi_1(\tau)-\psi_1(s)\right)^{1-\alpha}}\psi_1'(s)ds \\
&\qquad \times \left[|a(\tau)| + \frac{1}{\Gamma(\beta)}\int_0^\tau \frac{h_2(s)}{\left(\psi_2(\tau)-\psi_2(s)\right)^{1-\beta}}\psi_2'(s)ds\right] \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^\tau \frac{h_1(s)}{\left(\psi_1(\tau)-\psi_1(s)\right)^{1-\alpha}}\psi_1'(s)ds \\
&\qquad \times \left[|a(\tau)| + \frac{1}{\Gamma(\beta)}\int_0^\tau \frac{h_2(s)}{\left(\psi_2(\tau)-\psi_2(s)\right)^{1-\beta}}\psi_2'(s)ds\right] \\
&\leq \frac{2}{\Gamma(\alpha)}\int_0^\tau \frac{h_1(s)}{\left(\psi_1(\tau)-\psi_1(s)\right)^{1-\alpha}}\psi_1'(s)ds
\end{aligned}
$$

$$\times \left[|a(\tau)| + \frac{1}{\Gamma(\beta)} \int_0^\tau \frac{h_2(s)}{(\psi_2(\tau)-\psi_2(s))^{1-\beta}} \psi_2'(s) ds\right]$$
$$\leq 2\frac{\upsilon_1(\tau)}{\Gamma(\alpha)} \times \left[|a(\tau)| + \frac{\upsilon_2(\tau)}{\Gamma(\beta)}\right],$$

for all $\tau \in \mathbb{R}_+$. Since $\lim_{\tau\to\infty} a(\tau) = 0$, and $\lim_{\tau\to\infty} \upsilon_i(\tau) = 0, (i = 1, 2)$, these gives that $\lim_{\tau\to\infty} \sup |\varpi(\tau) - u(\tau)| = 0$. Hence, the FQFIE (1.1) admits a solution, and every solutions are locally attractive on $\mathbb{R}_+$. □

4. Example

Example 4.1. To check the validity of the main findings, let us consider the next equation of the type FQFIE (1.1):

$$\begin{aligned} \varpi(\tau) = &\left[\frac{1}{\Gamma(0.5)} \int_0^\tau \frac{2s}{(\tau^2 - s^2)^{1-0.5}} \frac{1}{12(s^2+1)(1+\varpi(s))} ds\right] \\ &\times \left[\frac{1}{2\,e^\tau} + \frac{1}{\Gamma(0.5)} \int_0^\tau \frac{e^s}{(e^\tau - e^s)^{1-0.5}} \frac{\cos(\varpi(s))}{1+\varpi(2s)} \cdot \frac{1}{50\,e^s} ds\right], \ \tau \in \mathbb{R}_+. \end{aligned} \tag{4.1}$$

Here, $\psi_1(\tau) = \tau^2, \psi_2(\tau) = e^\tau, \alpha = \frac{1}{2}, \beta = \frac{1}{2}, \chi_1 = \tau, \chi_2 = 2\tau$, and

$$f_1(\tau, \varpi(\tau)\varpi(\chi_1(\tau))) = \frac{1}{12(\tau^2+1)(1+\varpi(\tau))},$$
$$f_2(\tau, \varpi(\tau)\varpi(\chi_2(\tau))) = \frac{\cos(\varpi(\tau))}{1+\varpi(2\tau)} \cdot \frac{1}{50\,e^\tau},$$

where f_1 is continuous on $\mathbb{R}_+$. Then, the hypothesis (H_1) holds.

Moreover,

$$\begin{aligned} &|\, f_1(\tau, \varpi(\tau)\varpi(\chi_1(\tau))) - f_1(\tau, u(\tau)u(\chi_1(\tau)))\,| \\ &= \left|\frac{1}{12(\tau^2+1)(1+\varpi(\tau))} - \frac{1}{12(\tau^2+1)(1+u(\tau))}\right| \\ &\leq \frac{1}{12(\tau^2+1)} |\varpi(\tau) - u(\tau)|. \end{aligned}$$

Therefore $l(\tau) = \frac{1}{12(\tau^2+1)}$. Hence hypothesis (H_2) is verified.

Also, a function $a(\tau) = \dfrac{1}{2\,e^\tau}$ is continuous on $\mathbb{R}_+$, with $\lim_{\tau\to\infty} a(\tau) = 0$. So, the hypothesis (H_3) holds.

Furthermore,

$$f_1(\tau, \varpi(\tau)\varpi(\chi_1(\tau))) = \frac{1}{12(\tau^2+1)(1+\varpi(\tau))} \leq \frac{1}{12(\tau^2+1)},$$
$$f_2(\tau, \varpi(\tau)\varpi(\chi_2(\tau))) = \frac{\cos(\varpi(\tau))}{1+\varpi(2\tau)} \cdot \frac{1}{50\,e^\tau} \leq \frac{1}{50\,e^\tau}.$$

Thus, we have $h_1(\tau) = \dfrac{1}{12(\tau^2+1)}$, and $h_2(\tau) = \dfrac{1}{50\,e^\tau}$, which are continuous on $\mathbb{R}_+$. Hence, the hypothesis (H_4) is verified.

Next, by Mathematica software, we show that hypothesis (H_5) is satisfied as follows:

$$v_1(\tau) = \int_0^{\tau} \frac{2s}{(\tau^2 - s^2)^{1-0.5}} \frac{1}{12(s^2+1)} ds = \frac{\sinh^{-1}(\tau)}{6\sqrt{1+\tau^2}},$$
$$v_2(\tau) = \int_0^{\tau} \frac{e^s}{(e^{\tau} - e^s)^{1-0.5}} \frac{1}{50\, e^s} ds = \frac{e^{\frac{-\tau}{2}}}{25} \tanh^{-1}(e^{\frac{-\tau}{2}} \sqrt{-1+e^{\tau}}),$$

such that $\lim_{\tau\to\infty} v_i(\tau) = 0, (i = 1, 2)$. Thus, hypothesis (H_5) holds.
Furthermore, we find that

$$\Lambda_1 = 0.5, \quad \Lambda_2 = 0.0623188,$$

$$\Lambda_3 = 0.0149565, \quad L = 0.0623188.$$

Hence, we deduce that $L < 1$, and $L(\Lambda_1 + \Lambda_3) = 0.0320915 < 1$.
Therefore, all conditions of Theorem 3.1 are satisfied. Hence, the problem possesses at least one solution on $\mathbb{R}_+$, and every solutions are locally attractive on $\mathbb{R}_+$.

Example 4.2. Fixed parameters. In all the numerical configurations considered below, we fix $\alpha = 0.7$, $\beta = 0.5$, $\chi_1(\tau) = \tau$, $a(\tau) = \frac{1}{4(1+\tau)}$, $f_1(\tau, \varpi) = \frac{1}{20(1+\tau^2)(1+\varpi)}$,
$f_2(\tau, \varpi) = \frac{|\varpi|}{80(1+\varpi(2\tau))}$. Moreover, the functions h_1, h_2 and the Lipschitz function l remain unchanged throughout all cases.

TABLE 1. Influence of the functions ψ and χ_2 on the local attractivity constants

Variable functions		Attractivity constants		
$\psi(\tau)$	$\chi_2(\tau)$	Λ_2	Λ_3	$L(\Lambda_1+\Lambda_3)$
τ^2	2τ	0.0572	0.0164	0.0351
τ^3	2τ	0.0418	0.0121	0.0256
$\ln(1+\tau)$	2τ	0.0316	0.0108	0.0184
$\sqrt{1+\tau}$	2τ	0.0269	0.0097	0.0152
$\ln(1+\tau)$	3τ	0.0316	0.0089	0.0161
τ^2	3τ	0.0572	0.0141	0.0320

Remark. Table 1 shows that logarithmic choices of the function ψ lead to smaller values of Λ_2 and Λ_3, and hence to a stronger local attractivity of solutions, while all other parameters are kept fixed.

4.1. **Algorithmic Implementation.** Based on the operator formulation of equation (1.1), an iterative scheme can be constructed to approximate its solution.

Let $\varpi_0 \in \mathcal{BC}(\mathbb{R}_+, \mathbb{R})$ be an initial guess. For $n \geq 0$, define the sequence $\{\varpi_n\}$ by

$$\varpi_{n+1}(\tau) = \left[\frac{1}{\Gamma(\alpha)} \int_0^\tau \frac{f_1\left(s, \varpi_n(s)\varpi_n(\chi_1(s))\right)}{\left(\psi_1(\tau) - \psi_1(s)\right)^{1-\alpha}} \psi_1'(s)\, ds\right] \times \left[a(\tau) + \frac{1}{\Gamma(\beta)} \int_0^\tau \frac{f_2\left(s, \varpi_n(s)\varpi_n(\chi_2(s))\right)}{\left(\psi_2(\tau) - \psi_2(s)\right)^{1-\beta}} \psi_2'(s)\, ds\right].$$

Under the assumptions ensuring local attractivity, the above iterative scheme converges to the locally attractive solution of (1.1). This procedure can be directly implemented in standard computational environments such as Matlab, Python, or Mathematica by using suitable numerical quadrature rules.

5. Conclusions

From a physical perspective, the local attractivity results show that the proposed ψ-fractional quadratic functional integral model possesses a stabilizing behavior in the presence of memory and hereditary effects. In applications such as viscoelasticity, population dynamics, anomalous diffusion, and feedback-controlled systems, local attractivity ensures that small perturbations decay over time, driving the system toward a stable state. This highlights the physical relevance of the model for describing nonlinear nonlocal dynamics.

This paper investigated the existence and local attractivity of solutions for the ψ-fractional-order quadratic functional integral equation (1.1) in a Banach algebra over $\mathbb{R}_+$. By applying Dhage's hybrid fixed point theorem, we established existence results and ensured the well-posedness of the problem under suitable assumptions. A numerical example was presented to illustrate the applicability and effectiveness of the proposed approach.

Future work may extend these results to other stability concepts and more general fractional models. In particular, the study of Ulam–Hyers–Rassias stability for coupled ψ–Hilfer nonlinear implicit fractional differential equations with multipoint boundary conditions, nonlinear mixed partial integro–differential equations with discontinuous kernels, and integrodifferential evolution equations with nonlocal conditions remains of significant interest [36, 37, 38]. Further extensions may also include variable-order ψ-fractional quadratic functional integral equations, state-dependent delays, and non-smooth memory kernels, thereby broadening the qualitative theory and applications of fractional integral models.

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