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AN INITIAL-BOUNDARY VALUE PROBLEM FOR A STRONGLY DEGENERATE HIGHER-ORDER EVEN EQUATION

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Abstract. In the present paper we investigate an initial-boundary value problem for a degenerate partial differential equation of high even order, in which the differential expression with respect to the spatial variable carries paired (linked) boundary data given simultaneously at both endpoints of the interval in a rectangular domain. Employing the Fourier method, the problem is reduced to an associated spectral problem, whose Green's function is constructed in closed form and shown to be symmetric by establishing the self-adjointness of the corresponding operator. On the basis of the eigenfunction expansion, the existence, uniqueness and continuous dependence of the solution on the given data are proved.

Keywords. degenerate equation of high even order; paired boundary conditions; Fourier method; spectral problem; Green's function; self-adjoint operator; eigenfunction expansion; existence and uniqueness; stability estimate.
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1. Introduction and preliminaries

The theory of higher-order partial differential equations occupies a central place in modern mathematical physics, owing both to the depth of the analytical questions it raises and to the breadth of its applications. Among the most natural sources of such equations is the mechanics of deformable solids: the mathematical description of the oscillations and the equilibria of rods, beams, plates and shells—objects whose stability is of paramount importance in engineering—leads inevitably to differential equations of high *even* order and to the boundary value problems associated with them [9, 16, 17]. The simplest and most familiar representative of this class is the equation of transverse vibrations of a homogeneous beam,

$$w_{tt} + a^2 w_{xxxx} = F(x,t),$$

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in which the constant a^2 is proportional to the flexural rigidity of the beam. In practice, however, the rigidity is rarely constant: a beam of variable thickness, a tapering wing, or a plate with a sharpened edge has a rigidity that *varies along the rod and may even vanish at one of its points* [9]. At such a point the equation loses its uniform ellipticity in the spatial variable and becomes *degenerate*: the coefficient of the leading spatial derivative collapses on a line of the domain, and the very notion of a well-posed boundary value problem has to be reconsidered. Degenerate equations of high even order are not confined to elasticity alone; they arise as well in the modelling of diffusion in strongly inhomogeneous media, in the theory of the boundary layer, in degenerate heat conduction and in climate models of Budyko–Sellers type, where the diffusivity vanishes on part of the boundary [5, 7]. It is precisely the interplay between the high order of the equation and the degeneration of its leading coefficient that makes the subject both analytically delicate and physically meaningful.

The well-posedness of boundary value problems for *non-degenerate* equations of high even order has by now been studied in considerable depth. In the rectangle $\{0 < x < p,\ 0 < t < T\}$, Amanov and Ashyralyev [1] examined two problems for the $2k$-th order equation $\partial_x^{2k} u + u_{tt} = f$: the first under the conditions

$$\frac{\partial^{2m} u(0,t)}{\partial x^{2m}} = \frac{\partial^{2m} u(p,t)}{\partial x^{2m}} = 0, \quad m = \overline{0, k-1}, \qquad u(x,0) = u(x,T) = 0,$$

and the second under the same spatial conditions together with $u(x,0) = 0$, $u_t(x,0) = 0$; remarkably, they discovered that the well-posedness of these problems is governed by the *parity of* k. Earlier, Amanov and Yuldasheva [2] had treated several problems of Dirichlet and periodic type for $\partial_x^{2k} u - u_{tt} = f$ under conditions of the form $\partial_x^{2m} u(0,t) = \partial_x^{2m} u(p,t) = 0$ $(m = \overline{0, k-1})$, $u(x,0) = u(x,T) = 0$, establishing both regular and strong solvability together with the spectral properties of the underlying operator. This circle of ideas was carried further by Yuldasheva [22], who analysed $\partial_x^{2k} u - \partial_t^{2p} u = f$ under the conditions

$$\frac{\partial^{2m} u(0,t)}{\partial x^{2m}} = \frac{\partial^{2m} u(a,t)}{\partial x^{2m}} = 0, \ m = \overline{0, k-1},$$

$$\frac{\partial^{2m} u(x,0)}{\partial t^{2m}} = \frac{\partial^{2m} u(x,T)}{\partial t^{2m}} = 0, \ m = \overline{0, p-1},$$

and uncovered an unexpected phenomenon: in the even case of $k-p$ the solvability of the problem is dictated by *number-theoretic* (Liouville-type) properties of the ratio of the sides of the rectangle. In all of these investigations the leading coefficient is constant, the boundary conditions are local, and the principal tools are the method of separation of variables, energy (a priori) estimates, and spectral analysis—a methodological framework that the present paper also adopts, but now in a genuinely degenerate setting. Along with direct problems for higher-order equations, interest in studying inverse problems has also been growing. In particular, [13] studied an inverse boundary value problem in a rectangle for the equation $u_{tt} + u_{xxxx} = a(t)u + b(t)u_t + c(t)g + f$, recovering the coefficients $a(t)$, $b(t)$, and $c(t)$ together with $u(x,t)$. By applying the Fourier method, the problem was reduced to a system of integral equations and solved using the contraction mapping principle.

The beam equation and its degenerate generalizations have attracted no less attention. The Cauchy problem for the beam vibration equation was solved by Sabitov [14], while its degenerate analogue

$$u_{tt} + t^p u_{xxxx} + \lambda^2 t^p u = 0,$$

in which the degeneration occurs in the *time* variable, was treated by Sitnik, Karimov and Tulasheva [15]. Questions of a different nature—the boundary controllability and the stability of degenerate beam equations—were addressed by Camasta and Fragnelli [3, 4], who showed how the location and the strength of the degeneration influence the very possibility of steering or stabilizing the system. For equations that degenerate in the *spatial* variable and are of high even order, an initial–boundary value problem was studied by Urinov and Azizov [18] for the equation with the Bessel operator

$$B_t^{\gamma-1/2} u + (-1)^k \frac{\partial^k}{\partial x^k}\left(x^\alpha \frac{\partial^k u}{\partial x^k}\right) = f(x,t), \qquad B_t^{\gamma-1/2} = \frac{\partial^2}{\partial t^2} + \frac{2\gamma}{t}\frac{\partial}{\partial t},$$

with $0 \le \alpha < 1$, $0 \le \gamma < 1/2$, under the initial conditions $u(x,0) = \varphi_1(x)$, $u_t(x,0) = \varphi_2(x)$; a closely related problem for

$$\frac{\partial^{2n}}{\partial x^{2n}}\left(x^\alpha \frac{\partial^{2n} u}{\partial x^{2n}}\right) + u_{tt} + \frac{2\gamma}{t} u_t + bu = f(x,t), \qquad 0 < \alpha < 1,$$

was considered by Urinov and Oripov [20]. It is essential to observe that in both of these works the spatial degeneration is *weak*: the exponent satisfies $0 < \alpha < 1$, so that the degeneration affects only the lowest-order behaviour of the solution near $x = 0$ and leaves the full set of classical boundary conditions admissible.

In parallel, local and nonlocal problems for degenerate equations carrying *fractional* derivatives have become the subject of intensive study. Urinov and Mamanazarov [19] investigated a mixed problem for the time-fractional, space-degenerate beam equation $\partial_t^\alpha u + \partial_{xx}(x^\beta \partial_{xx} u) = f$ under conditions of the form

$$\frac{\partial^\nu u(0,t)}{\partial x^\nu} = 0, \ \nu = \overline{0, 1-[\beta]}, \qquad u(1,t) = 0, \ u_x(1,t) = 0,$$
$$u(x,0) = \varphi(x), \ u_t(x,0) = \psi(x),$$

and—most significantly for what follows—they showed that the *degree of degeneration determines not only the admissible boundary conditions but even the very nature of the solution*, that is, whether it is classical or only weak. Mamanazarov, Khalilov and Kodiraliev [10] studied direct and inverse source problems for the second-order, time- and space-degenerate equation $t^b\, {}^C D_{0t}^\gamma u - (x^\alpha(1-x)^\beta u_x)_x = f$ under the *nonlocal* boundary conditions

$$pu(0,t) = qu(1,t), \qquad q \lim_{x\to 0}\left(x^\alpha(1-x)^\beta u_x\right) = p \lim_{x\to 1}\left(x^\alpha(1-x)^\beta u_x\right),$$

which link the two endpoints of the interval, together with the initial condition $u(x,0) = \varphi(x)$ for the direct problem and the over-determination condition $u(x,T) = \psi(x)$ for the inverse one; there too, the spectral problem was reduced to a Fredholm integral equation with a symmetric kernel built from an explicitly constructed Green's function. In the same spirit, boundary value problems with conjugation conditions for degenerate equations with a Caputo fractional derivative were treated by Irgashev [6], and nonlocal initial problems for time-fractional, space-singular equations by Karimov, Mamchuev and Ruzhansky [8]. In [21] an initial-boundary value problem was studied in a rectangle for the high even-order equation

$$t^{\gamma}\, {}^{C}D_{0t}^{\delta}u + bu + (-1)^n \partial_x^n \left[x^{\alpha}(1-x)^{\beta}\partial_x^n u\right] = 0$$

with three lines of degeneration. The uniqueness of the solution was established by the method of energy identities, and the solution was constructed by the Fourier method using the Green's function of the corresponding spectral problem. However, the degeneration considered there is again weak ($0 \le \alpha < 1$, $0 \le \beta < 1$), and all the boundary conditions are local. Two features recur throughout this body of work and deserve emphasis: *nonlocal* (paired) conditions, which couple the two ends of the interval, have so far been analysed only for equations that are of the *second* order in the spatial variable; and the spatial degeneration, when present in a high even-order equation, has been kept *weak* ($0 < \alpha < 1$).

This is exactly the point at which the present work departs from its predecessors. To the best of our knowledge, an initial–boundary value problem *combining local and nonlocal conditions* for a *strongly* degenerate high even-order equation has not previously been considered. The equation we study,

$$\frac{\partial^2 u}{\partial t^2} + (-1)^n \frac{\partial^n}{\partial x^n}\left(x^{\alpha}\frac{\partial^n u}{\partial x^n}\right) = f(x,t), \tag{1.1}$$

carries a second-order time derivative together with the high even-order degenerate spatial operator $(-1)^n\partial_x^n(x^{\alpha}\partial_x^n u)$ of order $2n$, in which $u = u(x,t)$ is the unknown function, $f(x,t)$ is a given function, and the real number α satisfies

$$0 < \alpha < n, \qquad n \in \mathbb{N}, \qquad n \ge 2 \qquad \alpha = \alpha_0 + \varepsilon, \quad \alpha_0 = [\alpha], \quad 0 < \varepsilon < 1,$$

with $[\alpha]$ the integer part of α. The line $x = 0$ is the line of degeneration and α is its degree. In sharp contrast to the weakly degenerate case $0 < \alpha < 1$ treated in [18, 20], the admissible range $0 < \alpha < n$ allows the degree of degeneration to reach *half the order* $2n$ of the equation, so that (1.1) may be *strongly* degenerate. The local problem for (1.1) was considered in another work of ours, while the nonlocal conditions connecting the endpoints of the interval have so far been analyzed only for second-order (in a spatial variable) degenerate equations [10]. The principal aim and novelty of the present paper is the formulation and the analysis of a initial–boundary value problem, combining local and nonlocal conditions, for the strongly degenerate equation (1.1).

The decisive new feature, and the one that distinguishes the strongly degenerate regime from the weakly degenerate one, is the role played by the integer part $\alpha_0 = [\alpha]$. In the weak case $\alpha_0 = 0$, the factor x^{α} vanishes so mildly at $x = 0$ that the full set of classical boundary conditions may be imposed at both endpoints, and one is free to prescribe paired (nonlocal) data on all of them. As α increases and α_0 grows, this freedom is progressively lost: for the higher-order traces $j = \overline{n-\alpha_0, n-1}$ the degeneration is too severe for a Dirichlet-type relation to be meaningful at $x = 0$, and the only condition that survives there is the requirement that the trace remain *bounded*, $\left|\lim_{x\to 0}\partial_x^j u(x,t)\right| < \infty$. Thus α_0 acts as a precise counter: it is exactly the number of derivative orders (among $j=\overline{0,n-1}$) at the degenerate end $x=0$ at which a prescribed boundary value must be replaced by a boundedness condition; equivalently, it is the number of orders at which the flux $x^\alpha\partial_x^n u$ is forced to vanish at $x=0$, namely $\left(x^\alpha\partial_x^n u\right)^{(j)}\big|_{x=0} = 0$ for $j=\overline{0,\alpha_0-1}$. In this way the value of α_0 *shapes the very statement of the problem*: it partitions the n traces at the degenerate end into a lower block, on which paired (nonlocal) conditions coupling $x=0$ and $x=1$ are imposed, and an upper block, on which boundedness at $x=0$ and a local Dirichlet condition at $x=1$ take their place; symmetrically, it fixes how many of the flux traces are paired and how many must vanish. The same parameter governs the construction of the Green's function—it determines which of the coefficients in its representation are identically zero and the form of the recurrence relations that yield the rest—and it is precisely the boundedness conditions counted by α_0 that guarantee the convergence of the energy integral $\int_0^1 x^{-\alpha}\left[x^\alpha v^{(n)}(x)\right]^2 dx = \int_0^1 x^{\alpha_0-\varepsilon} f^2(x)\, dx$ on which the analysis of the spectral problem rests. In short, α_0 is the structural switch of the problem: a change of its value changes the number and the type of the admissible boundary conditions, the architecture of the Green's function, and the functional setting in which the solution is sought.

We now formulate the problem.

Problem N. *Find a function $u(x,t)$ with the following properties:*

(1) $u_t,\ \dfrac{\partial^j u}{\partial x^j},\ \dfrac{\partial^j}{\partial x^j}\left[x^\alpha \dfrac{\partial^n u}{\partial x^n}\right] \in C(\bar{\Omega}), \quad j=\overline{0,n-1};$
$\dfrac{\partial^n}{\partial x^n}\left[x^\alpha \dfrac{\partial^n u}{\partial x^n}\right],\ u_{tt} \in C(\Omega);$

(2) It satisfies equation (1.1) in the domain Ω;

(3) It satisfies the initial conditions

$$u(x,0)=\varphi_1(x), \quad u_t(x,0)=\varphi_2(x), \quad x\in[0,1]; \tag{1.2}$$

(4) It satisfies the boundary conditions

$$\begin{cases} \dfrac{\partial^j u(0,t)}{\partial x^j} + \dfrac{\partial^j u(1,t)}{\partial x^j} = 0, \quad j=\overline{0,n-\alpha_0-1}, \\ \left|\lim\limits_{x\to 0}\dfrac{\partial^j u(x,t)}{\partial x^j}\right| < \infty, \quad j=\overline{n-\alpha_0,n-1}; \\ \dfrac{\partial^j u(1,t)}{\partial x^j} = 0, \quad j=\overline{n-\alpha_0,n-1}, \\ \dfrac{\partial^j}{\partial x^j}\left(x^\alpha\dfrac{\partial^n u}{\partial x^n}\right)\Big|_{x=0} + \dfrac{\partial^j}{\partial x^j}\left(x^\alpha\dfrac{\partial^n u}{\partial x^n}\right)\Big|_{x=1} = 0, \quad j=\overline{\alpha_0,n-1}, \end{cases} \tag{1.3}$$

for $t\in[0,T]$, where $\varphi_1(x)$ and $\varphi_2(x)$ are given sufficiently smooth functions.

The first and third groups of relations in (1.3) are nonlocal, since they link the values of the corresponding quantities at $x=0$ and $x=1$, whereas the boundedness conditions near $x=0$ and the relations $\partial_x^j u(1,t)=0$ are local; the boundedness conditions reflect the strong degeneration of the equation, replacing ordinary Dirichlet-type conditions at the degeneration line $x=0$ for those orders j for which the latter cannot be imposed.

The paper is organized as follows. In Section 2 the spectral problem generated by Problem N is investigated: the Green's function is constructed, the self-adjointness of the operator is established, the problem is reduced to a Fredholm integral equation, and a number of auxiliary lemmas concerning the order of the Fourier coefficients are proved. In Section 3 the existence, uniqueness, and continuous dependence on the data of the solution of Problem N are established. Section 4 contains the concluding remarks.

2. Investigation of the spectral problem

A formal application of the Fourier method to the problem gives the spectral problem

$$M[v] \equiv (-1)^n \left(x^\alpha v^{(n)}(x)\right)^{(n)} = \lambda v(x), \quad 0<x<1. \tag{2.1}$$

This spectral problem is treated in the Hilbert space $H=L_2(0,1)$ equipped with the usual inner product $(u,v)=\int_0^1 u(x)v(x)\,dx$. We take the domain $D(M)$ of the operator M to consist of those functions that comply with the following requirements:

$$\begin{cases} v^{(j)}(x),\ \left(x^\alpha v^{(n)}(x)\right)^{(j)} \in C[0,1], \quad j=\overline{0,n-1};\\ v^{(j)}(0)+v^{(j)}(1)=0, \quad j=\overline{0,n-\alpha_0-1};\\ \left|\lim\limits_{x\to 0} v^{(j)}(x)\right|<\infty, \quad j=\overline{n-\alpha_0,n-1};\\ v^{(j)}(1)=0, \quad j=\overline{n-\alpha_0,n-1};\\ \left(x^\alpha v^{(n)}\right)^{(j)}\big|_{x=0} + \left(x^\alpha v^{(n)}\right)^{(j)}\big|_{x=1}=0, \quad j=\overline{\alpha_0,n-1}. \end{cases} \tag{2.2}$$

The boundedness requirement in (2.2) is equivalent to

$$\left(x^\alpha v^{(n)}\right)^{(j)}\big|_{x=0}=0, \quad j=\overline{0,\alpha_0-1}. \tag{2.3}$$

We are going to establish the existence of eigenvalues and eigenfunctions of problem $\{$(2.1), (2.2)$\}$. Suppose that such eigenfunctions and eigenvalues do exist. Under this hypothesis, multiplying equation (2.1) by $v(x)$, integrating the resulting identity over the segment $[0,1]$ and then carrying out integration by parts while taking the relations (2.2) into account, we are led to

$$\lambda \int_0^1 v^2(x)\,dx = \int_0^1 x^{-\alpha}\left[x^\alpha v^{(n)}(x)\right]^2\,dx. \tag{2.4}$$

The existence of the integral $\int_0^1 x^{-\alpha}\left[x^\alpha v^{(n)}(x)\right]^2 dx$ on the right-hand side of (2.4) is justified with the help of Taylor's formula with the remainder term written in

Lagrange's form. Combining Taylor's formula with conditions (2.3) yields the relation $x^{\alpha}v^{(n)}(x) = x^{\alpha_0}f(x)$, whence it follows that

$$\int_0^1 x^{-\alpha}\left[x^{\alpha}v^{(n)}(x)\right]^2 dx = \int_0^1 x^{\alpha_0-\varepsilon}f^2(x)\,dx,$$

and the last equality makes the convergence of the integral apparent. From (2.4) we infer that $\lambda \geq 0$.

Let us assume $\lambda = 0$. Then equation (2.4) forces $v^{(n)}(x) = 0$ for $0 < x < 1$. In view of the conditions $v^{(j)}(0) = 0$, $j = \overline{0, n-\alpha_0-1}$, and $v^{(j)}(1) = 0$, $j = \overline{n-\alpha_0, n-1}$, we then conclude that $v(x) \equiv 0$ on $0 \leqslant x \leqslant 1$. Accordingly, problem $\{(2.1), (2.2)\}$ may possess non-trivial solutions only when $\lambda > 0$.

Assuming henceforth that $\lambda > 0$, we shall demonstrate the existence of the eigenvalues of problem $\{(2.1), (2.2)\}$ by the Green's function technique. The Green's function $G(x,s)$ of this problem must have the properties listed below ([12], §3, p. 38):

(1) the functions $(\partial^j/\partial x^j)\,G(x,s)$, $j = \overline{0, n-1}$, and $(\partial^j/\partial x^j)\left[x^{\alpha}\left(\partial^n/\partial x^n\right)G(x,s)\right]$, $j = \overline{0, n-2}$, are continuous for all $x, s \in [0,1]$;

(2) on each of the intervals $[0,s)$ and $(s,1]$ the function $(\partial^{n-1}/\partial x^{n-1})\left[x^{\alpha}\left(\partial^n/\partial x^n\right)G(x,s)\right]$ admits a continuous derivative for $x \neq s$, and has a jump at $x = s$:

$$\left(\partial^{n-1}/\partial x^{n-1}\right)\left[x^{\alpha}\left(\partial^n/\partial x^n\right)G(x,s)\right]\Big|_{x=s-0}^{x=s+0} = (-1)^n; \tag{2.5}$$

(3) on each of the intervals $(0,s)$ and $(s,1)$ the function $MG(x,s)$ possesses a continuous derivative with respect to x, and $MG(x,s) = 0$;

(4) for $s \in (0,1)$ the function $G(x,s)$, regarded as a function of x, satisfies the boundary conditions

$$\begin{cases} \dfrac{\partial^j G(0,s)}{\partial x^j} + \dfrac{\partial^j G(1,s)}{\partial x^j} = 0, \quad j = \overline{0, n-\alpha_0-1}; \\ \left|\lim\limits_{x\to 0} G^{(j)}(x,s)\right| < \infty, \quad j = \overline{n-\alpha_0, n-1}; \\ G^{(j)}(1,s) = 0, \quad j = \overline{n-\alpha_0, n-1}; \\ \left(x^{\alpha}G^{(n)}(x,s)\right)^{(j)}\Big|_{x=0} + \left(x^{\alpha}G^{(n)}(x,s)\right)^{(j)}\Big|_{x=1} = 0, \quad j = \overline{\alpha_0, n-1}. \end{cases} \tag{2.6}$$

We now proceed to the construction of the Green's function of the problem. By the general solution of the equation $MG(x,s) = 0$, we look for $G(x,s)$, on the intervals $(0,s)$ and $(s,1)$, in the form

$$G(x,s) = \begin{cases} \displaystyle\sum_{j=1}^{n} \frac{a_j\,x^{2n-\alpha-j}}{(n-j)!\,(n-\alpha+1-j)_n} + \sum_{j=1}^{n} \frac{a_{n+j}\,x^{n-j}}{(n-j)!}, \quad 0 \leqslant x \leqslant s, \\ \displaystyle\sum_{j=1}^{n} \frac{b_j\,x^{2n-\alpha-j}}{(n-j)!\,(n-\alpha+1-j)_n} + \sum_{j=1}^{n} \frac{b_{n+j}\,x^{n-j}}{(n-j)!}, \quad s \leqslant x \leqslant 1, \end{cases} \tag{2.7}$$

where a_j and b_j, $j = \overline{1, 2n}$, are unknown functions of the variable s, while $(z)_n = z(z+1)(z+2)\ldots(z+n-1)$ denotes the Pochhammer symbol. Requiring the function (2.7) to fulfil properties 1) and 2) of the Green's function, we arrive at the following system of equations for the quantities $b_j - a_j$, $j = \overline{1, 2n}$:

$$\begin{cases} b_1 - a_1 = (-1)^n, \\ \displaystyle\sum_{j=1}^{m_1} \frac{s^{m_1-j}}{(m_1-j)!}(b_j - a_j) = 0, \quad m_1 = \overline{2, n}, \\ \displaystyle\sum_{j=1}^{n} \frac{s^{n-\alpha+m_2-j}(b_j - a_j)}{(n-j)!\,(n-\alpha-j+1)_{m_2}} + \sum_{j=1}^{m_2} \frac{s^{m_2-j}(b_{n+j} - a_{n+j})}{(m_2-j)!} = 0, \quad m_2 = \overline{1, n}. \end{cases}$$

This system has a unique solution [20]:

$$b_j - a_j = \frac{(-1)^{n+j-1} s^{j-1}}{(j-1)!}, \quad b_{n+j} - a_{n+j} = \frac{(-1)^{j-1} s^{n-1-\alpha+j}}{(j-1)!\,(j-\alpha)_n}, \quad j = \overline{1, n}. \tag{2.8}$$

Subjecting (2.7) to the second group of conditions in (2.6), we obtain

$$a_j = 0, \qquad j = \overline{n - \alpha_0 + 1, n}. \tag{2.9}$$

By means of these equalities, (2.8) implies

$$b_j = \frac{(-1)^{n+j-1} s^{j-1}}{(j-1)!}, \qquad j = \overline{n - \alpha_0 + 1, n}. \tag{2.10}$$

Next, imposing the paired part of the conditions, we find, for $j = 1, \dots, n - \alpha_0$,

$$R_j = a_j + b_j = -\sum_{m=1}^{j-1} \frac{b_m}{(j-m)!}, \tag{2.11}$$

and consequently

$$a_j = \frac{R_j}{2} - \frac{(-1)^{n+j-1} s^{j-1}}{2(j-1)!}, \qquad b_j = \frac{R_j}{2} + \frac{(-1)^{n+j-1} s^{j-1}}{2(j-1)!}. \tag{2.12}$$

For $j = 1$ we have $R_1 = 0$, so that $a_1 = \dfrac{(-1)^{n+1}}{2}$ and $b_1 = \dfrac{(-1)^n}{2}$. Turning to the third line of the conditions (2.6), we obtain the relation

$$b_{n+m} = -\sum_{j=1}^{n} \frac{b_j}{(n-j)!\,(n-\alpha+1-j)_m} - \sum_{j=1}^{m-1} \frac{b_{n+j}}{(m-j)!}, \qquad m = 1, \dots, \alpha_0, \tag{2.13}$$

which in turn gives, for the coefficients a_{n+m},

$$a_{n+m} = b_{n+m} - \frac{(-1)^{m-1} s^{n-1-\alpha+m}}{(m-1)!\,(m-\alpha)_n}, \qquad m = 1, \dots, \alpha_0. \tag{2.14}$$

Finally, from the first line of the conditions (2.6), for $k = n - \alpha_0 - 1, \dots, 0$, we have

$$R_{2n-k} = a_{2n-k} + b_{2n-k} = \\ -\sum_{j=1}^{n} \frac{b_j}{(n-j)!\,(n-\alpha+1-j)_{n-k}} - \sum_{j=1}^{n-k-1} \frac{b_{n+j}}{(n-j-k)!}, \tag{2.15}$$

whence the values of a_{2n-k} and b_{2n-k} are recovered as

$$\begin{aligned} a_{2n-k} &= \frac{R_{2n-k}}{2} - \frac{(-1)^{n-k-1} s^{2n-1-\alpha-k}}{2(n-k-1)!\,(n-k-\alpha)_n}, \\ b_{2n-k} &= \frac{R_{2n-k}}{2} + \frac{(-1)^{n-k-1} s^{2n-1-\alpha-k}}{2(n-k-1)!\,(n-k-\alpha)_n}. \end{aligned} \tag{2.16}$$

Substituting the quantities found in (2.9)–(2.16) into (2.7), the Green's function takes the form

$$G(x,s) = \begin{cases} \displaystyle\sum_{j=1}^{n-\alpha_0} \frac{a_j\, x^{2n-\alpha-j}}{(n-j)!\,(n-\alpha+1-j)_n} + \sum_{j=1}^{\alpha_0} \frac{a_{n+j}\, x^{n-j}}{(n-j)!} + \sum_{j=\alpha_0+1}^{n} \frac{a_{n+j}\, x^{n-j}}{(n-j)!}, \\ 0 \le x \le s; \\ \displaystyle\sum_{j=1}^{n-\alpha_0} \frac{b_j\, x^{2n-\alpha-j}}{(n-j)!\,(n-\alpha+1-j)_n} + \sum_{j=n-\alpha_0+1}^{n} \frac{b_j\, x^{2n-\alpha-j}}{(n-j)!\,(n-\alpha+1-j)_n} \\ \qquad \displaystyle + \sum_{j=1}^{\alpha_0} \frac{b_{n+j}\, x^{n-j}}{(n-j)!} + \sum_{j=\alpha_0+1}^{n} \frac{b_{n+j}\, x^{n-j}}{(n-j)!}, \quad s \le x \le 1. \end{cases} \tag{2.17}$$

In the particular case $n=2$, $\alpha_0=1$, formula (2.17) yields

$$G(x,s) = \begin{cases} \displaystyle -\frac{x^{3-\alpha}}{2(2-\alpha)_2} - \frac{x}{2(2-\alpha)} + \frac{sx}{1-\alpha} \\ \displaystyle -\frac{s^{2-\alpha}x}{(1-\alpha)_2} + \frac{s^{3-\alpha}}{2(2-\alpha)_2} - \frac{s}{2(2-\alpha)} + \frac{1}{4(3-\alpha)}, \quad x < s; \\ \displaystyle \frac{x^{3-\alpha}}{2(2-\alpha)_2} - \frac{sx^{2-\alpha}}{(1-\alpha)_2} - \frac{x}{2(2-\alpha)} \\ \displaystyle + \frac{sx}{1-\alpha} - \frac{s^{3-\alpha}}{2(2-\alpha)_2} - \frac{s}{2(2-\alpha)} + \frac{1}{4(3-\alpha)}, \quad x > s. \end{cases}$$

Consequently, the Green's function satisfying conditions 1)–4) exists and is unique

We shall now prove that it is symmetric in its arguments. Since establishing this fact directly from the representation (2.17) is awkward, we instead verify that the operator M is self-adjoint; the symmetry of the Green's function in its arguments will then follow from the general theory. Recall that M is symmetric provided that

$$(Mv, w) = (v, Mw), \qquad \forall\, v, w \in D(M).$$

Endowing $L^2(0,1)$ with the inner product $(v,w) = \int_0^1 v(x)w(x)\,dx$, we have

$$(Mv, w) = (-1)^n \int_0^1 \left(x^\alpha v^{(n)}(x)\right)^{(n)} w(x)\,dx.$$

Performing integration by parts repeatedly, we obtain

$$(Mv, w) = \sum_{k=0}^{n-1} (-1)^k \left[w^{(k)}(x) \left(x^\alpha v^{(n)}(x)\right)^{(n-1-k)} \right]\Big|_0^1 + \int_0^1 \left(x^\alpha w^{(n)}(x)\right)^{(n)} v(x)\,dx.$$

With the boundary conditions (2.2) taken into account, the vanishing of the boundary terms above requires every function $w \in D(M)$ to obey boundary conditions of exactly the same type as v, namely

$$\left.\begin{array}{l} \left(x^{\alpha} w^{(n)}(x)\right)^{(j)} \in C[0,1], \quad j=0,1,\ldots,n-1, \\ w^{(j)}(0)+w^{(j)}(1)=0, \quad j=0,1,\ldots,n-\alpha_0-1, \\ \left|\lim\limits_{x\to 0} w^{(j)}(x)\right|<\infty, \quad j=n-\alpha_0,\ldots,n-1, \\ w^{(j)}(1)=0, \quad j=n-\alpha_0,\ldots,n-1, \\ \left(x^{\alpha} w^{(n)}\right)^{(j)}\big|_{x=0}+\left(x^{\alpha} w^{(n)}\right)^{(j)}\big|_{x=1}=0, \quad j=\alpha_0,\ldots,n-1. \end{array}\right\}$$

Hence the operator M is self-adjoint in $L_2(0,1)$. Thus the spectral problem $\{(2.1), (2.2)\}$ is self-adjoint, and therefore its Green's function is symmetric in its arguments.

By the theorem on the equivalence between boundary value problems and integral equations (see [12], §3, pp.40-42), the boundary value problem $\{(2.1), (2.2)\}$ is equivalent to the following homogeneous Fredholm integral equation of the second kind

$$v(x)=\lambda \int_0^1 G(x,s) v(s)\, ds. \tag{2.18}$$

This equivalence rests on the fact that $G(x,s)$ is the Green's function of the differential operator M under the prescribed boundary conditions, which permits the differential operator to be realised as an inverse integral operator. Since the kernel $G(x,s)$ is continuous, symmetric and positive, the theory of symmetric integral equations (see [11], p. 140, Theorem 3) guarantees that the problem possesses a countable set of eigenvalues

$$0<\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \ldots \leqslant \lambda_k<\ldots, \quad \lambda_k \to +\infty,$$

whose associated system of eigenfunctions $v_1(x), v_2(x), v_3(x), \ldots, v_k(x), \ldots$ constitutes an orthonormal system in $L_2(0,1)$ [11]. Moreover, it is readily checked that the system of functions $s^{\alpha/2} v_k^{(n)}(s)/\sqrt{\lambda_k}$, $k=1,2,\ldots$, is likewise orthonormal in $L_2(0,1)$.

The lemma stated below indicates the class of functions that may be expanded in a series with respect to the eigenfunctions of problem $\{(2.1), (2.2)\}$.

Lemma 2.1. *Let the function $g(x)$ satisfy the conditions (2.2) and let $Mg(x) \in C(0,1) \cap L_2(0,1)$. Then $g(x)$ admits, on the segment $[0,1]$, an expansion into an absolutely and uniformly convergent series with respect to the system of eigenfunctions of problem $\{(2.1), (2.2)\}$.*

Proof. We begin by verifying that, under the hypotheses of Lemma 2.1, the following identity is valid:

$$\int_0^1 G(x,s)\, Mg(s)\, ds=(-1)^n \int_0^1 G(x,s)\left[s^{\alpha} g^{(n)}(s)\right]^{(n)} ds=g(x).$$

Indeed, having regard to the properties of the functions $G(x,s)$ and $g(x)$, we obtain

$$(-1)^n \int_0^1 G(x,s)\left[s^\alpha g^{(n)}(s)\right]^{(n)} ds = \left(G(x,s)\left[s^\alpha g^{(n)}(s)\right]^{(n-1)}\right)\Big|_0^1$$
$$-\left(G'_s(x,s)\left[s^\alpha g^{(n)}(s)\right]^{(n-2)}\right)\Big|_0^1 + \left(G''_s(x,s)\left[s^\alpha g^{(n)}(s)\right]^{(n-3)}\right)\Big|_0^1 - \dots$$
$$+(-1)^n\left(g(s)\left(s^\alpha G_s^{(n)}(x,s)\right)^{(n-1)}\right)\Big|_{s=0}^{s=1} + \int_0^1 g(s)\left(s^\alpha G_s^{(n)}(x,s)\right)^{(n)} ds.$$

By the properties of the Green's function this becomes

$$(-1)^n \int_0^1 G(x,s)\left[s^\alpha g^{(n)}(s)\right]^{(n)} ds = (-1)^n\left[\left(g(s)\left(s^\alpha G_s^{(n)}(x,s)\right)^{(n-1)}\right)\Big|_{s=0}^{s=x-0}\right] -$$
$$-(-1)^n\left[\left(g(s)\left(s^\alpha G_s^{(n)}(x,s)\right)^{(n-1)}\right)\Big|_{s=x+0}^{s=1}\right] + \int_0^1 g(s)\left(s^\alpha G_s^{(n)}(x,s)\right)^{(n)} ds = g(x),$$

since, by the second property of the Green's function,

$$\left(\left(s^\alpha G_s^{(n)}(x,s)\right)^{(n-1)}\right)\Big|_{s=x-0}^{s=x+0} = (-1)^n.$$

Therefore $g(x)$ is representable through the kernel $G(x,s)$. Furthermore, one verifies without difficulty the inequality

$$\int_0^1 G^2(x,s)\,ds \le C_3 = \text{const} < +\infty.$$

Hence, by the Hilbert–Schmidt theorem [11], the function $g(x)$ is expanded on $[0,1]$ into a uniformly convergent Fourier series with respect to the system of eigenfunctions $\{v_k(x)\}_{k=1}^{+\infty}$. Lemma 2.1 is proved. ☐

We now turn to the convergence of certain series built from the eigenvalues and eigenfunctions of the spectral problem $\{(2.1), (2.2)\}$.

Lemma 2.2. *The following series converge uniformly on the segment* $[0,1]$*:*

$$\sum_{k=1}^{+\infty}\frac{\left[v_k^{(j)}(x)\right]^2}{\lambda_k}, \qquad \sum_{k=1}^{+\infty}\frac{\left(\left[x^\alpha v_k^{(n)}(x)\right]^{(j)}\right)^2}{\lambda_k^2}, \qquad j=\overline{0,n-1}. \tag{2.19}$$

Proof. Using the equation (2.1) and the properties of the function $G(x,s)$, putting $v(x)\equiv v_k(x)$ in (2.18), we get for $j=\overline{0,n-1}$

$$v_k^{(j)}(x) = \lambda_k\int_0^1 \frac{\partial^j}{\partial x^j}G(x,s)v_k(s)\,ds = (-1)^n\int_0^1\left[s^\alpha v_k^{(n)}(s)\right]^{(n)}\frac{\partial^j}{\partial x^j}G(x,s)\,ds.$$

Applying integration by parts n times and then invoking the conditions (2.2), we find

$$v_k^{(j)}(x) = \int_0^1 s^\alpha v_k^{(n)}(s)\frac{\partial^{n+j}}{\partial x^j\partial s^n}G(x,s)\,ds, \quad j=\overline{0,n-1}.$$

Since $\lambda_k > 0$, this leads to

$$\frac{v_k^{(j)}(x)}{\sqrt{\lambda_k}} = \int_0^1 \left(s^{\alpha/2} \frac{\partial^{n+j}}{\partial x^j \partial s^n} G(x,s) \right) \left(\frac{s^{\alpha/2} v_k^{(n)}(s)}{\sqrt{\lambda_k}} \right) ds, \quad j = \overline{0, n-1}. \tag{2.20}$$

It follows from (2.20) that $v_k^{(j)}(x)/\sqrt{\lambda_k}$ is the Fourier coefficient of the function $s^{\alpha/2} \dfrac{\partial^{n+j}}{\partial x^j \partial s^n} G(x,s)$ relative to the orthonormal system $\left\{ s^{\alpha/2} v_k^{(n)}(s)/\sqrt{\lambda_k} \right\}_{k=1}^{+\infty}$. Hence, by Bessel's inequality [11],

$$\begin{aligned} \sum_{k=1}^{+\infty} \frac{\left[v_k^{(j)}(x) \right]^2}{\lambda_k} &\leqslant \int_0^1 s^{\alpha} \left[\frac{\partial^{n+j}}{\partial x^j \partial s^n} G(x,s) \right]^2 ds \\ &\leqslant \int_0^1 s^{-\alpha} \left[\frac{\partial^j}{\partial x^j} \left(s^{\alpha} \frac{\partial^n}{\partial s^n} G(x,s) \right) \right]^2 ds, \quad j = \overline{0, n-1}. \end{aligned} \tag{2.21}$$

Since $s^{\alpha} \dfrac{\partial^n G(x,s)}{\partial s^n} = s^{\alpha_0} O(1)$, the integral on the right-hand side of (2.21) is uniformly bounded for $j = \overline{0, n-1}$, which shows that the first series in (2.19) converges uniformly. The convergence of the remaining series is established in the same manner. Lemma 2.2 is proved. □

Lemma 2.3. *Suppose that $g^{(j)}(x) \in C[0,1]$, $j = \overline{0, n-1}$, $x^{\alpha/2} g^{(n)}(x) \in C(0,1) \cap L_2(0,1)$, $g^{(j)}(0) = 0$, $j = \overline{0, n-\alpha_0-1}$, $g^{(j)}(1) = 0$, $j = \overline{n-\alpha_0, n-1}$. Then*

$$\sum_{k=1}^{+\infty} \lambda_k g_k^2 \leqslant \int_0^1 x^{\alpha} \left[g^{(n)}(x) \right]^2 dx, \tag{2.22}$$

where $g_k = \int_0^1 g(x) v_k(x)\, dx$, $k \in \mathbb{N}$; in particular, the series on the left hand side of (2.22) converges.

Proof. Using equation (2.1), we have

$$\lambda_k^{1/2} g_k = \lambda_k^{1/2} \int_0^1 g(x) v_k(x)\, dx = (-1)^n \lambda_k^{-1/2} \int_0^1 g(x) \left[x^{\alpha} v_k^{(n)}(x) \right]^{(n)} dx.$$

Applying integration by parts n times and using the properties of $g(x)$ and $v_k(x)$, we obtain

$$\lambda_k^{1/2} g_k = \int_0^1 \left\{ x^{\alpha/2} g^{(n)}(x) \right\} \left\{ \lambda_k^{-1/2} x^{\alpha/2} v_k^{(n)}(x) \right\} dx.$$

Thus $\lambda_k^{1/2} g_k$ is the Fourier coefficient of $x^{\alpha/2} g^{(n)}(x)$ with respect to the orthonormal system $\left\{ x^{\alpha/2} v_k^{(n)}(x)/\sqrt{\lambda_k} \right\}_{k=1}^{+\infty}$. Bessel's inequality [11] now yields (2.22). Lemma 2.3 is proved. □

Lemma 2.4. *Let the function $g(x)$ satisfy condition (2.2) and let $Mg(x) \in C(0,1) \cap L_2(0,1)$. Then*

$$\sum_{k=1}^{+\infty} \lambda_k^2 g_k^2 \leqslant \int_0^1 [Mg(x)]^2\, dx. \tag{2.23}$$

Proof. Combining the formula for g_k with equation (2.1), we obtain

$$\lambda_k g_k = \lambda_k \int_0^1 g(x) v_k(x)\, dx = (-1)^n \int_0^1 g(x) \left[x^\alpha v_k^{(n)}(x) \right]^{(n)} dx.$$

Integrating by parts $2n$ times on the right-hand side and allowing for the properties of $g(x)$ and $v_k(x)$, we find

$$\lambda_k g_k = (-1)^n \int_0^1 \left[x^\alpha g^{(n)}(x) \right]^{(n)} v_k(x)\, dx = \int_0^1 [Mg(x)] v_k(x)\, dx.$$

Hence $\lambda_k g_k$ is the Fourier coefficient of $Mg(x)$ relative to the orthonormal system $\{v_k(x)\}_{k=1}^{+\infty}$, and Bessel's inequality [11] gives (2.23). The proof of Lemma 2.4 is complete. ☐

Similarly, the following lemma can be proved.

Lemma 2.5. *Let the function $g(x)$ satisfy conditions (2.2) and let $[Mg(x)]^{(j)} \in C[0,1]$, $j = \overline{0, n-1}$, $x^{\alpha/2}[Mg(x)]^{(n)} \in C(0,1) \cap L_2(0,1)$, $[Mg(x)]^{(j)}\big|_{x=0} = 0$, $j = \overline{0, n-\alpha_0-1}$, $[Mg(x)]^{(j)}\big|_{x=1} = 0$, $j = \overline{n-\alpha_0, n-1}$. Then*

$$\sum_{k=1}^{+\infty} \lambda_k^3 g_k^2 \leqslant \int_0^1 x^\alpha \left\{ [Mg(x)]^{(n)} \right\}^2 dx. \tag{2.24}$$

3. Existence, uniqueness and continuous dependence of the solution on the given data

In this section we prove the existence and uniqueness of the solution of the Problem N, together with its continuous dependence on the given data. The solution will be sought in the form

$$u(x,t) = \sum_{k=1}^{+\infty} u_k(t) v_k(x), \tag{3.1}$$

where $v_k(x)$, $k \in \mathbb{N}$, are the eigenfunctions of problem (2.1), (2.2)}, and $u_k(t)$, $k \in \mathbb{N}$, are functions yet to be determined. Substituting (3.1) into equation (1.1) and into the initial conditions (1.2), we have the following problem for $u_k(t)$:

$$u_k''(t) + \lambda_k u_k(t) = f_k(t), \quad t \in (0,T), \quad k \in \mathbb{N},$$
$$u_k(0) = \varphi_{1k}, \quad u_k'(0) = \varphi_{2k},$$

where

$$\varphi_{jk} = \int_0^1 \varphi_j(x) v_k(x)\, dx,\ j = \overline{1,2}; \qquad f_k(t) = \int_0^1 f(x,t) v_k(x)\, dx, \quad k \in \mathbb{N}.$$

As is well known, this problem has a unique solution, given by

$$\begin{aligned} u_k(t) &= \varphi_{1k} \cos\left(t\sqrt{\lambda_k}\right) + \varphi_{2k} \lambda_k^{-1/2} \sin\left(t\sqrt{\lambda_k}\right) \\ &\quad + \lambda_k^{-1/2} \int_0^t f_k(\tau) \sin\left[(t-\tau)\sqrt{\lambda_k}\right] d\tau, \quad 0 \leqslant t \leqslant T. \end{aligned} \tag{3.2}$$

From this the estimate

$$|u_k(t)| \leqslant |\varphi_{1k}| + \frac{1}{\sqrt{\lambda_k}}\,|\varphi_{2k}| + \frac{1}{\sqrt{\lambda_k}}\sqrt{\int_0^T f_k^2(\tau)\,d\tau}, \quad 0 \leqslant t \leqslant T, \tag{3.3}$$

is readily derived.

Theorem 3.1. *Suppose that $\varphi_1(x)$ satisfies the hypotheses of Lemma 2.5, $\varphi_2(x)$ satisfies those of Lemma 2.4, and $f(x,t)$ meets the hypotheses of Lemma 2.4 with respect to x uniformly in t. Then the series (3.1), with coefficients given by (3.2), furnishes the solution of the Problem N.*

Proof. It is required to prove the uniform convergence on $\overline{\Omega}$ of the series (3.1) and of the series formally obtained from it,

$$u_t(x,t) = \sum_{k=1}^{+\infty} u_k'(t)v_k(x), \quad \frac{\partial^j u(x,t)}{\partial x^j} = \sum_{k=1}^{+\infty} u_k(t)v_k^{(j)}(x), \quad j = \overline{1, n-1},$$

$$\frac{\partial^j}{\partial x^j}\left(x^\alpha \frac{\partial^n u(x,t)}{\partial x^n}\right) = \sum_{k=1}^{+\infty} u_k(t)\left(x^\alpha v_k^{(n)}(x)\right)^{(j)}, \quad j = \overline{0, n-1},$$

as well as the uniform convergence, on any compact set $\Omega_0 \subset \Omega$, of the series

$$\frac{\partial^n}{\partial x^n}\left(x^\alpha \frac{\partial^n u(x,t)}{\partial x^n}\right) = \sum_{k=1}^{+\infty} u_k(t)\left(x^\alpha v_k^{(n)}(x)\right)^{(n)}, \tag{3.4}$$

$$u_{tt}(x,t) = \sum_{k=1}^{+\infty} u_k''(t)v_k(x). \tag{3.5}$$

Consider first the series (3.1). By virtue of (3.3), for any $(x,t) \in \overline{\Omega}$,

$$|u(x,t)| \leqslant \sum_{k=1}^{+\infty} |u_k(t)|\,|v_k(x)| \leqslant \sum_{k=1}^{+\infty} \frac{|v_k(x)|}{\sqrt{\lambda_k}}\left(\sqrt{\lambda_k}\,|\varphi_{1k}| + |\varphi_{2k}| + \sqrt{\int_0^T f_k^2(\tau)\,d\tau}\right).$$

Applying the Cauchy–Schwarz inequality, we arrive at

$$|u(x,t)| \leqslant \sqrt{\sum_{k=1}^{+\infty} \frac{v_k^2(x)}{\lambda_k}}\left(\sqrt{\sum_{k=1}^{+\infty} \lambda_k \varphi_{1k}^2} + \sqrt{\sum_{k=1}^{+\infty} \varphi_{2k}^2} + \sqrt{\int_0^T \sum_{k=1}^{+\infty} [f_k(\tau)]^2\,d\tau}\right). \tag{3.6}$$

By the hypotheses of Theorem 3.1, and in view of Lemmas 2.2 and 2.3, the series on the right-hand side converge uniformly; consequently the series on the left, that is (3.1), converges uniformly on $\overline{\Omega}$.

Consider next the series (3.4). By equation (2.1), on any compact set Ω_0 the series (3.4) may be rewritten as

$$\sum_{k=1}^{+\infty} (-1)^n \lambda_k u_k(t)v_k(x). \tag{3.7}$$

To prove the uniform convergence of (3.7), in accordance with (3.3) it suffices to establish the absolute and uniform convergence of the series

$$\begin{gathered}\sum_{k=1}^{+\infty}(-1)^n\lambda_k\varphi_{1k}v_k(x), \qquad \sum_{k=1}^{+\infty}(-1)^n\sqrt{\lambda_k}\varphi_{2k}v_k(x),\\ \sum_{k=1}^{+\infty}(-1)^n\sqrt{\lambda_k}\sqrt{\int_0^T \left[f_k(\tau)\right]^2\,d\tau}\,v_k(x).\end{gathered} \tag{3.8}$$

For each of these, the Cauchy–Schwarz inequality on Ω_0 gives

$$\left|\sum_{k=1}^{+\infty}(-1)^n\lambda_k\varphi_{1k}v_k(x)\right| \leqslant \sum_{k=1}^{+\infty}\left|\sqrt{\lambda_k^3}\varphi_{1k}\frac{v_k(x)}{\sqrt{\lambda_k}}\right| \leqslant \left[\sum_{k=1}^{+\infty}\lambda_k^3\varphi_{1k}^2\sum_{k=1}^{\infty}\frac{v_k^2(x)}{\lambda_k}\right]^{1/2},$$

$$\left|\sum_{k=1}^{+\infty}(-1)^n\sqrt{\lambda_k}\varphi_{2k}v_k(x)\right| \leqslant \sum_{k=1}^{+\infty}\left|\lambda_k\varphi_{2k}\frac{v_k(x)}{\sqrt{\lambda_k}}\right| \leqslant \left[\sum_{k=1}^{+\infty}\lambda_k^2\varphi_{2k}^2\sum_{k=1}^{\infty}\frac{v_k^2(x)}{\lambda_k}\right]^{1/2},$$

$$\left|\sum_{k=1}^{+\infty}(-1)^n\sqrt{\lambda_k}\sqrt{\int_0^T \left[f_k(\tau)\right]^2\,d\tau}\,v_k(x)\right| \leqslant \sum_{k=1}^{+\infty}\left|\sqrt{\lambda_k^2\int_0^T \left[f_k(\tau)\right]^2\,d\tau}\frac{v_k(x)}{\sqrt{\lambda_k}}\right| \leqslant$$

$$\leqslant \left[\int_0^T\sum_{k=1}^{+\infty}\lambda_k^2\left[f_k(\tau)\right]^2\,d\tau\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}\right]^{1/2}.$$

Owing to the hypotheses of Theorem 3.1, and by Lemmas 2.3, 2.4 and 2.5, the series on the right-hand sides converge uniformly; hence the series on the left, namely (3.8), converge absolutely and uniformly on Ω_0. Therefore (3.7), and with it the series (3.4), converges uniformly on the compact set Ω_0. The uniform convergence of (3.5) on Ω_0 is a consequence of the convergence of (3.4) and of the validity of equation (1.1). The uniform convergence of the remaining series is proved in a similar way. Theorem 3.1 is proved. □

We next address the uniqueness of the solution of the Problem N.

Theorem 3.2. *The Problem N cannot possess more than one solution.*

Proof. Assume, on the contrary, that the Problem N admits two solutions $u_1(x,t)$ and $u_2(x,t)$, and denote their difference by $u(x,t)$. Then $u(x,t)$ satisfies equation (1.1) with $f(x,t)\equiv 0$, together with conditions (1.2) and (1.3) in which $\varphi_1(x)\equiv\varphi_2(x)\equiv 0$. Let $T_0\in(0,T]$ and $\Omega_1=\{(x,t): 0<x<1,\ 0<t<T_0\}$, so that $\overline{\Omega}_1\subset\overline{\Omega}$. Introduce the function

$$\omega(x,t) = -\int_t^{T_0} u(x,\xi)\,d\xi, \quad (x,t)\in\overline{\Omega}_1,$$

which has the properties:

(1) $\omega_t, \omega_{tt}, \dfrac{\partial^j\omega}{\partial x^j}, \dfrac{\partial^j}{\partial x^j}\left(x^\alpha\dfrac{\partial^{2n}\omega}{\partial x^{2n}}\right)\in C\left(\overline{\Omega}_1\right),\ j=\overline{0,n-1}$;
(2) it fulfils conditions (1.3) for $t\in[0,T_0]$.

Multiplying equation (1.1) (with $f(x,t)\equiv 0$) by $\omega(x,t)$ and integrating over Ω_1, we obtain

$$\int_{\Omega_1}\omega(x,t)\left\{u_{tt}(x,t)+(-1)^n\frac{\partial^n}{\partial x^n}\left[x^\alpha\frac{\partial^n u(x,t)}{\partial x^n}\right]\right\}dt\,dx=0,$$

which we recast as

$$\int_0^{T_0}dt\int_0^1(-1)^n\omega(x,t)\frac{\partial^n}{\partial x^n}\left[x^\alpha\frac{\partial^n u(x,t)}{\partial x^n}\right]dx+\int_0^1dx\int_0^{T_0}\omega(x,t)u_{tt}(x,t)\,dt=0.$$

Integration by parts then yields

$$\int_0^{T_0}\left[\omega(x,t)\frac{\partial^{2n-1}}{\partial x^{2n-1}}\left(x^\alpha\frac{\partial^{2n}u(x,t)}{\partial x^{2n}}\right)-\frac{\partial\omega(x,t)}{\partial x}\frac{\partial^{2n-2}}{\partial x^{2n-2}}\left(x^\alpha\frac{\partial^{2n}u(x,t)}{\partial x^{2n}}\right)+\dots\right.$$
$$\left.-\frac{\partial^{2n-1}\omega(x,t)}{\partial x^{2n-1}}\left(x^\alpha\frac{\partial^{2n}u(x,t)}{\partial x^{2n}}\right)\right]_{x=0}^{x=1}dt+\int_0^{T_0}dt\int_0^1x^\alpha\frac{\partial^{2n}\omega(x,t)}{\partial x^{2n}}\frac{\partial^{2n}u(x,t)}{\partial x^{2n}}dx$$
$$+\int_0^1\left[\omega(x,t)\frac{\partial u(x,t)}{\partial t}\bigg|_{t=0}^{t=T_0}-\int_0^{T_0}\frac{\partial\omega(x,t)}{\partial t}\frac{\partial u(x,t)}{\partial t}dt\right]dx=0.$$

On account of the properties of $\omega(x,t)$ and $u(x,t)$, the paired boundary conditions (1.3) cause the boundary terms in x to cancel pairwise at the two endpoints, so that

$$\int_0^{T_0}dt\int_0^1x^\alpha\frac{\partial^{2n}\omega(x,t)}{\partial x^{2n}}\frac{\partial^{2n}u(x,t)}{\partial x^{2n}}dx-\int_0^1dx\int_0^{T_0}\frac{\partial\omega(x,t)}{\partial t}\frac{\partial u(x,t)}{\partial t}dt=0.$$

Taking into account $u=\dfrac{\partial\omega}{\partial t}$ and $\dfrac{\partial^{2n}u}{\partial x^{2n}}=\dfrac{\partial^{2n+1}\omega}{\partial x^{2n}\partial t}$, we get

$$\int_0^1x^\alpha\,dx\int_0^{T_0}\frac{\partial^{2n}\omega(x,t)}{\partial x^{2n}}\frac{\partial^{2n+1}\omega(x,t)}{\partial x^{2n}\partial t}dt-\int_0^1dx\int_0^{T_0}u(x,t)\frac{\partial u(x,t)}{\partial t}dt=0.$$

Using the identities

$$u(x,t)\frac{\partial u(x,t)}{\partial t}=\frac{1}{2}\frac{\partial}{\partial t}[u(x,t)]^2,\qquad\frac{\partial^{2n}\omega(x,t)}{\partial x^{2n}}\frac{\partial^{2n+1}\omega(x,t)}{\partial x^{2n}\partial t}=\frac{1}{2}\frac{\partial}{\partial t}\left[\frac{\partial^{2n}\omega(x,t)}{\partial x^{2n}}\right]^2,$$

integrating by parts in t and recalling $\omega\left(x,T_0\right)=0$, $u(x,0)=0$, we obtain

$$\int_0^1u^2\left(x,T_0\right)dx+\int_0^1x^\alpha\left[\frac{\partial^{2n}\omega(x,t)}{\partial x^{2n}}\right]^2_{t=0}dx=0.$$

This forces $u\left(x,T_0\right)\equiv 0$ for $x\in[0,1]$. Since T_0 is an arbitrary point of $[0,T]$, it follows that $u(x,t)\equiv 0$ on $\overline{\Omega}$, that is $u_1(x,t)\equiv u_2(x,t)$ on $\overline{\Omega}$. Theorem 3.2 is proved. □

Finally, we establish certain estimates for the solution of the Problem N.

Theorem 3.3. *Let $\varphi_1(x)$, $\varphi_2(x)$ and $f(x,t)$ satisfy the hypotheses of Theorem 3.1. Then the estimates*

$$\|u(\cdot,t)\|^2_{L_2(0,1)}\leqslant K_0\left[\|\varphi_1\|^2_{L_2(0,1)}+\|\varphi_2\|^2_{L_2(0,1)}+\|f(\cdot,t)\|^2_{L_2(\Omega)}\right],\qquad(3.9)$$

$$\|u(x,t)\|_{C(\overline{\Omega})} \leqslant K_1 \left[\|\varphi_1^{(2n)}\|_{L_{2,r}(0,1)} + \|\varphi_2\|_{L_2(0,1)} + \|f(\cdot,t)\|_{L_2(\Omega)} \right] \tag{3.10}$$

are valid, where $\|\varphi_1\|_{L_{2,r}(0,1)} = \left[\int_0^1 x^\alpha \left[\varphi_1(x)\right]^2 dx \right]^{1/2}$, $r = r(x) = x^\alpha$, *and* K_0, K_1 *are certain positive constants.*

Proof. As the system $\{v_k(x)\}_{k=1}^{+\infty}$ is orthonormal, the inequality (3.3) together with (3.2) yields

$$\begin{aligned}
\|u(\cdot,t)\|^2_{L_2(0,1)} = \sum_{k=1}^{+\infty} u_k^2(t) &\leqslant \sum_{k=1}^{+\infty} \left[|\varphi_{1k}| + \frac{1}{\sqrt{\lambda_k}} |\varphi_{2k}| + \frac{1}{\sqrt{\lambda_k}} \|f_k(t)\|_{L_2(0,T)} \right]^2 \\
&\leqslant 3 \sum_{k=1}^{+\infty} \left[\varphi_{1k}^2 + \frac{1}{\lambda_k}\varphi_{2k}^2 + \frac{1}{\lambda_k} \|f_k(t)\|^2_{L_2(0,T)} \right] \\
&\leqslant 3 \sum_{k=1}^{+\infty} \left[\varphi_{1k}^2 + \frac{1}{\lambda_1}\varphi_{2k}^2 + \frac{1}{\lambda_1} \|f_k(t)\|^2_{L_2(0,T)} \right].
\end{aligned}$$

Hence, by Bessel's inequality,

$$\|u(\cdot,t)\|^2_{L_2(0,1)} \leqslant K_0 \left(\|\varphi_1\|^2_{L_2(0,1)} + \|\varphi_2\|^2_{L_2(0,1)} + \sum_{k=1}^{+\infty} \|f_k(t)\|^2_{L_2(0,T)} \right), \tag{3.11}$$

where $K_0 = 3C$, $C = \max(1, 1/\lambda_1)$. One readily verifies the equality

$$\|f(\cdot,t)\|^2_{L_2(\Omega)} = \sum_{k=1}^{+\infty} \|f_k(t)\|^2_{L_2(0,T)}.$$

Estimate (3.9) then follows from (3.11). Taking the statements of Lemmas 2.2 and 2.3 into account, (3.6) gives

$\|u(x,t)\|_{C(\overline{\Omega})} = \sup_{\overline{\Omega}} |u(x,t)| \leqslant$

$$\leqslant K_1 \left\{ \sqrt{\int_0^1 x^\alpha \left[\varphi_1^{(2n)}(x)\right]^2 dx} + \|\varphi_2(x)\|^2_{L_2(0,1)} + \sum_{k=1}^{+\infty} \|f_k(t)\|^2_{L_2(0,T)} \right\},$$

where $K_1 = \sup_{[0,1]} \sqrt{\sum_{k=1}^{+\infty} v_k^2(x)/\lambda_k}$. With the notation introduced, estimate (3.10) follows. Theorem 3.3 is proved. □

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