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INFINITE-CLASS MEAN PSEUDO-ALMOST AUTOMORPHIC SOLUTIONS

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ABSTRACT. This paper introduces and studies the concept of mean $(\eta_1;\eta_2)$-pseudo-almost automorphic solution of infinite class for stochastic processes using the measure theory. We present many interesting results on spaces, such as completeness and composition theorems. Our abstract results are then applied to the study of the existence and uniqueness of mean almost-pseudo automorphic solutions for the stochastic evolution equation. These results provide an effective framework for analyzing the existence and uniqueness of mean $(\eta_1;\eta_2)$-pseudo-almost automorphic solutions of infinite class of stochastic evolution equations, combining semigroup theory, fixed-point argument.

Keywords. Poisson process ; measure ; semigroup; pseudo-almost automorphic process ; stochastic evolution equations, Banach space, infinite class.
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1. Introduction

The study of pseudo-almost automorphic solutions plays a crucial role in the qualitative analysis of recurrent dynamical systems. In many applied fields, particularly in physics, biology, engineering and finance, such systems are often subject to random perturbations, which naturally leads to the study of stochastic evolution equations. Stochastic evolution differential equations arise in various areas, notably in neuroscience, where Poisson processes are widely used [11, 22]. However, classical notions of almost periodicity are not sufficient to capture the full complexity of these phenomena. This limitation motivates the development of more general frameworks, especially those based on measure theory. In the presence of Poisson noise, several challenges remain, including the proper definition of suitable functional spaces and the analysis of the existence and uniqueness of mild solutions, which require appropriate analytical tools. In this article, we investigate certain stochastic differential equations as well as pseudo-almost automorphic processes in a Banach space. The concept of almost automorphic processes was first established by S. Bochner in 1927 [7] as a generalization of almost periodic processes introduced by H. Bohr in 1925 [8]. We also study processes in

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the mean sense notably $(\eta_1;\eta_2)$-ergodic and $(\eta_1;\eta_2)$-pseudo-almost automorphic processes of infinite class . The notion of mean pseudo-almost automorphic processes and has relevant applications in mathematical physics and statistics [5]. In this paper, we study (1.1) in a Banach space $\mathbb{B}$, defined

$$dy(t) = [Ay(t) + L(y_t) + g(t)]dt + h(t)dN(t), \tag{1.1}$$

where $A : \mathbb{B} \to \mathbb{B}$ with domain $\mathbb{D}(A) \subset \mathbb{B}$, is a linear operator satisfying the Hille-Yosida condition; g, h: $\mathbb{R} \to L(\Omega, \mathbb{B})$ are two stochastic processes and $N(t)$ is a two-sided inhomogeneous Poisson process with intensity function λ^*, defined on the filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t^N\}_{t\geq 0})$ with natural filtration $\mathcal{F}_t^N = \sigma\{N(u) - N(v)|v, u \leq t\}$ and $L : \mathscr{B} \to L(\Omega, \mathbb{B})$ is a bounded linear operator, where $\mathscr{B}$ denotes the phase space which is a space of functions mapping $]-\infty; 0]$ into $\mathbb{X}$. For every $t \geq 0$, the history function $y_t \in \mathscr{B}$ corresponding to a stochastic process y defined by $y_t(\theta) = y(t+\theta)$ for $\theta \in]-\infty; 0]$. $L(\Omega, \mathbb{B})$ denotes the space of all $\mathbb{B}$-valued random variables y such that

$$\mathbb{E}\|y\| = \int_\Omega \|y\| d\mathbb{P} < \infty.$$

Our work is based on the studies established on Poisson processes in [17, 22] and on recent contributions concerning stochastic evolution differential equations similar to (1.1). As an illustration, in [21] the authors established the existence and uniqueness of mean $cl(\mu, \nu)$-pseudo-almost periodic solution of class r in a Banch space. In [5], the authors studied the concept of p-mean almost periodicity for stochastic processes and established the unique square-mean almost periodic solution of an equation similar to problem (1.1).
Our work is divided as follows: in the second section, we provide the spectral decomposition and study the mean $(\eta_1;\eta_2)$-ergodic process of infinite class; in the third section ; we establish mean $(\eta_1;\eta_2)$-pseudo-almost automorphic processes of infinite class and their properties; and the last section is devoted to an application.

2. Spectral decomposition and Mean (η_1, η_2)-ergodic processes

In this paper, we consider the space $(\mathscr{B}, |.|_{\mathscr{B}})$ is a normed linear space of functions mapping $]-\infty; 0]$ into $L(\Omega, \mathbb{B})$ and satisfying the following fundamental axioms.

$(\mathbf{A}_1)$: There exist a positive constant H and functions $K(.), M(.) : \mathbb{R}^+ \to \mathbb{R}^+$, with K continuous and M locally bounded, such that for any $\sigma \in \mathbb{R}$ and $a > 0$, if $v :]-\infty, a] \to L(\Omega, \mathbb{B}), v_\sigma \in \mathscr{B}$, and $v(.)$ is continuous on $[\sigma, \sigma + a]$, then for every $t \in [\sigma, \sigma + a]$ the following conditions hold :

- (i) $v_t \in \mathscr{B}$,
- (ii) $|v(t)| \leq H|v_t|_{\mathscr{B}}$, which is equivalent to $|\varphi(0)| \leq H|\varphi|_{\mathscr{B}}$ for every $\varphi \in \mathscr{B}$
- (iii) $|v_t|_{\mathscr{B}} \leq K(t-\sigma) \sup_{\sigma \leq s \leq t} |v(s)| + M(t-\sigma)|v_\sigma|_{\mathscr{B}}$

$(\mathbf{A}_2)$: For the function $v(.)$ in $(\mathbf{A}_1), t \mapsto v_t$ is a $\mathscr{B}-$valued continuous function for $t \in [\sigma, \sigma + a]$,

$(\mathbf{A}_3)$: The space $\mathscr{B}$ is a Banach space.

Now, we illustrate the concept of a normed linear space $(\mathscr{B}, |.|_{\mathscr{B}})$ with the following example

Example 2.1. $\mathscr{B} = \mathscr{C}_\gamma, \gamma > 0$ where

$$\mathscr{C}_\gamma = \left\{ \phi \in C(]-\infty;0], L(\Omega,\mathbb{B})) : \lim_{\theta\to-\infty} e^{\gamma\theta}\phi(\theta) \text{ exists in } L(\Omega,\mathbb{B}) \right\},$$

with the following norm

$$\|\phi\|_\gamma = \sup_{\theta\leq 0}\left(\mathbb{E}\|e^{\gamma\theta}\phi(\theta)\|\right).$$

We assume that :
$(\mathbf{C}_1)$: Let $(\varphi_n)_{n\geq 0} \subset \mathscr{B}$, if $\varphi_n \to 0$ in $\mathscr{B}$ as $n \to +\infty$, then sequence $(\varphi_n(\theta))_{n\geq 0}$ converges to 0 in $L(\Omega,\mathbb{B})$. Consider $C(]-\infty;0], L(\Omega,\mathbb{B}))$ the space of continuous functions from $]-\infty;0]$ to $L(\Omega,\mathbb{B})$. Now, we suppose the following assumptions :

$(\mathbf{C}_2)$: $\mathscr{B} \subset C(]-\infty;0], L(\Omega,\mathbb{B}))$;

$(\mathbf{B})$: There exists $\lambda_0 \in \mathbb{R}$ such that, for all $\lambda \in \mathbb{C}$ with $Re(\lambda) > \lambda_0$ and

$$y \in L(\Omega,\mathbb{B}), e^{\lambda\cdot}y \in \mathscr{B} \text{ and } K_0 = \sup_{\substack{Re(\lambda)>\lambda_0, y\in L(\Omega,\mathbb{B})\\ y\neq 0}} \frac{\left|e^{\lambda\cdot}y\right|_{\mathscr{B}}}{|y|} < \infty,$$

where $(e^{\lambda\cdot}y)(\theta) = e^{\lambda\theta}y$ for $\theta \in]-\infty,0]$ and $y \in L(\Omega,\mathbb{B})$.

Here, we associate to (1.1), the following initial value problem

$$\begin{cases} dv_t = [Av_t + L(v_t) + g(t)]\,dt + h(t)dN(t) \text{ for } t > 0 \\ v_0 = \phi \in \mathscr{B}, \end{cases} \tag{2.1}$$

where $g, h : \mathbb{R}^+ \to L(\Omega,\mathbb{B})$ are two continuous stochastic processes.

Definition 2.2. A continuous function $v :]-\infty;+\infty[\to L(\Omega,\mathbb{B})$ is an integral solution of (2.1), if the following conditions hold :

(1) $\int_0^t v(s)ds \in \mathbb{D}(A)$, for $t \geq 0$;
(2) $v(t) = \phi(0) + A\int_0^t v(s)ds + \int_0^t (L(v_s) + g(s))ds + \int_0^t h(s)dN(s)$, for $t \geq 0$;
(3) $v_0 = \phi$.

Remark 2.3. [20] If $\overline{\mathbb{D}(A)} = L(\Omega,\mathbb{B})$, the integral solution coincides with the mild solution.
From the closedness property of A, if $v(t)$ is an integral solution of (2.1), then $v(t) \in \overline{\mathbb{D}(A)}$ for all $t \geq 0$, in particular $\phi(0) \in \overline{\mathbb{D}(A)}$.

Now, we introduce part A_0 of the operator A in $\overline{\mathbb{D}(A)}$ which is defined by

$$\begin{cases} \mathbb{D}(A_0) = \left\{ y \in \mathbb{D}(A) : Ay \in \overline{\mathbb{D}(A)} \right\} \\ A_0 y = Ay \text{ for } y \in \mathbb{D}(A_0). \end{cases}$$

We assume

$$(\mathbf{H}_0) : A \text{ satisfies Hille-Yosida condition.}$$

Proposition 2.4. [1] *The part A_0 of A generates a C_0-semigroup $(T_0(t))_{t\geq 0}$ on $\overline{\mathbb{D}(A)}$.*

Introduce $\mathscr{B}_A$ of (2.1) which is defined by

$$\mathscr{B}_A = \{\varphi \in \mathscr{B} : \ \varphi(0) \in \overline{\mathbb{D}(A)}\}.$$

For $t \geq 0$, $\mathcal{V}(t)$ in $\mathscr{B}_A$ is defined by $\mathcal{V}(t)\varphi = u_t(.,\varphi)$ where $u(.,\varphi)$ is the solution of

$$\begin{cases} \dfrac{d}{dt} u_t = Au_t + L(u_t) \text{ for } t \geq 0 \\ u_0 = \varphi \in \mathscr{B}. \end{cases} \tag{2.2}$$

Proposition 2.5. [3] *Family $(\mathcal{V}(t))_{t\geq 0}$ is a C_0-semigroup of bounded linear operators on $\mathscr{B}_A$. Furthermore $(\mathcal{V}(t))_{t\geq 0}$ satisfies, all $t \geq 0$ and $\theta \leq 0$, the following translation property*

$$(\mathcal{V}(t)\varphi)(\theta) = \begin{cases} (\mathcal{V}(t+\theta)\varphi)(0), \forall t+\theta \geq 0 \\ \varphi(t+\theta), \ \ \forall t+\theta \leq 0. \end{cases}$$

Theorem 2.6. [3] *Let $\mathcal{A}_{\mathcal{V}}$ defined on $\mathscr{B}_A$ by*

$$\begin{cases} \mathbb{D}(\mathcal{A}_{\mathcal{V}}) = \left\{ \varphi \in C^1(]-\infty,0];\mathbb{X}) \cap \mathscr{B}_A; \ \ \phi' \in \mathscr{B}_A, \varphi(0) \in \overline{\mathbb{D}(A)} \ \textit{and} \ \ \varphi'(0) = A\varphi(0) + L(\varphi) \right\} \\ \mathcal{A}_{\mathcal{V}}\varphi = \varphi' \ \ \textit{for} \ \ \varphi \in \mathbb{D}(\mathcal{A}_{\mathcal{V}}), \end{cases}$$

then $\mathcal{A}_{\mathcal{V}}$ is the infinitesimal generator of the semigroup $(\mathcal{V}(t))_{t\geq 0}$ on $\mathscr{B}_A$.

Let $\langle X_0 \rangle$ the space defined by $\langle X_0 \rangle = \{X_0 y : \ \ y \in \mathbb{X}\}$ where function $X_0 y$ is defined by

$$(X_0 y)(\theta) = \begin{cases} 0 \text{ if } \theta \in]-\infty,0[\\ y \text{ if } \theta = 0. \end{cases} \tag{2.3}$$

The space $\mathscr{B}_A \oplus \langle X_0 \rangle$ is a Banach space with following norm
$\|z + X_0 p\|_{\mathscr{B}} = \|z\|_{\mathscr{B}} + \|p\|$ for $(z,p) \in \mathscr{B}_A \times \mathbb{X}$ and consider $\widetilde{\mathcal{A}}_{\mathcal{V}}$ the extension of the operator $\mathcal{A}_{\mathcal{V}}$ defined on $\mathscr{B}_A \oplus \langle X_0 \rangle$ by

$$\begin{cases} \mathbb{D}(\widetilde{\mathcal{A}}_{\mathcal{V}}) = \left\{ \varphi \in C^1(]-\infty,0];\mathbb{X}) : \varphi \in \mathbb{D}(A) \text{ and } \varphi' \in \overline{\mathbb{D}(A)} \right\} \\ \widetilde{\mathcal{A}}_{\mathcal{V}}\varphi = \varphi' + X_0 \left(A\varphi + L(\varphi) - \varphi' \right). \end{cases}$$

Lemma 2.7. [3] *Suppose that $\mathscr{B}$ satisfies $(\boldsymbol{A}_1), (\boldsymbol{A}_2), (\boldsymbol{B})$ and $(\boldsymbol{C}_1) - (\boldsymbol{C}_3)$. Then $\widetilde{\mathcal{A}}_{\mathcal{V}}$ satisfies the Hille-Yosida condition on $\mathscr{B}_A \oplus \langle X_0 \rangle$.*

Let C_{00} be the space of $\mathbb{X}$-valued continuous function on $]-\infty,0]$ with compact support. We assume that :
$(\mathbf{D})$: If $(\phi_n)_{n\geq 0}$ is a Cauchy sequence in $\mathscr{B}$ and converges compactly to ϕ on $]-\infty;0]$, then $\phi\in\mathscr{B}$ and $|\phi_n-\phi|\to 0$.

Proposition 2.8. [2] *Suppose that $\mathscr{B}$ satisfies $(\boldsymbol{A}_1),(\boldsymbol{A}_2),(\boldsymbol{B})$ and $(\boldsymbol{C}_1)-(\boldsymbol{C}_3)$. Assume that $(\boldsymbol{H}_0)$ holds. Then $\widetilde{\mathcal{A}}_{\mathcal{V}}$ satisfies the Hille-Yosida condition on $\mathscr{B}_A\oplus\langle X_0\rangle$, there exists $M\geq 0$, $\widetilde{\omega}\in\mathbb{R}$ such that*

$$]\widetilde{\omega},+\infty[\subset\rho(\widetilde{\mathcal{A}}_{\mathcal{V}}) \ \text{and} \ |(\lambda I-\widetilde{\mathcal{A}}_{\mathcal{V}})^{-n}|\leq\frac{M}{(\lambda-\widetilde{\omega})^n} \ \text{for} \ n\in\mathbb{N} \ \text{et} \ \lambda>\widetilde{\omega}.$$

Furthermore, the part $\widetilde{\mathcal{A}}_{\mathcal{V}}$ on $\mathbb{D}(\widetilde{\mathcal{A}}_{\mathcal{V}})=\mathscr{B}_A$ is exactly the operator $\widetilde{\mathcal{A}}_{\mathcal{V}}$.

Let $(S_0(t))_{t\geq 0}$ be the strongly continuous semigroup defined on the subspace $\mathscr{B}_0=\{\phi\in\mathscr{B}:\phi(0)=0\}$ by

$$(S_0(t)\varphi)(\theta)=\begin{cases}\phi(t+\theta), & \text{if } t+\theta\leq 0\\ 0, & \text{if } t+\theta\geq 0.\end{cases}$$

Definition 2.9. Assume that the space $\mathscr{B}$ satisfies axioms $(\mathbf{B})$ and $(\mathbf{D})$, $\mathscr{B}$ is said to be a fading memory space, if for all $\phi\in\mathscr{B}$,

$$|S_0(t)|\to 0 \ \text{as} \ t\to+\infty \ \text{in} \ \mathscr{B}_0.$$

Moreover, $\mathscr{B}$ is said to be a uniform fading memory space, if

$$|S_0(t)|\to 0 \ \text{as} \ t\to+\infty.$$

Lemma 2.10. [18]*(p.190) If $\mathscr{B}$ is a uniform fading memory space, then the function K may be taken as constant and the function M can be selected such that $M(t)\to 0$ as $t\to+\infty$.*

Proposition 2.11. [18] *If $\mathscr{B}$ is a fading memory space, then the Banach space $BC(]-\infty,0],\mathbb{X})$ of all bounded continuous $\mathbb{X}$-valued functions on $]-\infty,0]$ equipped with the supremum norm is continuous embedding in $\mathscr{B}$. In particular $\mathscr{B}$ satisfies $(\boldsymbol{C}_3)$, for $\lambda_0>0$.*

Now, we assume the following hypothesis

$$(\mathbf{H}_1): T_0(t) \ \text{is compact on} \ \overline{\mathbb{D}(A)}, \ \text{for all} \ t>0.$$

$$(\mathbf{E}): \mathscr{B} \ \text{is a uniform fading memory space.}$$

Theorem 2.12. [3] *Assume that $\mathscr{B}$ satisfies $(\boldsymbol{A}_1),(\boldsymbol{A}_2),(\boldsymbol{B}),(\boldsymbol{C}_1)$ and $(\boldsymbol{H}_0),(\boldsymbol{H}_1),(\boldsymbol{E})$ hold. Then $(\mathcal{V}(t))_{t\geq 0}$ can be decomposed on $\mathscr{B}_A$ as follows*

$$\mathcal{V}(t)=\mathcal{V}_1(t)+\mathcal{V}_2(t) \ \text{for} \ t\geq 0$$

where $(\mathcal{V}_1(t))_{t\geq 0}$ is an exponentially stable semigroup on $\mathscr{B}_A$, which means that there are constants $\alpha_0\geq 0$ and $N_0\geq 0$ such that

$$|\mathcal{V}_1(t)|\leq N_0e^{-\alpha_0 t}|\varphi| \ \text{for} \ t\geq 0 \ \text{and} \ \varphi\in\mathscr{B}_A$$

and $(\mathcal{V}_2(t))_{t\geq 0}$ is compact for $t>0$.

The result below concerns the spectral decomposition of $\mathscr{B}_A$.

Proposition 2.13. [3] *Assume that* $(\boldsymbol{A}_1), (\boldsymbol{A}_2), (\boldsymbol{C}_1)$ *and* $(\boldsymbol{H}_0), (\boldsymbol{H}_1), (\boldsymbol{E})$ *hold. Then the space* $\mathscr{B}_A$ *is decomposed as a direct sum*

$$\mathscr{B}_A = S \oplus U$$

of two $\mathcal{V}(t)$ *invariant closed subspaces* S *and* U *such that restricted semigroup on* $\mathcal{V}$ *is a group and there exist positive constant* M_0 *and* ω *such that*

$$|\mathcal{V}(t)\varphi| \leq M_0 e^{-\omega t}|\varphi| \ \text{ for } \ t \geq 0 \ \text{ and } \ \varphi \in S,$$
$$|\mathcal{V}(t)\varphi| \leq M_0 e^{\omega t}|\varphi| \ \text{ for } \ t \leq 0 \ \text{ and } \ \varphi \in U,$$

where S *and* U *are called the stable and unstable space respectively.*

Let $\mathcal{N}$ the Lebesgue σ-field of $\mathbb{R}$ and by $\mathcal{M}$ the set of all positive measures η on $\mathcal{N}$ satisfying $\eta(\mathbb{R}) = +\infty$ and $\eta([a, b]) < \infty$, for all $a, b \in \mathbb{R}, (a \leq b)$. The space $(L(\Omega, \mathbb{B}), \|.\|)$ is a Banach space.

Definition 2.14. Let $g : \mathbb{R} \to L(\Omega, \mathbb{B})$ a stochastic process.

1) If $\lim\limits_{t\to s} \mathbb{E}\|g(t) - g(s)\| = 0$ for all $s \in \mathbb{R}$, then g is continuous.
2) If there exists $C \in \mathbb{R}^+$ such that $\mathbb{E}\|g(t)\| \leq C, \ \forall t \in \mathbb{R}$, then g is bounded.

Denote $SBC(\mathbb{R}, L(\Omega, \mathbb{B}))$, the space of all bounded and continuous processes.

We can see that the space $SBC(\mathbb{R}, L(\Omega, \mathbb{B}))$ with the following norm $\|g\|_\infty = \sup\limits_{t\in\mathbb{R}} \mathbb{E}\|g(t)\|$ is a Banach space.

Definition 2.15. Let $\eta_1, \eta_2 \in \mathcal{M}$. A stochastic process $g : \mathbb{R} \to L(\Omega, \mathbb{B})$ is mean $(\eta_1; \eta_2)$-ergodic of infinite class if $g \in SBC(\mathbb{R}, L(\Omega, \mathbb{B}))$ and satisfies

$$\lim_{\zeta\to+\infty} \frac{1}{\eta_2([-\zeta, \zeta])} \int_{-\zeta}^{\zeta} \sup_{\theta\in]-\infty,t]} \mathbb{E}\|g(\theta)\| d\eta_1(t) = 0.$$

We also denote by $\mathcal{E}(\mathbb{R}, L(\Omega, \mathbb{B}), \eta_1, \eta_2, \infty)$, the space of all such processes.

For $\eta_1, \eta_2 \in \mathcal{M}$ and $a \in \mathbb{R}$, we denote by $\eta_{1,a}$ the positive measure on $(\mathbb{R}, \mathcal{N})$ defined by

$$\eta_{1,a}(\mathcal{S}) = \eta_1(\{a + b : b \in \mathcal{S}\}) \ \text{ for } \ \mathcal{S} \in \mathcal{N}. \tag{2.4}$$

From $\eta_1, \eta_2 \in \mathcal{M}$, we formulate the following hypothesis

$(\mathbf{H}_2)$: For all $a \in \mathbb{R}$, there exist $\beta > 0$ and a bounded interval $\mathcal{J}$ such that $\eta_{1,a}(\mathcal{S}) \leq \beta\eta_1(\mathcal{S})$ when $\mathcal{S} \in \mathcal{N}$ satisfies $\mathcal{S} \cap \mathcal{J} = \emptyset$.

$(\mathbf{H}_3)$: For all a, b, c and $\delta \in \mathbb{R}$, such that $0 \leq a < b \leq c$, there exist $\delta_0 > 0$ and $\alpha_0 > 0$ such that $|\delta| \geq \delta_0 \Rightarrow \eta_1(]a + \delta, b + \delta[) \geq \alpha_0\eta_1([\delta, c + \delta])$.

$(\mathbf{H}_4)$: Let $\eta_1, \eta_2 \in \mathcal{M}$ such that $\limsup\limits_{\zeta\to+\infty} \dfrac{\eta_1([-\tau, \tau])}{\eta_2([-\tau, \tau])} = \alpha < \infty$.

Remark 2.16. [19] Hypothesis $(\mathbf{H}_2)$ implies hypothesis $(\mathbf{H}_3)$.

Proposition 2.17. *Assume that* $(\boldsymbol{H}_2)$ *holds. Then* $\mathcal{E}(\mathbb{R}, L(\Omega, \mathbb{B}), \eta_1, \eta_2, \infty)$ *is a Banach space with the norm* $\|,\|_\infty$.

Proof. We can see that $\mathcal{E}(\mathbb{R}, L(\Omega, \mathbb{B}), \eta_1, \eta_2, \infty)$ is a vector subspace of $SBC(\mathbb{R}, L(\Omega, \mathbb{B}))$. To complete the proof, it is enough to prove that $\mathcal{E}(\mathbb{R}, L(\Omega, \mathbb{B}), \eta_1, \eta_2, \infty)$ is closed in $SBC(\mathbb{R}, L(\Omega, \mathbb{B}))$. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{E}(\mathbb{R}, L(\Omega, \mathbb{B}), \eta_1, \eta_2, \infty)$ such that $\lim_{n\to+\infty} g_n = g$ uniformly in $SBC(\mathbb{R}, L(\Omega, \mathbb{B}))$. From $\eta_2(\mathbb{R}) = +\infty$, it results that $\eta_2([-\zeta, \zeta]) > 0$ for ζ sufficiently large. $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ such that $\forall n \geq n_0, \|g_n - g\|_\infty < \varepsilon$. We have

$$\begin{aligned}
\frac{1}{\eta_2([-\zeta,\zeta])}\int_{-\zeta}^{\zeta}\sup_{\theta\in]-\infty,t]}\mathbb{E}\|g(\theta)\|d\eta_1(t) &\leq \frac{1}{\eta_2([-\zeta,\zeta])}\int_{-\zeta}^{\zeta}\sup_{\theta\in]-\infty,t]}\mathbb{E}\|g_n(\theta)-g(\theta)\|d\eta_1(t)\\
&+\frac{1}{\eta_2([-\zeta,\zeta])}\int_{-\zeta}^{\zeta}\sup_{\theta\in]-\infty,t]}\mathbb{E}\|g_n(\theta)\|d\eta_1(t)\\
&\leq \frac{1}{\eta_2([-\zeta,\zeta])}\int_{-\zeta}^{\zeta}\|g_n-g\|_\infty d\eta_1(t)\\
&+\frac{1}{\eta_2([-\zeta,\zeta])}\int_{-\zeta}^{\zeta}\sup_{\theta\in]-\infty,t]}\mathbb{E}\|g_n(\theta)\|d\eta_1(t),
\end{aligned}$$

It results that

$$\begin{aligned}
\limsup_{\zeta\to+\infty}\frac{1}{\eta_2([-\zeta,\zeta])}\int_{-\zeta}^{\zeta}\sup_{\theta\in]-\infty,t]}\mathbb{E}\|g(\theta)\|d\eta_1(t) &\leq \alpha\|g_n-g\|_\infty\\
&\leq \alpha\varepsilon,\ \ \forall\varepsilon>0.
\end{aligned}$$

Therefore $g \in \mathcal{E}(\mathbb{R}, L(\Omega, \mathbb{B}), \eta_1, \eta_2, \infty)$. ∎

The result stated below characterizes mean (η_1, η_2)-ergodic processes, including the possibility that $\mathcal{J} = \emptyset$.

Theorem 2.18. *Assume that* $(\boldsymbol{H}_4)$ *holds. Let* $g \in SBC(\mathbb{R}, L(\Omega, \mathbb{B}))$ *a process. Then the following assertions are equivalent :*
i) $g \in \mathcal{E}(\mathbb{R}, L(\Omega, \mathbb{B}), \eta_1, \eta_2, \infty)$;
ii) $\displaystyle\lim_{\zeta\to+\infty}\frac{1}{\eta_2([-\zeta,\zeta]\setminus\mathcal{J})}\int_{[-\zeta,\zeta]\setminus\mathcal{J}}\left(\sup_{\theta\in]-\infty,t]}\mathbb{E}\|g(\theta)\|\right)d\eta_1(t)=0$;
iii) For any $\varepsilon > 0$, $\displaystyle\lim_{\zeta\to+\infty}\frac{\eta_1\left(\left\{t\in[-\zeta,\zeta]\setminus\mathcal{J}:\sup_{\theta\in]-\infty,t]}\mathbb{E}\|g(\theta)\|>\varepsilon\right\}\right)}{\eta_2([-\zeta,\zeta]\setminus\mathcal{J})}=0.$

Proof. First, we show that i) $\Leftrightarrow$ ii).
We denote by $\mathcal{S} = \eta_2(\mathcal{J})$, $\mathcal{R} = \displaystyle\int_{\mathcal{J}}\sup_{\theta\in]-\infty,t]}\mathbb{E}\|g(\theta)\|d\eta_1(t)$. $\mathcal{S}$ and $\mathcal{R}$ belong to $\mathbb{R}$ because the interval $\mathcal{J}$ is bounded and the process g is stochastically continuous and bounded on $\mathbb{R}$. For $\zeta > 0$ such that $\mathcal{J} \subset [-\zeta, \zeta]$ and $\eta_2([-\zeta, \zeta] \setminus \mathcal{J}) > 0$, we have $\eta_2([-\zeta, \zeta]) < +\infty$ and $\mathcal{J} \subset [-\zeta, \zeta]$.

Hence $\eta_2([-\zeta,\zeta]\setminus\mathcal{J}) = \eta_2([-\zeta,\zeta]) - \eta_2(\mathcal{J})$. We have

$$\begin{aligned}
&\frac{1}{\eta_2([-\zeta,\zeta]\setminus\mathcal{J})}\int_{[-\zeta,\zeta]\setminus\mathcal{J}}\sup_{\theta\in]-\infty,t]}\mathbb{E}\|g(\theta)\|d\eta_1(t)\\
&=\frac{1}{\eta_2([-\zeta,\zeta])-\mathcal{S}}\left[\int_{-\zeta}^{\zeta}\sup_{\theta\in]-\infty,t]}\mathbb{E}\|g(\theta)\|d\eta_1(t)-\mathcal{R}\right];\\
&=\frac{\eta_2([-\zeta,\zeta])}{\eta_2([-\zeta,\zeta])-\mathcal{S}}\left[\frac{1}{\eta_2([-\zeta,\zeta])}\int_{-\zeta}^{\zeta}\sup_{\theta\in]-\infty,t]}\mathbb{E}\|g(\theta)\|d\eta_1(t)\right]\\
&+\frac{\eta_2([-\zeta,\zeta])}{\mathcal{S}-\eta_2([-\zeta,\zeta])}\times\frac{\mathcal{R}}{\eta_2([-\zeta,\zeta])};\\
&=\left(1+\frac{\mathcal{S}}{\eta_2([-\zeta,\zeta])-\mathcal{S}}\right)\times\frac{1}{\eta_2([-\zeta,\zeta])}\int_{-\zeta}^{\zeta}\sup_{\theta\in]-\infty,t]}\mathbb{E}\|g(\theta)\|d\eta_1(t)\\
&+\left(1+\frac{\mathcal{S}}{\mathcal{S}-\eta_2([-\zeta,\zeta])}\right)\times\frac{\mathcal{R}}{\eta_2([-\zeta,\zeta])}.
\end{aligned}$$

By passing to the limit and using the fact that $\eta_2(\mathbb{R}) = +\infty$, we obtain

$$\lim_{\zeta\to+\infty}\frac{1}{\eta_2([-\zeta,\zeta]\setminus\mathcal{J})}\int_{[-\zeta,\zeta]\setminus\mathcal{J}}\sup_{\theta\in]-\infty,t]}\mathbb{E}\|g(\theta)\|d\eta_1(t) = 0.$$

We deduce that i) is equivalent to ii).
Now, we show that iii) $\Rightarrow$ ii). Denote by $\mathcal{S}_\zeta^\varepsilon$ and $\mathcal{R}_\zeta^\varepsilon$, the following sets

$$\mathcal{S}_\zeta^\varepsilon = \left\{t\in[-\zeta,\zeta]\setminus\mathcal{J} : \sup_{\theta\in]-\infty,t]}\mathbb{E}\|g(\theta)\| > \varepsilon\right\};\ \mathcal{R}_\zeta^\varepsilon = \left\{t\in[-\zeta,\zeta]\setminus\mathcal{J} : \sup_{\theta\in]-\infty,t]}\mathbb{E}\|g(\theta)\| \leq \varepsilon\right\}.$$

Assume iii), then

$$\lim_{\zeta\to+\infty}\frac{\eta_1(\mathcal{S}_\zeta^\varepsilon)}{\eta_2([-\zeta,\zeta]\setminus I)} = 0. \tag{2.5}$$

We have $[-\zeta,\zeta]\setminus\mathcal{J} = \mathcal{S}_\zeta^\varepsilon\cup\mathcal{R}_\zeta^\varepsilon$, it follows that

$$\int_{[-\zeta,\zeta]\setminus\mathcal{J}}\sup_{\theta\in]-\infty,t]}\mathbb{E}\|g(\theta)\|d\eta_1(t) = \int_{\mathcal{S}_\zeta^\varepsilon}\sup_{\theta\in]-\infty,t]}\mathbb{E}\|g(\theta)\|d\eta_1(t) + \int_{\mathcal{R}_\zeta^\varepsilon}\sup_{\theta\in]-\infty,t]}\mathbb{E}\|g(\theta)\|d\eta_1(t).$$

For ζ sufficiently large, we obtain

$$\int_{[-\zeta,\zeta]\setminus\mathcal{J}}\sup_{\theta\in]-\infty,t]}\mathbb{E}\|g(\theta)\|d\eta_1(t) \leq \|g\|_\infty\eta_1(\mathcal{S}_\zeta^\varepsilon) + \varepsilon\eta_1(\mathcal{R}_\zeta^\varepsilon).$$

Hence

$$\begin{aligned}
&\limsup_{\zeta\to+\infty}\frac{1}{\eta_2([-\zeta,\zeta]\setminus\mathcal{J})}\int_{[-\zeta,\zeta]\setminus\mathcal{J}}\sup_{\theta\in]-\infty,t]}\mathbb{E}\|g(\theta)\|d\eta_1(t)\\
&\leq \|g\|_\infty\limsup_{\zeta\to+\infty}\frac{\eta_1(\mathcal{S}_\zeta^\varepsilon)}{\eta_2([-\zeta,\zeta]\setminus\mathcal{J})} + \varepsilon\limsup_{\zeta\to+\infty}\frac{\eta_1(\mathcal{R}_\zeta^\varepsilon)}{\eta_2([-\zeta,\zeta]\setminus\mathcal{J})}.
\end{aligned}$$

By using (2.5) and hypothesis $(\mathbf{H}_4)$, it follows that

$$\limsup_{\zeta\to+\infty}\frac{1}{\eta_2([-\zeta,\zeta]\setminus\mathcal{J})}\int_{[-\zeta,\zeta]\setminus\mathcal{J}}\sup_{\theta\in]-\infty,t]}\mathbb{E}\|g(\theta)\|d\eta_1(t)\le\alpha\varepsilon,\quad\forall\varepsilon>0.$$

As a result iii) implies ii).
Thus, we shall show that ii) implies iii). Assume that ii) holds. From the following inequality

$$\int_{[-\zeta,\zeta]\setminus\mathcal{J}}\sup_{\theta\in]-\infty,t]}\mathbb{E}\|g(\theta)\|d\eta_1(t)\ge\int_{\mathcal{S}_\zeta^\varepsilon}\sup_{\theta\in]-\infty,t]}\mathbb{E}\|g(\theta)\|d\eta_1(t);$$

$$\frac{1}{\varepsilon\eta_2([-\zeta,\zeta]\setminus\mathcal{J})}\int_{[-\zeta,\zeta]\setminus I}\sup_{\theta\in]-\infty,t]}\mathbb{E}\|g(\theta)\|d\eta_1(t)\ge\frac{\eta_1(\mathcal{S}_\zeta^\varepsilon)}{\eta_2([-\zeta,\zeta]\setminus\mathcal{J})},$$

it follows that $\lim\limits_{\zeta\to+\infty}\dfrac{\eta_1(\mathcal{S}_\zeta^\varepsilon)}{\eta_2([-\zeta,\zeta]\setminus\mathcal{J})}=0.$ □

Definition 2.19. Let $g\in SBC(\mathbb{R},L(\Omega,\mathbb{B}))$ and let $\zeta\in\mathbb{R}$. The translate g_ζ of g is defined by $g_\zeta(t)=g(t+\zeta)$ for all $t\in\mathbb{R}$. A subset $\mathfrak{D}\subseteq SBC(\mathbb{R},L(\Omega,\mathbb{B}))$ is translation invariant, if for all $g\in\mathfrak{D}$ implies $g_\zeta\in\mathfrak{D}$ for all $\zeta\in\mathbb{R}$.

Definition 2.20. Let $\eta_1,\eta_2\in\mathcal{M}$.
We say that η_1 and η_2 are equivalent ($\eta_1\sim\eta_2$), if there exist constants $\alpha,\beta>0$ and a bounded interval $\mathcal{J}$ (possibly empty) such that $\alpha\eta_1(\mathcal{S})\le\eta_2(\mathcal{S})\le\beta\eta_1(\mathcal{S})$ for $\mathcal{S}\in\mathcal{N}$ satisfying $\mathcal{S}\cap\mathcal{J}=\emptyset$.

Theorem 2.21. *Let $\eta_1,\eta_1^*,\eta_2,\eta_2^*\in\mathcal{M}$. If $\eta_1\sim\eta_1^*$ and $\eta_2\sim\eta_2^*$, then* $\mathcal{E}(\mathbb{R},L(\Omega,\mathbb{B}),\eta_1,\eta_2,\infty)=\mathcal{E}(\mathbb{R},L(\Omega,\mathbb{B}),\eta_1^*,\eta_2^*,\infty)$.

Proof. Given that $\eta_1\sim\eta_1^*$ and $\eta_2\sim\eta_2^*$, there exist some strictly positive constants $\alpha_1,\alpha_2,\beta_1,\beta_2$ and a bounded interval $\mathcal{J}$ (eventually $\mathcal{J}=\emptyset$) such that $\alpha_1\eta_1(\mathcal{S})\le\eta_1^*(\mathcal{S})\le\beta_1\eta_1(\mathcal{S})$ and $\alpha_2\eta_2(\mathcal{S})\le\eta_2^*(\mathcal{S})\le\beta_2\eta_2(\mathcal{S})$ for each $\mathcal{S}\in\mathcal{N}$ satisfies $\mathcal{S}\cap\mathcal{J}=\emptyset$. We obtain $\dfrac{\alpha_1\eta_1(\mathcal{S}_\zeta^\varepsilon)}{\beta_2\eta_2([-\zeta,\zeta]\setminus I)}\le\dfrac{\eta_1^*(\mathcal{S}_\zeta^\varepsilon)}{\eta_2^*([-\zeta,\zeta]\setminus\mathcal{J})}\le\dfrac{\beta_1\eta_1(\mathcal{S}_\zeta^\varepsilon)}{\alpha_2\eta_2([-\zeta,\zeta]\setminus\mathcal{J})}$, it follows that

$$\begin{aligned}&\lim_{\zeta\to+\infty}\frac{\alpha_1\eta_1\left(\left\{t\in[-\zeta,\zeta]\setminus\mathcal{J}:\sup\limits_{\theta\in]-\infty,t]}\mathbb{E}\|g(\theta)\|>\varepsilon\right\}\right)}{\beta_2\eta_2([-\zeta,\zeta]\setminus\mathcal{J})}\\&\le\lim_{\zeta\to+\infty}\frac{\eta_1^*\left(\left\{t\in[-\zeta,\zeta]\setminus\mathcal{J}:\sup\limits_{\theta\in]-\infty,t]}\mathbb{E}\|g(\theta)\|>\varepsilon\right\}\right)}{\eta_2^*([-\zeta,\zeta]\setminus\mathcal{J})};\\&\le\lim_{\zeta\to+\infty}\frac{\beta_1\eta_1\left(\left\{t\in[-\zeta,\zeta]\setminus\mathcal{J}:\sup\limits_{\theta\in]-\infty,t]}\mathbb{E}\|g(\theta)\|>\varepsilon\right\}\right)}{\alpha_2\eta_2([-\zeta,\zeta]\setminus\mathcal{J})}.\end{aligned}$$

Then, it results that

$$\lim_{\zeta\to+\infty} \frac{\eta_1\left(\mathcal{S}_\zeta^\varepsilon\right)}{\eta_2([-\zeta,\zeta]\setminus\mathcal{J})} = \lim_{\zeta\to+\infty} \frac{\eta_1^*\left(\mathcal{S}_\zeta^\varepsilon\right)}{\eta_2^*([-\zeta,\zeta]\setminus\mathcal{J})} = 0.$$

Therefore $\mathcal{E}(\mathbb{R}, L(\Omega,\mathbb{B}), \eta_1, \eta_2, \infty) = \mathcal{E}(\mathbb{R}, L(\Omega,\mathbb{B}), \eta_1^*, \eta_2^*, \infty)$. □

Let $\eta_1, \eta_2 \in \mathcal{M}$, we denote by $cl(\eta_1,\eta_2) = \{\omega_1,\omega_2 : \eta_1 \sim \omega_1; \eta_2 \sim \omega_2\}$.

Proposition 2.22. [6] *Let $\eta \in \mathcal{M}$. Then η satisfies $(\boldsymbol{H}_2)$ if and only if the measures η et η_ζ are equivalent for all $\zeta \in \mathbb{R}$.*

Proposition 2.23. [19] *For all $\sigma > 0$, $(\boldsymbol{H}_2)$ implies that* $\limsup_{\zeta\to+\infty} \dfrac{\eta_1([-\zeta-\sigma,\zeta+\sigma])}{\eta_2([-\zeta,\zeta])} < \infty.$

Theorem 2.24. *Let $\eta_1, \eta_2 \in \mathcal{M}$ satify $(\boldsymbol{H}_2)$. Then $\mathcal{E}(\mathbb{R}, L(\Omega,\mathbb{B}), \eta_1, \eta_2, \infty)$ is translation invariant.*

Proof. Assume that $(\mathbf{H}_2)$ holds. The proof of this theorem is also inspired by Theorem 3.1 in [6]. Let $g \in \mathcal{E}(\mathbb{R}, L(\Omega,\mathbb{B}), \eta_1, \eta_2, \infty)$ and $b \in \mathbb{R}$.
Since $\eta_2(\mathbb{R}) = +\infty$, there exists $b_0 > 0$ such that $\eta_2([-\zeta-|b|, \zeta+|b|]) > 0$ for $|b| > b_0$. We also denote by

$$\mathscr{R}_b(\zeta) = \frac{1}{\eta_{2,b}([-\zeta,\zeta])} \int_{[-\zeta,\zeta]} \sup_{\theta\in]-\infty,t]} \mathbb{E}\|g(\theta)\| d\eta_{1,b}(t), \;\; \forall \zeta > 0 \;\text{ and }\; b \in \mathbb{R},$$

where $\eta_{2,b}$ is the positive measure defined by (2.4). Assumption $(\mathbf{H}_2)$ being satisfied, using Proposition 2.22, we deduce that $\eta_2 \sim \eta_{2,b}$ and $\eta_1 \sim \eta_{1,b}$, using the Theorem 2.21, we obtain

$$\mathcal{E}(\mathbb{R}, L(\Omega,\mathbb{B}), \eta_{1,b}, \eta_{2,b}, \infty) = \mathcal{E}(\mathbb{R}, L(\Omega,\mathbb{B}), \eta_1, \eta_2, \infty).$$

Consequently $g \in \mathcal{E}(\mathbb{R}, L(\Omega,\mathbb{B}), \eta_{1,b}, \eta_{2,b}, \infty)$, hence $\lim_{\zeta\to+\infty} \mathscr{R}_b(\zeta) = 0$, $\forall b \in \mathbb{R}$. For all $\mathrm{B} \in \mathcal{N}$, we also denote by χ_{B} the characteric function of B. Using the definition of the measure $\eta_{1,b}$, we obtain

$$\int_{[-\zeta,\zeta]} \chi_{\mathrm{B}}(t) d\eta_{1,b}(t) = \int_{[-\zeta,\zeta]} \chi_{\mathrm{B}}(t) d\eta_1(t+b) = \int_{[-\zeta+b,\zeta+b]} \chi_{\mathrm{B}}(t-b) d\eta_1(t).$$

Given that the function $t \mapsto \sup_{\theta\in]-\infty,t]} \mathbb{E}\|g(\theta)\|$ can be expressed as the pointwise limit of an increasing sequence of linear combinations of functions, we deduce that

$$\int_{[-\zeta,\zeta]} \sup_{\theta\in]-\infty,t]} \mathbb{E}\|g(\theta)\| d\eta_{1,b}(t) = \int_{[-\zeta+b,\zeta+b]} \sup_{\theta\in]-\infty,t]} \mathbb{E}\|g(\theta)\| d\eta_1(t).$$

Let $b^+ := \max(b, 0)$ and $b^- := \max(-b, 0)$.
Then $[-\zeta + b - |b|, \zeta + b + |b|] = [-\zeta - 2b^-, \zeta + 2b^+]$ and we obtain

$$\begin{aligned}
\mathscr{R}_b(\zeta + |b|) &= \frac{1}{\eta_{2,b}([-\zeta - |b|, \zeta + |b|])} \int_{[-\zeta-|b|,\zeta+|b|]} \sup_{\theta\in]-\infty,t]} \mathbb{E}\|g(\theta)\| d\eta_{1,b}(t);\\
&= \frac{1}{\eta_2([-\zeta + b - |b|, \zeta + b + |b|])} \int_{[-\zeta-|b|,\zeta+|b|]} \sup_{\theta\in]-\infty,t]} \mathbb{E}\|g(\theta)\| d\eta_1(t+b);\\
&= \frac{1}{\eta_2([-\zeta - 2b^-, \zeta + 2b^+])} \int_{[-\zeta-2b^-,\zeta+2b^+]} \sup_{\theta\in]-\infty,t-b]} \mathbb{E}\|g(\theta)\| d\eta_1(t).
\end{aligned}$$

Then, we have

$$\begin{aligned}
\frac{1}{\eta_2([-\zeta, \zeta])} \int_{[-\zeta,\zeta]} \sup_{\theta\in]-\infty,t-b]} \mathbb{E}\|g(\theta)\| d\eta_1(t) &\leq \frac{1}{\eta_2([-\zeta, \zeta])} \int_{[-\zeta-2b^-,\zeta+2b^+]} \sup_{\theta\in]-\infty,t-b]} \mathbb{E}\|g(\theta)\| d\eta_1(t);\\
&\leq \frac{\eta_2([-\zeta - 2b^-, \zeta + 2b^+])}{\eta_2([-\zeta, \zeta])} \times \mathscr{R}_b(\zeta + |b|).
\end{aligned}$$

Using the Proposition 2.23, it results that $\lim_{\zeta\to+\infty} \frac{1}{\nu([-\zeta, \zeta])} \int_{[-\zeta,\zeta]} \sup_{\theta\in]-\infty,t]} \mathbb{E}\|g(\theta - b)\| d\mu(t) = 0$, that is $g_{-b} \in \mathcal{E}(\mathbb{R}, L(\Omega, \mathbb{B}), \eta_1, \eta_2, \infty)$. Therefore $\mathcal{E}(\mathbb{R}, L(\Omega, \mathbb{B}), \eta_1, \eta_2, \infty)$ is translation invariant. □

3. Mean (η_1, η_2)-pseudo-almost automorphic processes

In this part, we define mean (η_1, η_2)-pseudo-almost automorphic and their properties.

Definition 3.1. A continuous stochastic process $f : \mathbb{R} \to L(\Omega, \mathbb{B})$ is almost automorphic process in mean sense if for every sequence of real numbers $(s_m)_{m\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ and a stochastic process $g : \mathbb{R} \to L(\Omega, \mathbb{B})$ such that

$$\lim_{n\to\infty} \mathbb{E}\|f(t + s_n) - g(t)\| = 0$$

for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} \mathbb{E}\|g(t - s_n) - f(t)\| = 0$$

for each $t \in \mathbb{R}$.
Denote the space of all such stochastic processes by $SAA(\mathbb{R}, L(\Omega, \mathbb{B}))$.

Lemma 3.2. [10] *The space $SAA(\mathbb{R}, L(\Omega, \mathbb{B}))$ of mean almost automorphic stochastic processes equipped with the supremum norm $\|.\|_\infty$, constitutes a Banach space.*

Definition 3.3. Let $f : \mathbb{R} \to L(\Omega, \mathbb{B})$ be a bounded continuous stochastic process. f is mean compact almost automorphic process if for every sequence $(t'_n)_n \subset \mathbb{R}$, we can extract a subsequence $(t_n)_n$ such that, for some stochastic process $h : \mathbb{R} \to L(\Omega, \mathbb{B})$, we have

$$\lim_{n\to+\infty} \mathbb{E}\,\|f\,(t + t_n) - h(t)\| = 0 \text{ for all } t \in \mathbb{R}$$

and

$$\lim_{n\to+\infty} \mathbb{E}\left\|h\left(t-t_n\right)-f(t)\right\| = 0 \text{ for all } t\in\mathbb{R}$$

uniformly on compact subsets of $\mathbb{R}$. Such stochastic processes collectively constitute the space denoted $SAA_c\left(\mathbb{R}, L(\Omega,\mathbb{B})\right)$.

From [4], $AP\left(\mathbb{R},\mathbb{X}\right)\subset AA_c\left(\mathbb{R},\mathbb{X}\right)\subset AA\left(\mathbb{R},\mathbb{X}\right)$, thus we obtain the following remark.

Remark 3.4. $SAP\left(\mathbb{R}, L(\Omega,\mathbb{B})\right)\subset SAA_c\left(\mathbb{R}, L(\Omega,\mathbb{B})\right)\subset SAA\left(\mathbb{R}, L(\Omega,\mathbb{B})\right)$

Corollary 3.5. *$SAA_c\left(\mathbb{R}, L(\Omega,\mathbb{B})\right)$ equipped with the norm $\|\cdot\|_\infty$ is a Banach space.*

Definition 3.6. Let $\eta_1,\eta_2\in\mathcal{M}$. A bounded continuous stochastic process $f:\mathbb{R}\to L(\Omega,\mathbb{B})$ is (η_1,η_2)-pseudo-almost automorphic in mean, if it can be decomposed as follows

$$f=g+h,$$

where $g\in SAA_c(\mathbb{R}, L(\Omega,\mathbb{B}))$ and $h\in\mathcal{E}(\mathbb{R}, L(\Omega,\mathbb{B}),\eta_1,\eta_2)$.
We denote the space of all such stochastic processes by $SPAA_c(\mathbb{R}, L(\Omega,\mathbb{B}),\eta_1,\eta_2)$.

Proposition 3.7. [12] *Let $\eta_1,\eta_2\in\mathcal{M}$ satisfy $(\boldsymbol{H}_3)$ and let $f\in PAA(\mathbb{R};\mathbb{X},\eta_1,\eta_2)$ be such that $f=g+h$, where $g\in AA(\mathbb{R},\mathbb{X})$ and $h\in\mathcal{E}(\mathbb{R},\mathbb{X},\eta_1,\eta_2)$. Then the set $\{g(t):t\in\mathbb{R}\}$ is contained in the closure of the range of f, that is*

$$\{g(t):t\in\mathbb{R}\}\subset\overline{\{f(t):t\in\mathbb{R}\}}$$

Proposition 3.8. [23] *Assume that $(\boldsymbol{H}_3)$ holds. Then the decomposition of (η_1,η_2)-pseudo-almost automorphic function in form $f=g+h$, where $g\in SAA(\mathbb{R},\mathbb{X})$ and $h\in\mathcal{E}(\mathbb{R},\mathbb{X},\eta_1,\eta_2,\infty)$ is unique.*

Remark 3.9. For $\mathbb{X}=L(\Omega,\mathbb{H})$, Proposition 3.7 and 3.8 are always holds.

Theorem 3.10. *Let $\eta_1,\eta_2\in\mathcal{M}$. Assume that $SPAA\left(\mathbb{R}, L(\Omega,\mathbb{B}),\eta_1,\eta_2,\infty\right)$ is translation invariant. Then $(SPAA(\mathbb{R}, L(\Omega,\mathbb{B}),\eta_1,\eta_2,\infty),\|\cdot\|_\infty)$ is a Banach space.*

Proof. Let $\eta_1,\eta_2\in\mathcal{M}$ and (f_n) be a Cauchy sequence in $SPAA(\mathbb{R}, L(\Omega,\mathbb{B}),\eta_1,\eta_2,\infty)$. For each n, we decompose $f_n=g_n+\phi_n$, where $g_n\in SAA\left(\mathbb{R}, L(\Omega,\mathbb{B})\right)$ and $\phi_n\in\mathcal{E}\left(\mathbb{R}, L(\Omega,\mathbb{B}),\eta_1,\eta_2,\infty\right)$. By Proposition 3.7 and Remark 3.9 , we have $\|g_n-g_m\|_\infty\leq\|f_n-f_m\|_\infty$, which implies that (g_n) is a Cauchy sequence in the Banach space $\left(SAA\left(\mathbb{R}, L(\Omega,\mathbb{B})\right);\|\cdot\|_\infty\right)$. Since $\phi_n=f_n-g_n$, then $\|\phi_n-\phi_m\|_\infty\leq\|f_n-f_m\|_\infty+\|g_n-g_m\|_\infty$. It follows that (ϕ_n) is also Cauchy sequence in the Banach space $\left(\mathcal{E}\left(\mathbb{R}, L(\Omega,\mathbb{B}),\eta_1,\eta_2,\infty\right),\|\cdot\|_\infty\right)$. By Lemma 3.2 and Proposition 2.17, there exist processes $g\in SAA\left(\mathbb{R}, L(\Omega,\mathbb{B})\right)$ and $\phi\in\mathcal{E}\left(\mathbb{R}, L(\Omega,\mathbb{B}),\eta_1,\eta_2,\infty\right)$ such that $\lim\limits_{n\to+\infty} g_n=g$ and $\lim\limits_{n\to+\infty}\phi_n=\phi$. Consequently $\lim\limits_{n\to+\infty} f_n=g+\phi\in SPAA(\mathbb{R}, L(\Omega,\mathbb{B}),\eta_1,\eta_2,\infty)$, which proves that $SPAA(\mathbb{R}, L(\Omega,\mathbb{B}),\eta_1,\eta_2,\infty)$ is complete with respect to the supremum norm. $\square$

Now, we study the composition properties of mean (η_1,η_2)-pseudo-almost automorphic processes.

Theorem 3.11. *Let $f : \mathbb{R} \times L(\Omega, \mathbb{B}) \to L(\Omega, \mathbb{B}), (t, x) \mapsto f(t, x)$. Suppose that f is mean almost automorphic in $t \in \mathbb{R}$ for each $x \in L(\Omega, \mathbb{B})$ and satisfies the following Lipschitz condition :*

$$\mathbb{E}\|f(t,x) - f(t,y)\| \leqslant L.\mathbb{E}\|x - y\|$$

for all $x, y \in L(\Omega, \mathbb{B})$ and for each $t \in \mathbb{R}$, where $L > 0$ does not depend on t. Then, for any mean almost automorphic process $x : \mathbb{R} \to L(\Omega, \mathbb{B})$, the process $F : \mathbb{R} \to L(\Omega, \mathbb{B})$ defined by $F(t) := f(t, x(t))$ is also mean almost automorphic.

Proof. The proof of this theorem is similar to the proof of theorem 2.6 in [16]. □

Lemma 3.12. . *Assume $(\boldsymbol{H}_3)$ holds and let $g \in SBC\left(\mathbb{R}; L(\Omega, \mathbb{B})\right)$. Then $g \in \mathcal{E}\left(\mathbb{R}; L(\Omega, \mathbb{B}), \eta_1, \eta_2, \infty\right)$ if and only if for any $\varepsilon > 0$,*

$$\lim_{\zeta \to +\infty} \frac{\eta_1\left(\mathcal{K}^{\zeta,\varepsilon}(g)\right)}{\eta_2([-\zeta, \zeta])} = 0$$

where

$$\mathcal{K}^{\zeta,\varepsilon}(g) = \left\{ t \in [-\zeta, \zeta] : \sup_{\theta \in]-\infty, t]} \mathbb{E}\|g(\theta)\| \geq \varepsilon \right\}$$

Proof. On the one hand, suppose that $g \in \mathcal{E}\left(\mathbb{R}; L(\Omega, \mathbb{B}), \eta_1, \eta_2, \infty\right)$, it follows that

$$\begin{aligned}
\frac{1}{\eta_2([-\zeta,\zeta])} \int_{-\zeta}^{\zeta} \sup_{\theta \in]-\infty,t]} \mathbb{E}\|g(\theta)\| d\eta_1(t) &= \frac{1}{\eta_2([-\zeta,\zeta])} \int_{\mathcal{K}^{\zeta,\varepsilon}(g)} \left(\sup_{\theta \in]-\infty,t]} \mathbb{E}\|g(\theta)\| \right) d\eta_1(t) \\
&+ \frac{1}{\eta_2([-\zeta,\zeta])} \int_{[-\zeta,\zeta] \setminus \mathcal{K}^{\zeta,\varepsilon}(g)} \left(\sup_{\theta \in]-\infty,t]} \mathbb{E}\|g(\theta)\| \right) d\eta_1(t) \\
&\geq \frac{1}{\eta_2([-\zeta,\zeta])} \int_{\mathcal{K}^{\zeta,\varepsilon}(g)} \left(\sup_{\theta \in]-\infty,t]} \mathbb{E}\|g(\theta)\| \right) d\eta_1(t) \\
&\geq \frac{\varepsilon \eta_1(\mathcal{K}^{\zeta,\varepsilon}(g))}{\eta_2([-\zeta,\zeta])}.
\end{aligned}$$

Then

$$\lim_{\zeta \to +\infty} \frac{\eta_1\left(\mathcal{K}^{\zeta,\varepsilon}(g)\right)}{\eta_2([-\zeta, \zeta])} = 0.$$

On the other hand, Suppose that $g \in SBC\left(\mathbb{R}; L(\Omega, \mathbb{B})\right)$ such that,

$$\lim_{\zeta \to +\infty} \frac{\eta_1\left(\mathcal{K}^{\zeta,\varepsilon}(g)\right)}{\eta_2([-\zeta, \zeta])} = 0.$$

Since $g \in SBC\left(\mathbb{R}; L(\Omega,\mathbb{B}\right)$, for all $t\in\mathbb{R}$, there exists a constant $k_0>0$ such that $\mathbb{E}\|g(t)\|\leq k_0$, then using $(\boldsymbol{H}_3)$, it follows that

$$\begin{aligned}\frac{1}{\eta_2([-\zeta,\zeta])}\int_{-\zeta}^{\zeta}\sup_{\theta\in]-\infty,t]}\mathbb{E}\|g(\theta)\|d\eta_1(t) &= \frac{1}{\eta_2([-\zeta,\zeta])}\int_{\mathcal{K}^{\zeta,\varepsilon}(g)}\left(\sup_{\theta\in]-\infty,t]}\mathbb{E}\|g(\theta)\|\right)d\eta_1(t)\\ &\quad+\frac{1}{\eta_2([-\zeta,\zeta])}\int_{[-\zeta,\zeta]\setminus\mathcal{K}^{\zeta,\varepsilon}(g)}\left(\sup_{\theta\in]-\infty,t]}\mathbb{E}\|g(\theta)\|\right)d\eta_1(t);\\ &\leq \frac{k_0\eta_1(\mathcal{K}^{\zeta,\varepsilon}(g))}{\eta_2([-\zeta,\zeta])}+\frac{\varepsilon\eta_1([-\zeta,\zeta])}{\eta_2([-\zeta,\zeta])}.\end{aligned}$$

Which implies that

$$\lim_{\zeta\to+\infty}\frac{1}{\eta_2([-\zeta,\zeta])}\int_{-\zeta}^{\zeta}\sup_{\theta\in]-\infty,t]}\mathbb{E}\|g(\theta)\|d\eta_1(t)\leq\alpha\varepsilon,\quad\forall\varepsilon>0.$$

Then $g\in\mathcal{E}\left(\mathbb{R}; L(\Omega,\mathbb{B}),\eta_1,\eta_2,\infty\right)$. □

Theorem 3.13. *Let $\eta_1,\eta_2\in\mathcal{M}$, $\Psi=\Psi_1+\Psi_2\in SPAA\left(\mathbb{R}\times L(\Omega,\mathbb{B}); L(\Omega,\mathbb{B}),\eta_1,\eta_2,\infty\right)$ with $\Psi_1\in SAA\left(\mathbb{R}\times L(\Omega,\mathbb{B}); L(\Omega,\mathbb{B})\right)$, $\Psi_2\in\mathcal{E}\left(\mathbb{R}\times L(\Omega,\mathbb{B}); L(\Omega,\mathbb{B}),\eta_1,\eta_2,\infty\right)$ and $h\in SPAA\left(\mathbb{R}; L(\Omega,\mathbb{B}),\eta_1,\eta_2,\infty\right)$. Assume:*

i) *$\Psi_1(t,x)$ is uniformly continuous on any bounded subset uniformly for $t\in\mathbb{R}$.*

ii) *there exists a nonnegative function $L_\Psi\in L^p(\mathbb{R}),(1\leq p\leq\infty)$ such that*

$\mathbb{E}\left\|\Psi\left(t,x_1\right)-\Psi\left(t,x_2\right)\right\|\leq L_\Psi(t)\mathbb{E}\left\|x_1-x_2\right\|$, *for all $t\in\mathbb{R}$ and for all $x_1,x_2\in L(\Omega,\mathbb{B})$.*

If

$$\beta=\lim_{\zeta\to+\infty}\frac{1}{\eta_2([-\zeta,\zeta])}\int_{-\zeta}^{\zeta}\left(\sup_{\theta\in]-\infty,t]}L_\Psi(\theta)\right)d\eta_1(t)<\infty$$

then the function $t\mapsto f(t,h(t))$ belongs to $SPAA\left(\mathbb{R}; L(\Omega,\mathbb{B}),\eta_1,\eta_2,\infty\right)$.

Proof. The proof of this theorem follows by the same reasoning as in Lemma 3.2 in [24]. Let $h\in SPAA(\mathbb{R},L(\Omega,\mathbb{B}),\eta_1,\eta_2,\infty)$, we can write h as $h=h_1+h_2$, where $h_1\in SAA(\mathbb{R},L(\Omega,\mathbb{B}))$ and $h_2\in\mathcal{E}(\mathbb{R},L(\Omega,\mathbb{B}),\eta_1,\eta_2,\infty)$. Since $\Psi\in SPAA(\mathbb{R},L(\Omega,\mathbb{B}),\eta_1,\eta_2,\infty)$, then $\Psi_1+\Psi_2$, where $\Psi_1\in SAA(\mathbb{R}\times L(\Omega,\mathbb{B}),L(\Omega,\mathbb{B}))$ and $\Psi_2\in\mathcal{E}(\mathbb{R}\times L(\Omega,\mathbb{B}),L(\Omega,\mathbb{B}),\eta_1,\eta_2,\infty)$. So, stochastic process Ψ can be decomposed as follows
$\Psi(t,h(t))=\Psi_1(t,h_1(t))+[\Psi(t,h(t))-\Psi(t,h_1(t))]+[\Psi(t,h_1(t))-\Psi_1(t,h_1(t))$.
Since $\Psi_2(t,h_1(t))=\Psi(t,h_1(t))-\Psi_1(t,h_1(t))$, then

$$\Psi(t,h(t))=\Psi_1(t,h_1(t))+[\Psi(t,h(t))-\Psi(t,h_1(t))]+\Psi_2(t,h_1(t)).$$

Using Theorem 3.11, we obtain $t\mapsto\Psi_1(t,h_1(t))\in SAA(\mathbb{R},L(\Omega,\mathbb{B}))$. Now, it remains to show that the functions $t\mapsto[\Psi(t,h(t))-\Psi(t,h_1(t))]$ and $t\mapsto\Psi_2(t,h_1(t))$ belong to $\mathcal{E}(\mathbb{R},L(\Omega,\mathbb{B}),\eta_1,\eta_2,\infty)$. The function $t\mapsto[\Psi(t,h(t))-\Psi(t,h_1(t))]$ is bounded and continuous. Suppose that $\mathbb{E}\|\Psi(t,h(t))-\Psi(t,h_1(t))\|\leq N_0,\forall t\in\mathbb{R}$. Since $t\mapsto h(t)$ and $t\mapsto h_1(t)$ are bounded, choose a bounded subset $B\subset\mathbb{R}$ such that $h(\mathbb{R}),h_1(\mathbb{R})\subset B$. Under assumption (ii), for given $\varepsilon>0,\mathbb{E}\|x_1-x_2\|\leq\varepsilon$,

implies that $\mathbb{E}\left\|\Psi\left(t, x_1\right)-\Psi\left(t, x_2\right)\right\| \leq \varepsilon L_{\Psi}(t)$, for all $t \in \mathbb{R}$.
For $g \in \mathcal{E}(\mathbb{R} \times L(\Omega, \mathbb{B}), L(\Omega, \mathbb{B}), \eta_1, \eta_2, \infty)$, using Lemma 3.12, it follows that

$$\lim_{\zeta \rightarrow+\infty} \frac{\eta_1\left(\mathcal{K}^{\zeta, \varepsilon}(g)\right)}{\eta_2([-\zeta, \zeta])}=0.$$

Consequently

$$\begin{aligned}
& \frac{1}{\eta_2([-\zeta, \zeta])} \int_{-\zeta}^{\zeta}\left(\sup_{\theta \in]-\infty, t]} \mathbb{E}\|\Psi(\theta, h(\theta))-\Psi(\theta, h_1(\theta))\|\right) d \eta_1(t) \\
& =\frac{1}{\eta_2([-\zeta, \zeta])} \int_{\mathcal{K}^{\zeta, \varepsilon}(g)}\left(\sup_{\theta \in]-\infty, t]} \mathbb{E}\|\Psi(\theta, h(\theta))-\Psi(\theta, h_1(\theta))\|\right) d \eta_1(t) \\
& +\frac{1}{\eta_2([-\zeta, \zeta])} \int_{[-\zeta, \zeta] \backslash \mathcal{K}^{\zeta, \varepsilon}(g)}\left(\sup_{\theta \in]-\infty, t]} \mathbb{E}\|\Psi(\theta, h(\theta))-\Psi(\theta, h_1(\theta))\|\right) d \eta_1(t) ; \\
& \leq \frac{N_0}{\eta_2([-\zeta, \zeta])} \int_{\mathcal{K}^{\zeta, \varepsilon}(g)} d \eta_1(t)+\frac{\varepsilon}{\eta_2([-\zeta, \zeta])} \int_{[-\zeta, \zeta] \backslash \mathcal{K}^{\zeta, \varepsilon}(g)}\left(\sup_{\theta \in]-\infty, t]}\left|L_{\Psi}(\theta)\right|\right) \eta_1(t) ; \\
& \leq \frac{N_0 \eta_1(\mathcal{K}^{\zeta, \varepsilon}(g))}{\eta_2([-\zeta, \zeta])}+\frac{\varepsilon}{\eta_2([-\zeta, \zeta])} \int_{[-\zeta, \zeta]}\left(\sup_{\theta \in]-\infty, t]}\left|L_{\Psi}(\theta)\right|\right) d \eta_1(t),
\end{aligned}$$

which implies that

$$\lim_{\zeta \rightarrow+\infty} \frac{1}{\eta_2([-\zeta, \zeta])} \int_{-\zeta}^{\zeta}\left(\sup_{\theta \in]-\infty, t]} \mathbb{E}\|\Psi(\theta, h(\theta))-\Psi(\theta, h_1(\theta))\|\right) d \eta_1(t) \leq \varepsilon \beta, \quad \forall \varepsilon>0.$$

Then function $t \mapsto[\Psi(t, h(t))-\Psi(t, h_1(t))]$ belongs to $\mathcal{E}(\mathbb{R}, L(\Omega, \mathbb{B}), \eta_1, \eta_2, \infty)$. To complete the proof, it is enough to prove that $t \mapsto \Psi_2(t, h_1(t))$ belongs to $\mathcal{E}(\mathbb{R}, L(\Omega, \mathbb{B}), \eta_1, \eta_2, \infty)$. Since Ψ_2 is uniformly continuous on the compact set $\mathbb{K}=\overline{\{h_1(t): t \in \mathbb{R}\}}$ with respect to the second variable x, it follows that for given $\varepsilon>0$, there exists $\delta>0$ such that , for all $t \in \mathbb{R}$, $\xi_1, \xi_2 \in \mathbb{K}$, one has

$$\mathbb{E}\|\xi_1-\xi_2\| \leq \delta \Rightarrow \mathbb{E}\|\Psi(t, \xi_1(t))-\Psi(t, \xi_2(t))\| \leq \varepsilon.$$

Therefore, there exists $n(\varepsilon)$ and $\{z_i\}_{i=1}^{n(\varepsilon)} \subset \mathbb{K}$ such that

$$\mathbb{K} \subset \bigcup_{i=1}^{n(\varepsilon)} B_\delta\left(z_i, \delta\right).$$

Let $t \in \mathbb{R}$ and $z \in \mathbb{K}$, there exists $i_0 \in\{1, \ldots, n(\varepsilon)\}$ such that
$\mathbb{E}\|\Psi_2(t, z)-\Psi_2(t, z_{i_0}(t))\| \leq \varepsilon$,
it results that

$$\mathbb{E}\|\Psi_2(t, h_1(t))\| \leq \mathbb{E}\|\Psi_2(t, h_1(t))-\Psi_2(t, z_{i_0}(t))\|+\mathbb{E}\|\Psi_2(t, z_{i_0}(t))\|$$

$$\mathbb{E}\|\Psi_2(t, h_1(t))\| \leq \varepsilon+\mathbb{E}\|\Psi_2(t, z_{i_0}(t))\| \leq \varepsilon+\sum_{i=1}^{n(\varepsilon)} \mathbb{E}\|\Psi_2(t, z_{i_0}(t))\|.$$

It follows that

$$\begin{aligned}&\frac{1}{\eta_2([-\zeta,\zeta])}\int_{-\zeta}^{\zeta}\sup_{\theta\in]-\infty,t]}\mathbb{E}\|\Psi_2(\theta,h_1(\theta))\|d\eta_1(t)\\&\leq\frac{\varepsilon\eta_1([-\zeta,\zeta])}{\eta_2([-\zeta,\zeta])}+\sum_{i=1}^{n(\varepsilon)}\frac{1}{\eta_2([-\zeta,\zeta])}\int_{-\zeta}^{\zeta}\sup_{\theta\in]-\infty,t]}\mathbb{E}\|\Psi_2(\theta,z_i(\theta))\|d\eta_1(t).\end{aligned}$$

In fact

$$\forall i\in\{1,\dots,n(\varepsilon)\},\sum_{i=1}^{n(\varepsilon)}\frac{1}{\eta_2([-\zeta,\zeta])}\int_{-\zeta}^{\zeta}\sup_{\theta\in]-\infty,t]}\mathbb{E}\|\Psi_2(\theta,z_i(\theta))\|d\eta_1(t)=0,$$

then, using $(\mathbf{H}_4)$ we deduce that

$$\limsup_{\zeta\to+\infty}\frac{1}{\eta_2([-\zeta,\zeta])}\int_{-\zeta}^{\zeta}\sup_{\theta\in]-\infty,t]}\mathbb{E}\|\Psi_2(\theta,h_1(\theta))\|\eta_1(t)\leq\varepsilon\alpha,\quad\forall\varepsilon>0.$$

It results that

$$t\mapsto\Psi_2(t,h_1(t))\in\mathcal{E}(\mathbb{R},L(\Omega,\mathbb{B}),\eta_1,\eta_2,\infty).$$

□

4. Existence and uniqueness of solution

Proposition 4.1. [17] *Suppose that $\{N(t):t\geq 0\}$ inhomogeneous Poisson process with intensity function $\{\lambda^*(t):t\geq 0\}$. Let $h:\mathbb{R}\to L(\Omega,\mathbb{B}),s\mapsto h(s)$ be the mean square integrability on $0\leq t_0\leq t$ and $\mathcal{F}_t^N$-predictable process such that*

$$\mathbb{E}\left[\int_{t_0}^{t}\|h^2(s)\|ds\right]<\infty\ \text{ and }\ \mathbb{E}[dN(s)]=\lambda^*(s)ds.$$

Then

$$\mathbb{E}\left[\left\|\int_{t_0}^{t}h(s)dN(s)\right\|\right]\leq\int_{t_0}^{t}\mathbb{E}[\|h(s)\|]\lambda^*(s)ds.$$

Assume the following $(\mathbf{H}_5)$: h is a bounded process.

Theorem 4.2. *Let $\mathscr{B}$ be the phase space satisfying conditions $(\boldsymbol{A}_1),(\boldsymbol{A}_2),(\boldsymbol{B}),(\boldsymbol{C}_1),(\boldsymbol{H}_0),(\boldsymbol{H}_1),(\boldsymbol{E})$ and $(\boldsymbol{H}_5)$. Assume in addition that the family $(\mathcal{V}(t))_{t\geq0}$ possesses a hyperbolic structure. If g is bounded and continuous on $\mathbb{R}$, then there exists a unique bounded solution v of (1.1) on $\mathbb{R}$ given by*

$$\begin{aligned}v_t=&\left[\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{V}^S(t-s)\Gamma^S\widetilde{D}_\lambda(X_0g(s))ds+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{V}^U(t-s)\Gamma^U\widetilde{D}_\lambda(X_0g(s))ds\right](0)\\&+\left[\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{V}^S(t-s)\Gamma^S\widetilde{D}_\lambda(X_0h(s))dN(s)+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{V}^U(t-s)\Gamma^U\widetilde{D}_\lambda(X_0h(s))dN(s)\right](0),\end{aligned}$$

$\forall t\geq 0$ with $\widetilde{D}_\lambda=\lambda\left(\lambda I-\widetilde{\mathcal{A}}_{\mathcal{V}}\right)^{-1}$, Γ^S and Γ^U are the projections of $\mathscr{B}$ on the stable and unstable subspaces.

Proof. Let $v_t = w_1(t) + w_2(t)$, where

$$w_1(t) = \lim_{\lambda\to+\infty}\int_{-\infty}^{t} \mathcal{V}^S(t-s)\Gamma^S \widetilde{D}_\lambda(X_0 g(s)(0))ds + \lim_{\lambda\to+\infty}\int_{+\infty}^{t} \mathcal{V}^U(t-s)\Gamma^U \widetilde{D}_\lambda(X_0 g(s)(0))ds$$

and

$$\begin{aligned} w_2(t) = & \lim_{\lambda\to+\infty}\int_{-\infty}^{t} \mathcal{V}^S(t-s)\Gamma^S \widetilde{D}_\lambda(X_0 h(s)(0))dN(s) \\ & + \lim_{\lambda\to+\infty}\int_{+\infty}^{t} \mathcal{V}^U(t-s)\Gamma^U \widetilde{D}_\lambda(X_0 h(s)(0))dN(s), \ \ \forall t \geq 0. \end{aligned}$$

Firstly, we are going to prove that v_t exists.
On the one hand, we show the existence of $w_1(t)$. Let $w_1(t) = \lim_{\lambda\to+\infty} \alpha(t)$, where

$$\alpha(t) = \int_{-\infty}^{t} \mathcal{V}^S(t-s)\Gamma^S \widetilde{D}_\lambda(X_0 g(s)(0))ds + \int_{+\infty}^{t} \mathcal{V}^U(t-s)\Gamma^U \widetilde{D}_\lambda(X_0 g(s)(0))ds.$$

Using the triangular inequality and Bochner's Theorem, we obtain

$$\|\alpha(t)\| \leq \int_{-\infty}^{t} \|\mathcal{V}^S(t-s)\Gamma^S \widetilde{D}_\lambda(X_0 g(s)(0))\|ds + \int_{t}^{+\infty} \|\mathcal{V}^U(t-s)\Gamma^U \widetilde{D}_\lambda(X_0 g(s)(0))\|ds.$$

By using the Propositions 2.8 and 2.13, it follows that

$$\mathbb{E}\|\alpha(t)\| \leq MM_0|\Gamma^S|\sum_{n=1}^{\infty}\int_{t-n}^{t-n+1} e^{-\omega(t-s)}\mathbb{E}\|g(s)\|ds + MM_0|\Gamma^U|\sum_{n=1}^{\infty}\int_{t+n-1}^{t+n} e^{\omega(t-s)}\mathbb{E}\|g(s)\|ds.$$

Function g being bounded on $\mathbb{R}$, there exists a positive real β such that
$\mathbb{E}\|g(s)\| \leq \beta$, for all $s \in \mathbb{R}$. Hence

$$\mathbb{E}\|\alpha(t)\| \leq \frac{1}{\omega}(e^\omega - 1)\beta MM_0(|\Gamma^S| + |\Gamma^U|)\sum_{n=1}^{\infty} e^{-\omega n}.$$

Since $\sum\limits_{n=1}^{\infty} e^{-\omega n} < \infty$, we deduce that $w_1(t)$ exists and is bounded on $\mathbb{R}$.
On the other hand, we will show that $w_2(t)$ exists.
First we show that $\lim\limits_{\lambda\to+\infty}\int_{-\infty}^{t} \mathcal{V}^S(t-s)\Gamma^S \widetilde{D}_\lambda(X_0 h(s)(0))dN(s)$ exists.
Since $\{N(t) : t \geq 0\}$ is inhomogeneous Poisson process with intensity function

$\{\lambda^*(t) : t \geq 0\}$, by using the Propositions 2.8, 2.13 and 4.1, we obtain

$$\begin{aligned}
&\mathbb{E}\left\| \int_{-\infty}^{t} \mathcal{V}^S(t-s)\Gamma^S \widetilde{D}_\lambda(X_0 h(s)(0))dN(s)\right\| \\
&\leq MM_0|\Gamma^S| \left[\int_{-\infty}^{t} e^{-\omega(t-s)}\lambda^*(s)\mathbb{E}\|h(s)\|ds\right] \\
&\leq MM_0|\Gamma^S| \left[\sum_{n=1}^{\infty}\int_{t-n}^{t-n+1} e^{-\omega(t-s)} \sup_{s\in]-\infty,t]}(\lambda^*(s))\mathbb{E}\|h(s)\|ds\right] \\
&\leq MM_0|\Gamma^S| \sup_{s\in]-\infty,t]}(\lambda^*(s))\sum_{n=1}^{\infty}\int_{t-n}^{t-n+1} e^{-\omega(t-s)}\mathbb{E}\|h(s)\|ds.
\end{aligned}$$

Using $(\mathbf{H}_5)$, there exists positive constant $\mathcal{C}$ such that $\mathbb{E}\|h(s)\| \leq \mathcal{C}$, it follows that

$$\begin{aligned}
\mathbb{E}\left\| \int_{-\infty}^{t} \mathcal{V}^S(t-s)\Gamma^S \widetilde{D}_\lambda(X_0 h(s)(0))dN(s)\right\| &\leq \mathcal{C}MM_0|\Gamma^S| \sup_{s\in]-\infty,t]}(\lambda^*(s))\sum_{n=1}^{\infty}\int_{t-n}^{t-n+1} e^{-\omega(t-s)}ds \\
&\leq \mathcal{C}MM_0|\Gamma^S| \sup_{s\in]-\infty,t]}(\lambda^*(s))\frac{1}{\omega}(e^{\omega}-1)\sum_{n=1}^{\infty} e^{-\omega n}.
\end{aligned}$$

Since $\sum\limits_{n=1}^{\infty} e^{-\omega n} < \infty$, then there exists positive constant c such that $\sum\limits_{n=1}^{\infty} e^{-\omega n} \leq c$. It results that

$$\mathbb{E}\left\| \int_{-\infty}^{t} \mathcal{V}^S(t-s)\Gamma^S \widetilde{D}_\lambda(X_0 h(s)(0))dN(s)\right\| \leq \gamma, \tag{4.1}$$

where $\gamma = \mathcal{C}MM_0|\Gamma^S|\frac{c}{\omega}(e^{\omega}-1) \sup\limits_{s\in]-\infty,t]}(\lambda^*(s))$.

Let $H_1(n,s,t) = \mathcal{V}^S(t-s)\Gamma^S \widetilde{D}_n(X_0 h(s)(0))$, for $n \in \mathbb{N}$ and $s \leq t$. For n large enough and $\sigma \leq t$, we have

$$\mathbb{E}\left\| \int_{-\infty}^{\sigma} H_1(n,s,t)dN(s)\right\| \leq \mathcal{C}MM_0|\Gamma^S| \sup_{s\in]-\infty,t]}(\lambda^*(s))\sum_{n=1}^{\infty}\int_{t-n}^{t-n+1} e^{-\omega(t-s)}ds \leq \gamma e^{-\omega(t-\sigma)},$$

where $\gamma = \mathcal{C}MM_0|\Gamma^S|\frac{c}{\omega}(e^{\omega}-1) \sup\limits_{s\in]-\infty,t]}(\lambda^*(s))$. Thus, it results

$$\begin{aligned}
&\mathbb{E}\left\| \int_{-\infty}^{t} H_1(n,s,t)dN(s) - \int_{-\infty}^{t} H_1(m,s,t)dN(s)\right\| \\
&\leq \mathbb{E}\left\| \int_{-\infty}^{\sigma} H_1(n,s,t)dN(s)\right\| + \mathbb{E}\left\| \int_{-\infty}^{\sigma} H_1(m,s,t)dN(s)\right\| \\
&+ \mathbb{E}\left\| \int_{\sigma}^{t} H_1(n,s,t)dN(s) - \int_{\sigma}^{t} H_1(m,s,t)dN(s)\right\| \\
&\leq 2\gamma e^{-\omega(t-\sigma)} + \mathbb{E}\left\| \int_{\sigma}^{t} H_1(n,s,t)dN(s) - \int_{\sigma}^{t} H_1(m,s,t)dN(s)\right\|.
\end{aligned}$$

It follows that

$$\limsup_{n,m\to+\infty}\mathbb{E}\left\|\int_{-\infty}^{t} H_1(n,s,t)dN(s)-\int_{-\infty}^{t} H_1(m,s,t)dN(s)\right\|\le 2\gamma e^{-\omega(t-\sigma)}. \quad (4.2)$$

If $\sigma\to-\infty$, using inequality (4.2), we obtain

$$\limsup_{n,m\to+\infty}\mathbb{E}\left\|\int_{-\infty}^{t} H_1(n,s,t)dN(s)-\int_{-\infty}^{t} H_1(m,s,t)dN(s)\right\|=0.$$

Therefore, we deduce that function

$$t\mapsto \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{V}^S(t-s)\Gamma^S\widetilde{D}_\lambda(X_0h(s)(0))dN(s)$$

exists and is bounded on $\mathbb{R}$. Similarly, we can show that function

$$t\mapsto \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{V}^U(t-s)\Gamma^U\widetilde{D}_\lambda(X_0h(s)(0))dN(s)$$

is also well defined and bounded on $\mathbb{R}$. Thus, v_t exists and is bounded on $\mathbb{R}$.
Finally, we will prove the uniqueness of v_t.
since v_t is a bounded solution on $\mathbb{R}$ of equation (1.1), suppose that there exists another bounded solution $\widetilde{v}_t$ on $\mathbb{R}$ of equation (1.1).
Let $t\ge\sigma$, $v_t-\widetilde{v}_t=\mathcal{V}(t-\sigma)(v_\sigma-\widetilde{v}_\sigma)$ and $|\Gamma^S(v_t-\widetilde{v}_t)|\le M_0e^{-\omega(t-\sigma)}|\Gamma^S(v_\sigma-\widetilde{v}_\sigma)|$.
Since $v_\sigma-\widetilde{v}_\sigma$ is bounded on $\mathbb{R}$, when $\sigma\to-\infty$, we obtain $\Gamma^S(v_t-\widetilde{v}_t)=0$. Hence $(v_t-\widetilde{v}_t)\in S$. Given that $\mathcal{V}^U(t)$ is group, then for $t\le\sigma$, we have
$v_t-\widetilde{v}_t=\mathcal{V}^U(t-\sigma)(v_\sigma-\widetilde{v}_\sigma)$. Therefore $|v_t-\widetilde{v}_t|\le M_0e^{\omega(t-\sigma)}|v_\sigma-\widetilde{v}_\sigma|$. When $\sigma\to+\infty$, we obtain $v_t=\widetilde{v}_t,\ \ \forall t\in\mathbb{R}$. Hence the uniqueness of v_t. ☐

Theorem 4.3. *Assume that* $(\boldsymbol{H}_3)$ *holds. Let* $\eta_1,\eta_2\in\mathcal{M}$ *and* $\phi\in SPAA_c(\mathbb{R},L(\Omega,\mathbb{B}),\eta_1,\eta_2,\infty)$ *then the function* $t\mapsto\phi_t$, *belongs to* $SPAA_c(C(]-\infty,0],L(\Omega,\mathbb{B})),L(\Omega,\mathbb{B}),\eta_1,\eta_2,\infty)$.

Proof. Let $L>0$ and assume that $\phi=g+h$, where $g\in SAA_c(\mathbb{R},L(\Omega,\mathbb{B}))$ and $h\in\mathcal{E}(\mathbb{R},L(\Omega,\mathbb{B}),\eta_1,\eta_2,\infty)$. We have $\phi_t=g_t+h_t$. Firstly, we show that $g_t\in SAA_c(\mathbb{R},L(\Omega,\mathbb{B}))$. Let $(s_m)_{m\in\mathbb{N}}$ of real numbers, fix a subsequence $(s_n)_{n\in\mathbb{N}}$ and $w\in SBC(\mathbb{R},L(\Omega,\mathbb{B}))$ such that $g(s+s_n)\longrightarrow w(s)$ uniformly on compact subsets of $\mathbb{R}$. Let $\mathbb{K}\subset[-L;L]$. For $\varepsilon>0$ fix $N_{\varepsilon,L}\in\mathbb{N}$ such that $\mathbb{E}\|g(s+s_n)-w(s)\|\leqslant\varepsilon$ for $s\in[-L;L]$. Whenever $n\geqslant N_{\varepsilon,L}$. For $t\in\mathbb{K}$ and $n\geqslant N_{\varepsilon,L}$ we have

$$\begin{aligned}\mathbb{E}\left\|g_{t+s_n}-w_t\right\| &\leqslant \sup_{\theta\in[-L;L]}\mathbb{E}\left\|g(\theta+s_n)-w(\theta)\right\|\\ &\leqslant\varepsilon\end{aligned}$$

then, g_{t+s_n} converges to w_t uniformly in $\mathbb{K}$. Similarly, we can show prove that w_{t-s_n} converges to g_t uniformly in $\mathbb{K}$. Now, we show that $h_t\in\mathcal{E}(\mathbb{R},L(\Omega,\mathbb{B}),\eta_1,\eta_2,\infty)$, we denote by

$$\mathscr{R}_\alpha=\frac{1}{\eta_{2,\alpha}([-\zeta,\zeta])}\int_{-\zeta}^{\zeta}\left(\sup_{\theta\in]-\infty,t]}\mathbb{E}\|h(\theta)\|\right)d\eta_{1,\alpha}(t)$$

where $\eta_{2,\alpha}$ is the positive measure defined by (2.4). Assumption $(\mathbf{H}_2)$ being satisfied, using the

Proposition 2.22, we deduce that $\eta_2 \sim \eta_{2,\alpha}$ and $\eta_1 \sim \eta_{1,\alpha}$, using the Theorem 2.21, we obtain

$$\mathcal{E}(\mathbb{R}, L(\Omega,\mathbb{B}), \eta_{1,\alpha}, \eta_{2,\alpha}, \infty) = \mathcal{E}(\mathbb{R}, L(\Omega,\mathbb{B}), \eta_1, \eta_2, \infty).$$

Consequently $h \in \mathcal{E}(\mathbb{R}, L(\Omega,\mathbb{B}), \eta_{1,\alpha}, \eta_{2,\alpha}, \infty)$, hence $\lim\limits_{\zeta\to+\infty} \mathscr{R}_\alpha(\zeta) = 0,\ \forall\alpha \in \mathbb{R}$. Let

$$\mathcal{Q} = \frac{1}{\eta_2([-\zeta,\zeta])}\int_{-\zeta}^{\zeta} \sup_{\theta\in]t-r,t]} \mathbb{E}\|h(\theta)\| d\eta_1(t),$$

for $r > 0$, it follows that

$$\begin{aligned}
&\frac{1}{\eta_2([-\zeta,\zeta])}\int_{-\zeta}^{\zeta} \sup_{\theta\in]-\infty,t]} \left(\sup_{\eta\in]-\infty,0]} \mathbb{E}\|h(\theta)\|\right) d\eta_1(t)\\
&\leq \frac{1}{\eta_2([-\zeta,\zeta])}\int_{-\zeta}^{\zeta} \left(\sup_{\theta\in]-\infty,t-r]} \mathbb{E}\|h(\theta)\| + \sup_{\theta\in]t-r,t]} \mathbb{E}\|h(\theta)\|\right) d\eta_1(t);\\
&\leq \frac{1}{\eta_2([-\zeta,\zeta])}\int_{-\zeta}^{\zeta} \sup_{\theta\in]-\infty,t-r]} \mathbb{E}\|h(\theta)\| d\eta_1(t) + \mathcal{Q};\\
&\leq \frac{1}{\eta_2([-\zeta,\zeta])}\int_{-\zeta-r}^{\zeta+r} \sup_{\theta\in]-\infty,t]} \mathbb{E}\|h(\theta)\| d\eta_1(t+r) + \mathcal{Q};\\
&\leq \frac{\eta_1([-\zeta-r,\zeta+r])}{\eta_2([-\zeta,\zeta])} \times \mathscr{R}_b(\zeta+r) + \mathcal{Q},
\end{aligned}$$

using Proposition 2.23, we deduce that $h_t \in \mathcal{E}\left(\mathbb{R}, L(\Omega,\mathbb{B}), \eta_1, \eta_2, \infty\right)$.
Then function $t \mapsto \phi_t$ belongs to $SPAA(C(]-\infty,0]), L(\Omega,\mathbb{B}), \eta_1, \eta_2, \infty)$. □

Theorem 4.4. *Let $g, h \in SAA_c(\mathbb{R}, L(\Omega,\mathbb{B}))$ and Γ be the mapping defined, for $t \in \mathbb{R}$ by*

$$\begin{aligned}
&\Gamma(g,h)(t)\\
&= \Bigg[\lim_{\lambda\to+\infty}\int_{-\infty}^{t} \mathcal{V}^S(t-s)\Gamma^S\widetilde{D}_\lambda(X_0 g(s)ds + \lim_{\lambda\to+\infty}\int_{+\infty}^{t} \mathcal{V}^U(t-s)\Gamma^U\widetilde{D}_\lambda(X_0 g(s))ds\\
&+ \lim_{\lambda\to+\infty}\int_{-\infty}^{t} \mathcal{V}^S(t-s)\Gamma^S\widetilde{D}_\lambda(X_0 h(s))dN(s) + \lim_{\lambda\to+\infty}\int_{+\infty}^{t} \mathcal{V}^U(t-s)\Gamma^U\widetilde{D}_\lambda(X_0 h(s))dN(s)\Bigg](0).
\end{aligned}$$

Then $\Gamma(g,h) \in SAA_c(\mathbb{R}, L(\Omega,\mathbb{B}))$.

Proof. For given $(s_m)_{m\in\mathbb{N}}$ of real numbers, fix a subsequence $(s_n)_{n\in\mathbb{N}}$ and $g, h \in SBC\left(\mathbb{R}, L(\Omega,\mathbb{B})\right)$ such that $f(t+s_n)$ converges to $g(t)$ and $g(t-s_n)$ converges to $f(t)$,and also $v(t+s_n)$ converges to $h(t)$ and $h(t-s_n)$ converges to $v(t)$ uniformly on compact subsets of $\mathbb{R}$. By using Propositions 2.8 and 2.13, we get the following estimates

$$\lim_{\lambda\to+\infty} \|\mathcal{V}^S(t-s)\Gamma^S\widetilde{D}_\lambda(X_0 g(s))(0)\| \leq MM_0|\Gamma^S|e^{-\omega(t-s)}\|g(s)\| \tag{4.3}$$

$$\lim_{\lambda\to+\infty} \|\mathcal{V}^U(t-s)\Gamma^U\widetilde{D}_\lambda(X_0 g(s))(0)\| \leq MM_0|\Gamma^U|e^{\omega(t-s)}\|g(s)\| \tag{4.4}$$

$$\lim_{\lambda\to+\infty}\left\|\mathcal{V}^S(t-s)\Gamma^S\widetilde{D}_\lambda(X_0h(s))(0)\right\| \leq MM_0|\Gamma^S|e^{-\omega(t-s)}\|h(s)\| \tag{4.5}$$

and

$$\lim_{\lambda\to+\infty}\left\|\mathcal{V}^U(t-s)\Gamma^U\widetilde{D}_\lambda(X_0h(s))(0)\right\| \leq MM_0|\Gamma^U|e^{\omega(t-s)}\|h(s)\| \tag{4.6}$$

It results that, if

$$\begin{aligned}
z(t+s_n) = \Bigg[&\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{V}^S(t-s)\Gamma^S\widetilde{D}_\lambda(X_0f(s+s_n)ds + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{V}^U(t-s)\Gamma^U\widetilde{D}_\lambda(X_0f(s+s_n))ds\\
&+\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{V}^S(t-s)\Gamma^S\widetilde{D}_\lambda(X_0v(s+s_n))dN(s)\\
&+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{V}^U(t-s)\Gamma^U\widetilde{D}_\lambda(X_0v(s+s_n))dN(s)\Bigg](0).
\end{aligned}$$

Then, by using (4.3) − (4.6) and the Lebesgue dominated convergence theorem, we have $z\,(t+s_n)$ that converges to $w(t)$.

$$\begin{aligned}
w(t) = \Bigg[&\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{V}^S(t-s)\Gamma^S\widetilde{D}_\lambda(X_0g(s))ds + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{V}^U(t-s)\Gamma^U\widetilde{D}_\lambda(X_0g(s))ds\\
&+\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{V}^S(t-s)\Gamma^S\widetilde{D}_\lambda(X_0h(s))dN(s) + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{V}^U(t-s)\Gamma^U\widetilde{D}_\lambda(X_0h(s))dN(s)\Bigg](0).
\end{aligned}$$

It remains to prove that the convergence is uniform on all compact subset of $\mathbb{R}$. Let $\mathbb{K}\subset\mathbb{R}$ be an arbitrary compact and let $\varepsilon>0$. We fix $L>0$ and $N_\varepsilon\in\mathbb{N}$ such that $\mathbb{K}\subset\left[\frac{-L}{2};\frac{L}{2}\right]$ with,

$$\int_{\frac{L}{2}}^{+\infty}e^{-\omega s}ds<\varepsilon$$

$$\mathbb{E}\left\|f\left(s+s_n\right)-g(s)\right\| \leq \varepsilon \text{ for } n\geqslant N_\varepsilon \text{ and } s\in[-L,L] \tag{4.7}$$

and

$$\mathbb{E}\left\|v\left(s+s_n\right)-h(s)\right\| \leq \varepsilon \text{ for } n\geqslant N_\varepsilon \text{ and } s\in[-L,L]. \tag{4.8}$$

Then, for each $t \in \mathbb{K}$, using Propositions 2.8, 2.13 and 4.1, we it results that

$$\begin{aligned}
&\mathbb{E}\|z(t+s_n)-w(t)\| \\
&\leq MM_0\Bigg[|\Gamma^S|\left(\int_{-\infty}^{-L}\mathbb{E}\|f(s+s_n)-g(s)\|e^{-\omega(t-s)}ds+\int_{-L}^{t}\mathbb{E}\|f(s+s_n)-g(s)\|e^{-\omega(t-s)}ds\right) \\
&+|\Gamma^U|\int_t^{+\infty}\mathbb{E}\|f(s+s_n)-g(s)\|e^{-\omega(s-t)}ds \\
&+|\Gamma^S|\sup_{s\in\mathbb{R}}(\lambda^*(s))\left(\int_{-\infty}^{-L}\mathbb{E}\|v(s+s_n)-h(s)\|e^{-\omega(t-s)}ds+\int_{-L}^{t}\mathbb{E}\|v(s+s_n)-h(s)\|e^{-\omega(t-s)}ds\right) \\
&+|\Gamma^U|\sup_{s\in\mathbb{R}}(\lambda^*(s))\int_t^{+\infty}\mathbb{E}\|v(s+s_n)-h(s)\|e^{-\omega(s-t)}ds\Bigg].
\end{aligned}$$

By using (4.7) and (4.8), it follows that

$$\begin{aligned}
&\mathbb{E}\|z(t+s_n)-w(t)\| \\
&\leq MM_0\Bigg[|\Gamma^S|\left(\varepsilon\int_{-\infty}^{-L}e^{-\omega(t-s)}ds+\varepsilon\int_{-L}^{t}e^{-\omega(t-s)}ds\right)+|\Gamma^U|\varepsilon\int_t^{+\infty}e^{-\omega(s-t)}ds \\
&+|\Gamma^S|\sup_{s\in\mathbb{R}}(\lambda^*(s))\left(\varepsilon\int_{-\infty}^{-L}e^{-\omega(t-s)}ds+\varepsilon\int_{-L}^{t}e^{-\omega(t-s)}ds\right)+|\Gamma^U|\sup_{s\in\mathbb{R}}(\lambda^*(s))\varepsilon\int_t^{+\infty}e^{-\omega(s-t)}ds\Bigg]; \\
&\leq MM_0\Bigg[|\Gamma^S|\left(\varepsilon\int_{t+L}^{+\infty}e^{-\omega s}ds+\varepsilon\int_{-L}^{t}e^{-\omega(t-s)}ds\right)+|\Gamma^U|\varepsilon\int_0^{+\infty}e^{-\omega s}ds \\
&+|\Gamma^S|\sup_{s\in\mathbb{R}}(\lambda^*(s))\left(\varepsilon\int_{t+L}^{+\infty}e^{-\omega s}ds+\varepsilon\int_{-L}^{t}e^{-\omega(t-s)}ds\right)+|\Gamma^U|\sup_{s\in\mathbb{R}}(\lambda^*(s))\varepsilon\int_0^{+\infty}e^{-\omega s}ds\Bigg],
\end{aligned}$$

Since $\int_{-L}^{t}e^{-\omega(t-s)}ds<+\infty$, there exists $\alpha\in\mathbb{R}^+$ such that $\int_{-L}^{t}e^{-\omega(t-s)}ds\leq\alpha$, we obtain

$$\begin{aligned}
&\mathbb{E}\|z(t+s_n)-w(t)\| \\
&\leq MM_0\Bigg[|\Gamma^S|\left(\varepsilon\int_{\frac{L}{2}}^{+\infty}e^{-\omega s}ds+\varepsilon\alpha\right)+|\Gamma^U|\frac{\varepsilon}{\omega} \\
&+|\Gamma^S|\sup_{s\in\mathbb{R}}(\lambda^*(s))\left(\varepsilon\int_{\frac{L}{2}}^{+\infty}e^{-\omega s}ds+\varepsilon\alpha\right)+|\Gamma^U|\sup_{s\in\mathbb{R}}(\lambda^*(s))\frac{\varepsilon}{\omega}\Bigg]; \\
&\leq MM_0\left[|\Gamma^S|\left((\varepsilon+\alpha)(1+\sup_{s\in\mathbb{R}}(\lambda^*(s))\right)+\frac{1}{\omega}|\Gamma^U|\left(1+\sup_{s\in\mathbb{R}}(\lambda^*(s))\right)\right]\varepsilon.
\end{aligned}$$

This shows that the convergence is uniform on $\mathbb{K}$, since the last estimate is independent of $t\in\mathbb{K}$. Proceeding in the same way as before, one can similarly show that $z(t-s_n)$ converges to w uniformly on compact subsets of $\mathbb{R}$. This completes the proof. □

Theorem 4.5. *Assume that* $(\boldsymbol{H}_5)$ *holds. Let* $g, h \in \mathcal{E}(\mathbb{R}, L(\Omega, \mathbb{B}), \eta_1, \eta_2, \infty)$ *then* $\Gamma(g, h) \in \mathcal{E}(\mathbb{R}, L(\Omega, \mathbb{B}), \eta_1, \eta_2, \infty)$.

Proof. Assume that $(\mathbf{H}_5)$ holds. Let $t \in \mathbb{R}$, $\delta \in]-\infty, t]$ and $g, h \in \mathcal{E}(\mathbb{R}, L(\Omega, \mathbb{B}), \eta_1, \eta_2, \infty)$, we have

$$\frac{1}{\eta_2([-\zeta,\zeta])}\int_{-\zeta}^{\zeta}\sup_{\delta\in]-\infty,t]}\mathbb{E}\left\|\Gamma(g,h)(\delta)\right\|d\eta_1(t)$$
$$\leq \frac{MM_0}{\eta_2([-\zeta,\zeta])}\int_{-\zeta}^{\zeta}\sup_{\delta\in]-\infty,t]}\mathbb{E}\Big[|\Gamma^S|\int_{-\infty}^{\delta}e^{\omega s}\left(\|g(s)\|+\sup_{s\in\mathbb{R}}(\lambda^*(s))\|h(s)\|\right)ds\Big]d\eta_1(t)$$
$$+\frac{MM_0}{\eta_2([-\zeta,\zeta])}\int_{-\zeta}^{\zeta}\sup_{\delta\in]-\infty,t]}\mathbb{E}\Big[|\Gamma^U|\int_{\delta}^{+\infty}e^{-\omega s}\left(\|g(s)\|+\sup_{s\in\mathbb{R}}(\lambda^*(s))\|h(s)\|\right)ds\Big]d\eta_1(t);$$
$$\leq \frac{MM_0}{\eta_2([-\zeta,\zeta])}\int_{-\zeta}^{\zeta}\sup_{\delta\in]-\infty,t]}\mathbb{E}\Big[|\Gamma^S|\int_{-\infty}^{t}e^{\omega s}\left(\|g(s)\|+\sup_{s\in\mathbb{R}}(\lambda^*(s))\|h(s)\|\right)ds\Big]d\eta_1(t)$$
$$+\frac{MM_0}{\eta_2([-\zeta,\zeta])}\int_{-\zeta}^{\zeta}\sup_{\delta\in]-\infty,t]}\mathbb{E}\Big[|\Gamma^U|\int_{t-r}^{+\infty}e^{-\omega s}\left(\|g(s)\|+\sup_{s\in\mathbb{R}}(\lambda^*(s))\|h(s)\|\right)ds\Big]d\eta_1(t).$$

Using Fubini's Theorem, we obtain

$$\frac{1}{\eta_2([-\zeta,\zeta])}\int_{-\zeta}^{\zeta}\sup_{\delta\in]-\infty,t]}\mathbb{E}\left\|\Gamma(g,h)(\delta)\right\|d\eta_1(t)$$
$$\leq MM_0|\Gamma^S|\int_{-\infty}^{t}e^{\omega s}\Big[\frac{1}{\eta_2([-\zeta,\zeta])}\int_{-\zeta}^{\zeta}\sup_{\delta\in]-\infty,t]}\mathbb{E}\|g(s)\|d\eta_1(t)$$
$$+\frac{\sup_{s\in\mathbb{R}}(\lambda^*(s))}{\eta_2([-\zeta,\zeta])}\int_{-\zeta}^{\zeta}\sup_{\delta\in]-\infty,t]}\mathbb{E}\|h(s)\|d\eta_1(t)\Big]ds$$
$$+MM_0|\Gamma^U|\int_{t-r}^{+\infty}e^{-\omega s}\Big[\frac{1}{\eta_2([-\zeta,\zeta])}\int_{-\zeta}^{\zeta}\sup_{\delta\in]-\infty,t]}\mathbb{E}\|g(s)\|d\eta_1(t)$$
$$+\frac{\sup_{s\in\mathbb{R}}(\lambda^*(s))}{\eta_2([-\zeta,\zeta])}\int_{-\zeta}^{\zeta}\sup_{\delta\in]-\infty,t]}\mathbb{E}\|h(s)\|d\eta_1(t)\Big]ds.$$

Since $g, h \in \mathcal{E}(\mathbb{R}, L(\Omega, \mathbb{B}), \eta_1, \eta_2, \infty)$, we deduce that

$$\lim_{\zeta\to+\infty}\frac{1}{\eta_2([-\zeta,\zeta])}\int_{-\zeta}^{\zeta}\sup_{\delta\in]-\infty,t]}\mathbb{E}\left\|\Gamma(g,h)(\delta)\right\|d\eta_1(t)=0.$$

□

Proposition 4.6. *Assume that* $(\boldsymbol{H}_5)$ *holds.*
Let $g, h \in SPAA_c(\mathbb{R}, L(\Omega, \mathbb{B}), \eta_1, \eta_2, \infty)$ *then* $\Gamma(g, h) \in SPAA_c(\mathbb{R}, L(\Omega, \mathbb{B}), \eta_1, \eta_2, \infty)$.

Proof. This Proposition is the consequence of Theorem 4.4 and Theorem 4.5. □

For the existence of mean (η_1, η_2)-pseudo-almost automorphic solution of infinite class, we make the following assumption.
$(\mathbf{H}_6)$: $g, h : \mathbb{R} \to L(\Omega, \mathbb{B})$ are mean compact $cl(\eta_1, \eta_2)$-pseudo-almost automorphic of infinite class.

Theorem 4.7. *Assume that $\mathscr{B}$ satisfies $(\boldsymbol{A}_1), (\boldsymbol{A}_2), (\boldsymbol{B})$ and $(\boldsymbol{C}_1)$. Suppose that $(\boldsymbol{H}_0), (\boldsymbol{H}_1), (\boldsymbol{E}), (\boldsymbol{H}_5)$ and $(\boldsymbol{H}_6)$ hold. Then (1.1) has a unique mean compact $cl(\eta_1, \eta_2)$-pseudo-almost automorphic solution of infinite class.*

Proof. Using Theorem 4.2 and Proposition 4.6 , we obtain the desired result. □

Our next objective is to establish the existence of mean compact (η_1, η_2)-pseudo-almost automorphic of infinite class for the following problem

$$dz(t) = [Az(t) + L(z_t) + g(t, z_t)]dt + h(t, z_t)dN(t), \text{ for } t \in \mathbb{R}, \tag{4.9}$$

where $g : \mathbb{R} \times \mathscr{B} \to L(\Omega, \mathbb{B})$ and $h : \mathbb{R} \times \mathscr{B} \to L(\Omega, \mathbb{B})$ are two stochastic continuous processes.
To our result, we formulate the following assumptions

$(\mathbf{H}_7)$: Let $\eta_1, \eta_2 \in \mathcal{M}$ and $g : \mathbb{R} \times \mathscr{B} \to L(\Omega, \mathbb{B})$
mean compact $cl(\eta_1, \eta_2)$-pseudo-almost automorphic process of infinite class
such that there exists a function L_g; such that $\mathbb{E}\|g(t, \vartheta_1) - g(t, \vartheta_2)\| \leq L_g(t)\mathbb{E}\|\vartheta_1 - \vartheta_2\|$,
: for all $t \in \mathbb{R}$ and $\vartheta_1, \vartheta_2 \in \mathscr{B}$.

$(\mathbf{H}_8)$: Let $\eta_1, \eta_2 \in \mathcal{M}$ and $h : \mathbb{R} \times \mathscr{B} \to L(\Omega, \mathbb{B})$
mean compact $cl(\eta_1, \eta_2)$-pseudo-almost automorphic process of infinite class
such that there exists a function L_h; such that $\mathbb{E}\|h(t, \vartheta_1) - h(t, \vartheta_2)\| \leq L_h(t)\mathbb{E}\|\vartheta_1 - \vartheta_2\|$,
for all $t \in \mathbb{R}$ and $\vartheta_1, \vartheta_2 \in \mathscr{B}$, where $L_g, L_h \in L^p(\mathbb{R}, \mathbb{R}^+), (1 \leq p < \infty)$,

$(\mathbf{H}_9)$: The unstable space $\mathrm{U} \equiv \{0\}$.

Theorem 4.8. *Assume that $\mathscr{B}$ satisfies $(\boldsymbol{A}_1), (\boldsymbol{A}_2), (\boldsymbol{B}); (\boldsymbol{C}_1); (\boldsymbol{E})$ and ; $(\boldsymbol{H}_0), (\boldsymbol{H}_1)$ and $(\boldsymbol{H}_5) - (\boldsymbol{H}_9)$ hold. Then (4.9) has a unique mean compact $cl(\eta_1, \eta_2)$-pseudo-almost automorphic solution of infinite class.*

Proof. Let $y \in SPAA_c(\mathbb{R}, L(\Omega, \mathbb{B}), \eta_1, \eta_2, \infty)$ from Theorem 4.3, the function $t \mapsto y_t$ belongs to $SPAA_c(C(]-\infty, 0], L(\Omega, \mathbb{B})), \eta_1, \eta_2, \infty)$. Thus Theorem 3.13 implies that function $\phi(.) := g(., y_.)$ is in $SPAA_c(\mathbb{R}, L(\Omega, \mathbb{B}, \eta_1, \eta_2, \infty)$. Given that the instable space $U \equiv \{0\}$, then $\Gamma^U = 0$.
Consider the following operator

$$\mathcal{L} : SPAA_c(\mathbb{R}, L(\Omega, \mathbb{B}), \eta_1, \eta_2, \infty) \to SPAA_c(\mathbb{R}, L(\Omega, \mathbb{B}), \eta_1, \eta_2, \infty)$$

defined for $t \in \mathbb{R}$ by

$$\begin{aligned}(\mathcal{L}y)(t) = \Bigg[&\lim_{\lambda \to +\infty} \int_{-\infty}^{t} \mathcal{V}^S(t-s)\Gamma^S \widetilde{D}_\lambda(X_0(g(s, y_s))ds \\ &+ \lim_{\lambda \to +\infty} \int_{-\infty}^{t} \mathcal{V}^S(t-s)\Gamma^S \widetilde{D}_\lambda(X_0(h(s, y_s))dN(s) \Bigg](0).\end{aligned}$$

From Theorem 4.2 and Proposition 4.6, it is now sufficient to show that the operator $\mathcal{L}$ has a single fixed point in $SPAA_c(\mathbb{R}, L(\Omega, \mathbb{B}), \eta_1, \eta_2, \infty)$. Since $\mathscr{B}$ is a uniform fading memory space, by the Lemma 2.10, we can choose the function K constant and the function M such that $M(t) \to 0$ as $t \to +\infty$. Let

$$C = \max(\sup_{t\in\mathbb{R}} M(t), \sup_{t\in\mathbb{R}} K(t)) \quad \text{and} \quad k = \max(L_g, L_h)$$

.

<u>**Step 1**</u> : $L_g, L_h \in L^1(\mathbb{R}, \mathbb{R}^+), (p = 1)$.
Let $y_1, y_2 \in SPAA_c(\mathbb{R}, L(\Omega, \mathbb{B}), \eta_1, \eta_2, \infty)$, we have

$$\begin{aligned}\mathbb{E}\|(\mathcal{L}y_1)(t) - (\mathcal{L}y_2)(t)\| \leq \mathbb{E}\Big[&\Big\|\lim_{\lambda\to+\infty}\int_{-\infty}^{t} \mathcal{V}^S(t-s)\Gamma^S\widetilde{D}_\lambda(X_0(g(s,y_{1s}) - g(s,y_{2s}))(0)ds\Big\| \\ &+ \Big\|\lim_{\lambda\to+\infty}\int_{-\infty}^{t} \mathcal{V}^S(t-s)\Gamma^S\widetilde{D}_\lambda(X_0(h(s,y_{1s}) - h(s,y_{2s}))(0)dN(s)\Big\|\Big]\end{aligned}$$

Hence

$$\begin{aligned}\mathbb{E}\|(\mathcal{L}y_1)(t) - (\mathcal{L}y_2)(t)\| \leq \mathbb{E}\Big[&\lim_{\lambda\to+\infty}\int_{-\infty}^{t} \|\mathcal{V}^S(t-s)\Gamma^S\widetilde{D}_\lambda(X_0(g(s,y_{1s}) - g(s,y_{2s}))(0)\|ds \\ &+ \lim_{\lambda\to+\infty}\int_{-\infty}^{t} \|\mathcal{V}^S(t-s)\Gamma^S\widetilde{D}_\lambda(X_0(h(s,y_{1s}) - h(s,y_{2s}))(0)\|dN(s)\Big].\end{aligned}$$

By using Proposition 2.8 and 2.13, it follows that

$$\begin{aligned}&\mathbb{E}\|(\mathcal{L}y_1)(t) - (\mathcal{L}y_2)(t)\| \\ &\leq MM_0|\Gamma^S|\int_{-\infty}^{t} e^{-\omega(t-s)}\mathbb{E}\|g(s,y_{1s}) - g(s,y_{2s})\|ds; \\ &+ MM_0|\Gamma^S|\int_{-\infty}^{t} e^{-\omega(t-s)}\mathbb{E}\|h(s,y_{1s}) - h(s,y_{2s})\| \sup_{s\in]-\infty,t]}\lambda^*(s)ds; \\ &\leq MM_0|\Gamma^S|\int_{-\infty}^{t} e^{-\omega(t-s)}\left(L_g(s) + L_h(s)\sup_{s\in\mathbb{R}}\lambda^*(s)\right)\mathbb{E}\|y_{1s} - y_{2s}\|_{\mathscr{B}}ds; \\ &\leq MM_0|\Gamma^S|\left(1 + \sup_{s\in\mathbb{R}}\lambda^*(s)\right)\int_{-\infty}^{t} k(s)e^{-\omega(t-s)}\left(K(s)\sup_{0\leq s\leq t}\mathbb{E}\|y_{1s} - y_{2s}\| + M(s)\mathbb{E}\|y_{1_0} - y_{2_0}\|_{\mathscr{B}}\right)ds; \\ &\leq MM_0|\Gamma^S|\left(1 + \sup_{s\in\mathbb{R}}\lambda^*(s)\right)\int_{-\infty}^{t} k(s)e^{-\omega(t-s)}\left(K(s) + M(s)\right)ds\|y_1 - y_2\|_\infty; \\ &\leq 2CMM_0|\Gamma^S|\left(1 + \sup_{s\in\mathbb{R}}\lambda^*(s)\right)\int_{-\infty}^{t} k(s)e^{-\omega(t-s)}ds\|y_1 - y_2\|_\infty; \\ &\leq 2CMM_0|\Gamma^S|\left(1 + \sup_{s\in\mathbb{R}}\lambda^*(s)\right)\left(\int_{-\infty}^{t} k(s)ds\right)\|y_1 - y_2\|_\infty,\end{aligned}$$

then

$$\mathbb{E}\|(\mathcal{L}y_1)(t) - (\mathcal{L}y_2)(t)\| \leq 2CMM_0|\Gamma^S|\left(1 + \sup_{s\in\mathbb{R}}\lambda^*(s)\right)|k|_{L^1(\mathbb{R})}\|y_1 - y_2\|_\infty,$$

$$\begin{aligned}\mathbb{E}\|(\mathcal{L}^2y_1)(t)-(\mathcal{L}^2y_2)(t)\| &\leq \mathbb{E}\Big[\Big\|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{V}^S(t-s)\Gamma^S\widetilde{D}_\lambda(X_0(g(s,(\mathcal{L}y_1)s)-g(s,(\mathcal{L}y_2)s)))(0)ds\Big\|\\ &+\Big\|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{V}^S(t-s)\Gamma^S\widetilde{D}_\lambda(X_0(h(s,(\mathcal{L}y_1)s)-h(s,(\mathcal{L}y_2)s)))(0)dN(s)\Big\|\Big];\\ &\leq \left[2CMM_0|\Gamma^S|\left(1+\sup_{s\in\mathbb{R}}\lambda^*(s)\right)|k|_{L^1(\mathbb{R})}\right]^2\|y_1-y_2\|_\infty.\end{aligned}$$

By induction on n, we obtain the following inequality,

$$\mathbb{E}\|(\mathcal{L}^ny_1)(t)-(\mathcal{L}^ny_2)(t)\|\leq\left[2CMM_0|\Gamma^S|\left(1+\sup_{s\in\mathbb{R}}\lambda^*(s)\right)|k|_{L^1(\mathbb{R})}\right]^n\|y_1-y_2\|_\infty.$$

Consequently

$$\|(\mathcal{L}^ny_1)-(\mathcal{L}^ny_2)\|_\infty\leq\left[2CMM_0|\Gamma^S|\left(1+\sup_{s\in\mathbb{R}}\lambda^*(s)\right)|k|_{L^1(\mathbb{R})}\right]^n\|y_1-y_2\|_\infty.$$

Let n_0 be such that $\left[2CMM_0|\Gamma^S|\left(1+\sup_{s\in\mathbb{R}}\lambda^*(s)\right)|k|_{L^1(\mathbb{R})}\right]^{n_0}<1$, by Banach's fixed point theorem, we deduce that $\mathcal{L}$ has a unique fixed point and this satisfies the following integral equation

$$v_t=\left[\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{V}^S(t-s)\Gamma^S\widetilde{D}_\lambda(X_0g(s))ds+\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{V}^S(t-s)\Gamma^S\widetilde{D}_\lambda(X_0h(s))dN(s)\right](0).$$

Step 2 : $L_g, L_h\in L^p(\mathbb{R},\mathbb{R}^+),(1<p<+\infty \quad \text{and} \quad \frac{1}{p}+\frac{1}{q}=1)$.
First, put

$$\mu(t)=\int_{-\infty}^{t}(k(s))^pds$$

Then we define an equivalent norm over $SPAA_c(R,L(\Omega,\mathbb{B}),\eta_1,\eta_2,\infty)$ as follows

$$\|g\|_c=\sup_{t\in\mathbb{R}}\left(e^{-c\mu(t)}\mathbb{E}\|g(t)\|\right),$$

where c is a fixed positive number to be precised later. Using the Holder inequality, it follows that

$$\begin{aligned}
&\mathbb{E}\|(\mathcal{L}y_1)(t)-(\mathcal{L}y_2)(t)\| \\
&\leq MM_0|\Gamma^S|\int_{-\infty}^{t} e^{-\omega(t-s)}\left(L_g(s)+L_h(s)\sup_{s\in\mathbb{R}}\lambda^*(s)\right)\mathbb{E}\|y_{1s}-y_{2s}\|_{\mathscr{B}}ds; \\
&\leq MM_0|\Gamma^S|\left(1+\sup_{s\in\mathbb{R}}\lambda^*(s)\right)\int_{-\infty}^{t} k(s)e^{-\omega(t-s)}\left(K(s)\sup_{0\leq z\leq s}\mathbb{E}\|y_{1z}-y_{2z}\|+M(s)\mathbb{E}\|y_{1_0}-y_{2_0}\|_{\mathscr{B}}\right)ds; \\
&\leq 2CMM_0|\Gamma^S|\left(1+\sup_{s\in\mathbb{R}}\lambda^*(s)\right)\int_{-\infty}^{t} k(s)e^{-\omega(t-s)}e^{c\mu(s)}\sup_{s\in\mathbb{R}}\left(e^{-c\mu(s)}\mathbb{E}\|y_{1s}-y_{2s}\|\right)ds; \\
&\leq 2CMM_0|\Gamma^S|\left(1+\sup_{s\in\mathbb{R}}\lambda^*(s)\right)\int_{-\infty}^{t} e^{-\omega(t-s)}k(s)e^{c\mu(s)}ds\|y_1-y_2\|_c; \\
&\leq 2CMM_0|\Gamma^S|\left(1+\sup_{s\in\mathbb{R}}\lambda^*(s)\right)\left(\int_{-\infty}^{t} e^{-q\omega(t-s)}ds\right)^{\frac{1}{q}}\left(\int_{-\infty}^{t} k^p(s)e^{cp\mu(s)}ds\right)^{\frac{1}{p}}\|y_1-y_2\|_c; \\
&\leq 2CMM_0|\Gamma^S|\left(1+\sup_{s\in\mathbb{R}}\lambda^*(s)\right)\frac{1}{(\omega q)^{\frac{1}{q}}}\left(\int_{-\infty}^{t} \mu'(s)e^{cp\mu(s)}ds\right)^{\frac{1}{p}}\|y_1-y_2\|_c,
\end{aligned}$$

hence

$$e^{-c\mu(t)}\mathbb{E}\|(\mathcal{L}y_1)(t)-(\mathcal{L}y_2)(t)\| \leq 2CMM_0|\Gamma^S|\left(1+\sup_{s\in\mathbb{R}}\lambda^*(s)\right)\frac{1}{(\omega q)^{\frac{1}{q}}}\times\frac{1}{(cp)^{\frac{1}{p}}}\|y_1-y_2\|_c.$$

Then

$$\|(\mathcal{L}y_1)-(\mathcal{L}y_2)\|_c \leq 2CMM_0|\Gamma^S|\left(1+\sup_{s\in\mathbb{R}}\lambda^*(s)\right)\frac{1}{(\omega q)^{\frac{1}{q}}}\times\frac{1}{(cp)^{\frac{1}{p}}}\|y_1-y_2\|_c.$$

Fix $c>0$ so large, then the function $c\mapsto\frac{1}{(pc)^{\frac{1}{p}}}$ converges to 0 when c converges to $+\infty$. It follows that for $c>0$ so large enough, we have

$$2CMM_0|\Gamma^S|\left(1+\sup_{s\in\mathbb{R}}\lambda^*(s)\right)\frac{1}{(\omega q)^{\frac{1}{q}}}\times\frac{1}{(cp)^{\frac{1}{p}}}<1.$$

Therefore $\mathcal{L}$ is strict contraction, we conclude that there is a $cl(\eta_1,\eta_2)$-unique pseudo-almost automorphic integral solution to equation (4.9), which ends the proof.

□

Corollary 4.9. *Assume that $\mathscr{B}$ is uniform fading space and $(\boldsymbol{A}_1),(\boldsymbol{A}_2),(\boldsymbol{B})$ and $(\boldsymbol{C}_1),(\boldsymbol{E})$; $(\boldsymbol{H}_0),(\boldsymbol{H}_1)$ and $(\boldsymbol{H}_5)-(\boldsymbol{H}_9)$ hold, g, h are Lipschitz continuous processes with respect the second argument. If*

$$k=\max(Lip(g),Lip(h))<\frac{\omega}{2CMM_0|\Gamma^S|\left(1+\sup_{s\in\mathbb{R}}\lambda^*(s)\right)},$$

then equation (4.9) has unique mean $cl(\eta_1,\eta_1)$-pseudo-almost automorphic solution of infinite class , where $Lip(g)$ and $Lip(h)$ are respectively the Lipschitz constants of g and h.

Proof. Let us pose $k = \max(Lip(g), Lip(h))$, it follows that

$$\mathbb{E}\|(\mathcal{L}y_1)(t) - (\mathcal{L}y_2)(t)\| \leq 2\frac{k}{\omega}CMM_0|\Gamma^S|\left(1 + \sup_{s\in\mathbb{R}}\lambda^*(s)\right)\|y_1 - y_2\|_\infty.$$

Therefore $\mathcal{L}$ is strict contraction if

$$k < \frac{\omega}{2CMM_0|\Gamma^S|\left(1 + \sup_{s\in\mathbb{R}}\lambda^*(s)\right)}.$$

□

5. Application

To illustrate our results, we consider the following model

$$\begin{cases} dv(t,x) = \left[\frac{\partial^2}{\partial x^2}v(t,x) + \int_{-\infty}^{0}\eta(\theta)v(t+\theta,x)d\theta\right. \\ \qquad\left. + \sin\left(\frac{1}{2+\cos(t)+\cos(\sqrt{2}t)}\right) + \arctan(t) + \int_{-\infty}^{0}e^{\omega\theta}q(t,v(t+\theta,x))d\theta\right]dt \\ \qquad + \left[\sin\left(\frac{1}{2+\cos(t)+\cos(\sqrt{3}t)}\right) + \cos(t) + \int_{-\infty}^{0}e^{\omega\theta}q(t,v(t+\theta,x))d\theta\right]dN(t), \\ \qquad -\infty \leq \theta \leq 0, t\in\mathbb{R}, x\in[0,\pi] \\ v(t,0) = v(t,\pi) = 0 \text{ for } t\in\mathbb{R}. \end{cases} \tag{5.1}$$

where $\eta :]-\infty, 0] \to \mathbb{R}$ define by $\eta(\theta) = e^{(\gamma+1)\theta}$ is continuous function, $q :]-\infty,0]\times\mathbb{R} \to \mathbb{R}$ is Lipschitz continuous with respect to the second argument. For example take $q(\theta,x) = \theta^2 - \sin(\frac{x}{4})$ for $(\theta,x) \in]-\infty,0]\times\mathbb{R}$ and using the fact that the sine is Lipschitz continous with Lipschitz constant 1, it follows that

$$\mathbb{E}\|q(\theta,x_1) - q(\theta,x_1)\| \leq \frac{1}{4}\mathbb{E}\|x_1 - x_2\|$$

which implies that $q :]-\infty,0]\times\mathbb{R} \to \mathbb{R}$ is Lipschitzian with respect to the second argument. $N(t)$ is a two-sided inhomogeneous Poisson process with intensity function λ^*, defined on the filtered probability space $(\Omega,\mathcal{F},P,\{\mathcal{F}_t^N\}_{t\geq 0})$ with natural filtration $\mathcal{F}_t^N = \sigma\{N(u) - N(v)|v,u \leq t\}$. We consider $\mathscr{B} = \mathscr{C}_\gamma, \gamma > 0$ in example 2.1, to rewrite system (5.1) in the abstract form, we introduce $\mathbb{B} = C_0([0,\pi];\mathbb{R})$ of continuous functions from $[0,\pi]$ to $\mathbb{R}^+$ equipped with the uniform topology. Let $A : \mathbb{D}(A) \to C_0([0,\pi];\mathbb{R})$ defined by

$$\begin{cases} \mathbb{D}(A) = \{y \in C_0([0,\pi];\mathbb{R}) \cap C^2([0,\pi];\mathbb{R}), y'' \in C_0([0,\pi];\mathbb{R})\} \\ Ay = y''. \end{cases}$$

From [25] A satisfied the Hille-Yosida condition in $C_0([0,\pi];\mathbb{R})$. Moreover the part A_0 in $\overline{\mathbb{D}(A)}$ is the generator of strongly continuous compact semigroup $(T_0(t))_{t\geq 0}$ on $\overline{\mathbb{D}(A)}$. Then it follows that $(\mathbf{H}_0)$ and $(\mathbf{H}_1)$ are satisfied. We also define

$g : \mathbb{R} \times \mathscr{B} \to L(\Omega, C_0([0,\pi];\mathbb{R}))$, $h : \mathbb{R} \times \mathscr{B} \to L(\Omega, C_0([0,\pi];\mathbb{R}))$ and $L : \mathscr{B} \to L(\Omega, C_0([0,\pi];\mathbb{R}))$ as follows

$$g(t,v)(x) = \sin\left(\frac{1}{2+\cos(t)+\cos(\sqrt{2}t)}\right) + \arctan(t) + \int_{-\infty}^{0} e^{\omega\theta} q(t, v(t+\theta, x))d\theta,$$

$$h(t,v)(x) = \sin\left(\frac{1}{2+\cos(t)+\cos(\sqrt{3}t)}\right) + \cos(t) + \int_{-\infty}^{0} e^{\omega\theta} q(t, v(t+\theta, x))d\theta,$$

$$L(v)(x) = \int_{-\infty}^{0} \eta(\theta) v(t+\theta, x)d\theta, x \in [0,\pi].$$

Let $z(t) = v(t,x)$ be, we obtain the abstract form of equation (5.1)

$$dz(t) = [Az(t) + L(z_t) + g(t,z_t)]dt + h(t,z_t)dN(t), \text{ for } t \in \mathbb{R}.$$

Consider the measures η_1 and η_2 where its Radon-Nikodyn derivative are respectively $\beta_1, \beta_2 : \mathbb{R} \to \mathbb{R}$ defined by

$$\beta_1(t) = \begin{cases} 1, & \text{for } t > 0 \\ e^{nt}, & \text{for } t \leq 0\ , \quad n \in \mathbb{N}^* \end{cases} \quad \text{and } \beta_2(t) = |t| \text{ for all } t \in \mathbb{R}.$$

Then $d\eta_1(t) = \beta_1(t)dt$ and $d\eta_2(t) = \beta_2(t)dt$ where dt denotes the Lebesgue measure on $\mathbb{R}$ and $\eta_1(B) = \displaystyle\int_B \beta_1(t)dt$, $\eta_2(B) = \displaystyle\int_B \beta_2(t)dt$, for $B \in \mathcal{B}$. We have

$$\eta_1([-\zeta,\zeta]) = \int_{-\zeta}^{0} e^{nt}dt + \int_0^{\zeta} dt = \frac{1-e^{\zeta}}{n\zeta^2} + \zeta \text{ and } \eta_2([-\zeta,\zeta]) = 2\int_0^{\zeta} tdt = \zeta^2.$$

Thus $\displaystyle\limsup_{\zeta\to+\infty} \frac{\eta_1([-\zeta,\zeta])}{\eta_2([-\zeta,\zeta])} = \limsup_{\zeta\to+\infty} \left(\frac{1-e^{-nt}}{n\zeta} + \frac{1}{\zeta}\right) = 0 < \infty$, it results that $(\mathbf{H}_2)$ and $(\mathbf{H}_4)$ are satisfied. From[13, 14, 9], both functions

$$t \mapsto \sin\left(\frac{1}{2+\cos(t)+\cos(\sqrt{2}t)}\right) \text{ and } t \mapsto \sin\left(\frac{1}{2+\cos(t)+\cos(\sqrt{3}t)}\right)$$

are almost automorphic functions.
For all $t \in \mathbb{R}, |\arctan(t)| \leq \dfrac{\pi}{2}$, we have

$$\frac{1}{\eta_2([-\zeta,\zeta])} \int_{-\zeta}^{\zeta} \sup_{\theta\in]-\infty,t]} \mathbb{E}\left|\arctan(\theta)\right| d\eta_1(t) \leq \frac{\pi}{2n\zeta^2}\left(1-e^{-n\zeta}\right) \to 0 \text{ as } \zeta \to +\infty,$$

we deduce that
$\displaystyle\lim_{\zeta\to+\infty} \frac{1}{\eta_2([-\zeta,\zeta])} \int_{-\zeta}^{\zeta} \sup_{\theta\in]-\infty,t]} \mathbb{E}\left|\arctan(\theta)\right| d\eta_1(t) = 0$. Then $t \mapsto \arctan(t)$ is (η_1,η_2)-ergodic of infinite class and for all $t \in \mathbb{R}, |\cos(t)| \leq 1$, we have

$$\frac{1}{\eta_2([-\zeta,\zeta])} \int_{-\zeta}^{\zeta} \sup_{\theta\in]-\infty,t]} \mathbb{E}\left|\cos(\theta)\right| d\eta_1(t) \leq \frac{\eta_1([-\zeta,\zeta])}{\eta_2([-\zeta,\zeta])} \to 0 \text{ as } \zeta \to \infty,$$

we conclude that
$\displaystyle\lim_{\zeta\to+\infty} \frac{1}{\eta_2([-\zeta,\zeta])} \int_{-\zeta}^{\zeta} \sup_{\theta\in]-\infty,t]} \mathbb{E}\left|\cos(\theta)\right| d\eta_1(t) = 0$. Then $t \mapsto \cos(t)$ is (η_1,η_2)-ergodic of infinite class. consequently, g and h are uniformly mean (η_1,η_2)-pseudo-almost automorphic of infinite class.

For $\phi \in \mathscr{C}_\gamma$, $\phi \in C(]-\infty;0], L(\Omega,\mathbb{B}))$ and $\lim_{\theta\to-\infty} e^{\gamma\theta}\phi(\theta) = x_0$ exists in $L(\Omega, C_0([0,\pi];\mathbb{R}))$, then there exists a constant $\beta > 0$ such that $\mathbb{E}\|e^{\gamma\theta}\phi(\theta)\| \leq \beta$ for $\theta \in]-\infty, 0]$.

$$\begin{aligned}\mathbb{E}\|L(\phi)(x)\| &= \mathbb{E}\left\|\int_{-\infty}^{0} \eta(\theta)\phi(\theta)(x)d\theta\right\| \\ &\leq \int_{-\infty}^{0} e^{(\gamma+1)\theta}\mathbb{E}\left\|e^{-\gamma\theta}e^{\gamma\theta}\phi(\theta)(x)\right\| d\theta \\ &\leq \beta < \infty\end{aligned}$$

Moreover,

$$\begin{aligned}\mathbb{E}\|L(\phi)(x)\| &\leq \left(\int_{-\infty}^{0} e^{\theta}d\theta\right)\sup_{\theta\leq 0}\mathbb{E}\left\|e^{\gamma\theta}\phi(\theta)(x)\right\| \\ &\leq \|\phi\|_{\mathscr{B}}\end{aligned}$$

Then L is well defined and L is bounded linear operator from $\mathscr{B}$ to $L(\Omega, C_0([0,\pi];\mathbb{R}))$.

$$\begin{aligned}\mathbb{E}\|g(t,\phi_1)(x) - g(t,\phi_2)(x)\| &= \mathbb{E}\left\|\int_{-\infty}^{0} e^{\omega\theta}\left[q(\theta,\phi_1(\theta)(x)) - q(\theta,\phi_2(\theta)(x))\right]\right\| d\theta \\ &\leq \frac{1}{4}\int_{-\infty}^{0} e^{\omega\theta}e^{-\gamma\theta}e^{\gamma\theta}\mathbb{E}\|\phi_1(\theta)(x) - \phi_2(\theta)(x)\|d\theta \\ &\leq \frac{1}{4}\left(\int_{-\infty}^{0} e^{(\omega-\gamma)\theta}d\theta\right)\|\phi_1 - \phi_2\|_{\mathscr{B}}\end{aligned}$$

We conclude that g and h are continuous Lipschitzian functions and $cl(\eta_1,\eta_2)$-pseudo-almost automorphic of infinite class.
Moreover, given that q is stochastically bounded, then

$$\forall (t,x) \in \mathbb{R}\times[0,\pi], \theta \in]-\infty,0], \exists\beta > 0, \mathbb{E}\left\|q(t, v(t+\theta, x))\right\| \leq \beta.$$

And, we obtain

$$\mathbb{E}\|h(t,v)(x)\| \leq 1 + \frac{\pi}{2} + \int_{-\infty}^{0}\mathbb{E}\|q(t,v(t+\theta,x))\|d\theta \leq 1 + \frac{\pi}{2} + \beta,$$

this implies that $(\mathbf{H}_5)$ is satisfied.

Proposition 5.1. [15] *If* $\int_{-\infty}^{0}\eta(\theta)d\theta < 1$, *then the semigroup* $(\mathcal{V}(t))_{t\geq 0}$ *is hyperbolic and the instable space* $U \equiv \{0\}$.

Observe that

$$\int_{-\infty}^{0}\eta(\theta)d\theta = \lim_{r\to+\infty}\int_{-r}^{0}\eta(\theta)e^{(\gamma+1)\theta}d\theta = \lim_{r\to+\infty}\left[\frac{1}{\gamma+1}e^{(\gamma+1)\theta}\right]_{-r}^{0} = \frac{1}{\gamma+1} < 1,$$

then $(\mathbf{H}_8)$ holds. Then by using corollary 4.9 we deduce the following result.

Theorem 5.2. *Under the above assumptions, the equation* (5.1) *has a unique* $cl(\eta_1,\eta_2)$*-pseudo-almost automorphic solution* v *of infinite class.*

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