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APPROXIMATE SOLUTION FOR A BIHARMONIC PROBLEM OUTSIDE A QUARTER CIRCLE

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ABSTRACT. This paper investigates an inverse boundary value problem for the biharmonic equation in the exterior of a quarter circle. The problem is shown to be ill-posed in the sense of Hadamard. Using the Almansi decomposition, it is reduced to two harmonic problems, which are solved by separation of variables and Fourier series expansion. An explicit solution representation is derived, and a truncated Fourier regularization method is proposed for stable reconstruction from noisy boundary data. A logarithmic stability estimate and a convergence result are established.

Keywords. Ill-posed problem, conditional well-posedness, biharmonic equation, stability estimate, Tikhonov regularization, Fourier method.
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1. INTRODUCTION AND PRELIMINARIES

The biharmonic equation is one of the fundamental equations of mathematical physics. It arises naturally in the description of physical processes such as the deformation of thin elastic plates, the flow of viscous fluids at low Reynolds numbers, and the theory of elasticity [1, 2]. In particular, the equilibrium state of a thin clamped plate under transverse loading is governed by the biharmonic equation with appropriate boundary conditions.

Many boundary value problems for the biharmonic equation are ill-posed in the sense of Hadamard: at least one of the three requirements for well-posedness - existence, uniqueness, and continuous dependence on the data - fails to hold [8, 15, 16]. Typically it is the continuous-dependence condition that fails, so that arbitrarily small perturbations of the boundary data can produce arbitrarily large changes in the solution. Problems of this type require special regularization methods in order to construct stable approximate solutions..

The study of ill-posed problems was initiated by Hadamard, who demonstrated the ill-posedness of certain problems for the Laplace and biharmonic equations. Subsequently, Tikhonov, Lavrentyev, and others developed the theory of conditionally well-posed problems [16, 9, 10]. Tikhonov regularization, one of the most

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widely used approaches for solving ill-posed problems, involves constructing an auxiliary problem whose solution converges to the solution of the original problem as the regularization parameter tends to zero.

Several regularization techniques have been proposed specifically for the biharmonic equation. Chapko and Johansson [3] developed an iterative regularization method for the biharmonic equation with incomplete boundary data. Stability estimates play a crucial role in the theory of ill-posed problems, as they provide quantitative information about the sensitivity of the solution to perturbations in the input data. Choudhury and Krishnan [5] investigated logarithmic-type stability estimates for inverse boundary value problems for the biharmonic operator. In [14], investigated a regularization method for the bilateral obstacle problem, establishing convergence results together with a priori and a posteriori error estimates. In [12], studied the simultaneous reconstruction of boundary coefficients in a parabolic system using Tikhonov regularization and a Levenberg-Marquardt optimization algorithm. Their numerical results demonstrated the accuracy, stability, and computational efficiency of the proposed methods.

This paper considers an interior boundary value problem for the biharmonic equation in a domain exterior to a quarter circle. Such domains are of interest in applications involving corner singularities and exterior problems in elasticity and fluid mechanics. The main contributions of this work are:

- Demonstration of Hadamard ill-posedness for the problem under consideration;
- Formulation of conditional well-posedness with appropriate a priori constraints;
- Derivation of a logarithmic-type stability estimate;
- Construction of a regularized approximate solution using Fourier series truncation;
- Proof of convergence of the approximate solution to the exact solution as the data error tends to zero;
- Numerical validation of the theoretical results.

2. Problem formulation

Let $\Omega \subset \mathbb{R}^2$ be the domain defined in polar coordinates (r, θ) as

$$\Omega = \left\{ (r, \theta) : r > 1, \ 0 < \theta < \frac{\pi}{2} \right\}.$$

Thus, Ω is the exterior of a quarter-circle of radius 1 centered at the origin. The boundary of Ω consists of three parts:

- $\Gamma_1 = \{(1, \theta) : 0 \leq \theta \leq \pi/2\}$;
- $\Gamma_2 = \{(r, 0) : r > 1\}$;
- $\Gamma_3 = \{(r, \pi/2) : r > 1\}$.

Consider the following interior boundary value problem: find a function $u(r, \theta)$ satisfying

$$\Delta^2 u = 0 \quad \text{in } \Omega, \tag{2.1}$$

$$u(1,\theta)=f(\theta),\quad 0\le\theta\le\frac{\pi}{2}, \tag{2.2}$$

$$\frac{\partial u}{\partial r}(1,\theta)=g(\theta),\quad 0\le\theta\le\frac{\pi}{2}, \tag{2.3}$$

$$u(r,0)=0,\quad r>1, \tag{2.4}$$

$$u\left(r,\frac{\pi}{2}\right)=0,\quad r>1, \tag{2.5}$$

$$\lim_{r\to\infty}u(r,\theta)=0\quad\text{uniformly in }\theta\in[0,\pi/2]. \tag{2.6}$$

Here $\Delta=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$ is the Laplace operator in polar coordinates, $f(\theta)$ and $g(\theta)$ are given functions on $[0,\pi/2]$, and (2.6) imposes decay at infinity.

Remark 2.1. Conditions (2.4) and (2.5) are homogeneous Dirichlet boundary conditions on the two rays. These are natural for problems with symmetry or for modeling domains with fixed boundaries.

3. ILL-POSEDNESS HARMONIC REPRESENTATION

We first demonstrate that problem (2.1) - (2.6) is ill-posed in the sense of Hadamard. Specifically, we show that the solution does not depend continuously on the boundary data.

Lemma 3.1. *Let*

$$v_{p,n}(r,\theta)=r^p\sin(2n\theta),\qquad n\in\mathbb{N}, \tag{3.1}$$

where $p\in\mathbb{R}$. Then

$$\Delta^2 v_{p,n}(r,\theta)=\left(p^2-4n^2\right)\left((p-2)^2-4n^2\right)r^{p-4}\sin(2n\theta).$$

Consequently, $v_{p,n}$ is biharmonic if and only if

$$p\in\{-2n,\,2n,\,2-2n,\,2+2n\}.$$

Proof. Recall that the Laplace operator in polar coordinates is given by

$$\Delta=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}.$$

For

$$v_{p,n}(r,\theta)=r^p\sin(2n\theta),$$

we have

$$\frac{\partial v_{p,n}}{\partial r}=pr^{p-1}\sin(2n\theta),$$

and

$$\frac{\partial^2 v_{p,n}}{\partial r^2}=p(p-1)r^{p-2}\sin(2n\theta).$$

Moreover,

$$\frac{\partial^2 v_{p,n}}{\partial\theta^2}=-4n^2r^p\sin(2n\theta).$$

Therefore,

$$\begin{aligned}\Delta v_{p,n} &= \frac{\partial^2 v_{p,n}}{\partial r^2} + \frac{1}{r}\frac{\partial v_{p,n}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_{p,n}}{\partial \theta^2} \\ &= p(p-1)r^{p-2}\sin(2n\theta) + pr^{p-2}\sin(2n\theta) - 4n^2 r^{p-2}\sin(2n\theta) \\ &= \left(p^2 - 4n^2\right) r^{p-2}\sin(2n\theta).\end{aligned}$$

Applying the Laplace operator once more, we obtain

$$\begin{aligned}\Delta^2 v_{p,n} &= \Delta\left(\left(p^2-4n^2\right) r^{p-2}\sin(2n\theta)\right) \\ &= \left(p^2-4n^2\right)\Delta\left(r^{p-2}\sin(2n\theta)\right) \\ &= \left(p^2-4n^2\right)\left((p-2)^2-4n^2\right) r^{p-4}\sin(2n\theta).\end{aligned}$$

Hence, $v_{p,n}$ is biharmonic if and only if

$$\left(p^2-4n^2\right)\left((p-2)^2-4n^2\right) = 0.$$

The first factor gives

$$p^2 - 4n^2 = 0,$$

and therefore

$$p = \pm 2n.$$

The second factor gives

$$(p-2)^2 - 4n^2 = 0,$$

and consequently

$$p = 2 \pm 2n.$$

Thus, the admissible exponents are

$$p \in \{-2n,\, 2n,\, 2-2n,\, 2+2n\}.$$

This completes the proof. □

Now consider the boundary data for the original problem. For arbitrarily large n, the boundary functions

$$f_n(\theta) = \sin(2n\theta), \quad g_n(\theta) = (4n+2)\sin(2n\theta)$$

satisfy

$$\|f_n\|_{L_2(0,\pi/2)} = \sqrt{\frac{\pi}{4}}, \quad \|g_n\|_{L_2(0,\pi/2)} = (4n+2)\sqrt{\frac{\pi}{4}}.$$

Thus, the boundary data are bounded (for f_n) and grow linearly with n (for g_n). However, the corresponding solutions u_n grow without bound as $r \to \infty$, indicating instability. Therefore, there is no continuous dependence of the solution on the boundary data. This demonstrates Hadamard ill-posedness. To restore well-posedness we introduce, following Tikhonov, a set of correctness obtained by restricting the solution to a compact subset of the solution space.

Definition 3.2. Define the set of correctness

$$\mathcal{M}_{M_1,M_2} = \left\{u \in H^2(\Omega) \cap L_2(\Omega) : \|\Delta u\|_{L_2(\Omega)} \le M_1,\ \|u\|_{L_2(\Omega)} \le M_2\right\},$$

where $M_1 > 0$ and $M_2 > 0$ are given constants.

The constants M_1 and M_2 are a priori bounds that are assumed to be known. In physical applications, such bounds often follow from energy considerations or from additional measurements.

Remark 3.3. The set $\mathcal{M}_{M_1,M_2}$ is compact in $L_2(\Omega)$ under appropriate conditions (by the Rellich–Kondrachov embedding theorem). This compactness is what makes the problem well-posed once it is restricted to this set; we show below that on $\mathcal{M}_{M_1,M_2}$ the solution depends continuously on the boundary data, with an explicit, quantitative stability estimate.

We will prove that on $\mathcal{M}_{M_1,M_2}$, the solution depends continuously on the boundary data, and a quantitative stability estimate holds.

Lemma 3.4 (Biharmonic representation)**.** *Any biharmonic function u in a simply connected domain can be represented as*

$$u(r,\theta) = u_1(r,\theta) + r^2 u_2(r,\theta), \tag{3.2}$$

where u_1 and u_2 are harmonic functions, i.e., $\Delta u_1 = 0$ and $\Delta u_2 = 0$.

Proof. This is a standard result. If $\Delta^2 u = 0$, let $v = \Delta u$. Then $\Delta v = 0$, so v is harmonic. The existence of a harmonic function u_2 such that $\Delta(r^2 u_2) = v$ and then setting $u_1 = u - r^2 u_2$ yields the representation. □

3.1. **Representation of the harmonic components.** According to Lemma 3.4, every biharmonic function admits the representation

$$u(r,\theta) = u_1(r,\theta) + r^2 u_2(r,\theta), \tag{3.3}$$

where both u_1 and u_2 are harmonic functions in the domain Ω.

Consequently, it is sufficient to determine harmonic functions satisfying the boundary conditions inherited from the original biharmonic problem. To derive the general form of these harmonic functions, we apply the method of separation of variables. Assume

$$u_i(r,\theta) = R(r)\Theta(\theta), \qquad i = 1,2.$$

Substituting this expression into the Laplace equation

$$\Delta u_i = 0$$

and using the polar-coordinate representation of the Laplace operator,

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2},$$

we obtain

$$r^2 R''(r) + rR'(r) + \frac{\Theta''(\theta)}{\Theta(\theta)} R(r) = 0.$$

Dividing by $R(r)\Theta(\theta)$ yields

$$\frac{r^2 R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda,$$

where λ is the separation constant. Hence the angular and radial equations become

$$\Theta'' + \lambda\Theta = 0, \tag{3.4}$$

and

$$r^2 R'' + rR' - \lambda R = 0. \tag{3.5}$$

The homogeneous boundary conditions

$$u(r,0) = 0, \qquad u\Big(r, \frac{\pi}{2}\Big) = 0$$

imply

$$\Theta(0) = 0, \qquad \Theta\Big(\frac{\pi}{2}\Big) = 0.$$

Therefore, problem (3.4) is a Sturm–Liouville eigenvalue problem, whose eigenvalues are

$$\lambda = (2n)^2, \qquad n = 1, 2, \dots,$$

with corresponding eigenfunctions

$$\Theta_n(\theta) = \sin(2n\theta).$$

Substituting these eigenvalues into equation (3.5), we obtain the Euler equation

$$r^2 R'' + rR' - 4n^2 R = 0.$$

Seeking solutions in the form

$$R(r) = r^{\alpha},$$

gives the characteristic equation

$$\alpha(\alpha - 1) + \alpha - 4n^2 = 0,$$

or equivalently,

$$\alpha^2 - 4n^2 = 0.$$

Hence

$$\alpha = \pm 2n.$$

Since the domain is exterior to the unit circle and the solution is required to satisfy the decay condition

$$\lim_{r\to\infty} u(r,\theta) = 0,$$

only the decaying radial solution

$$R_n(r) = r^{-2n}$$

is admissible.

Therefore, every harmonic function satisfying the homogeneous boundary conditions on the rays and the decay condition at infinity admits the Fourier representation

$$u_i(r,\theta) = \sum_{n=1}^{\infty} A_n^{(i)} r^{-2n} \sin(2n\theta), \qquad i = 1, 2,$$

where the coefficients $A_n^{(i)}$ are determined from the boundary conditions prescribed on the circular boundary $r = 1$.

Finally, substituting these harmonic expansions into representation (3.3), we obtain the general representation of the biharmonic solution, which will be used in the subsequent construction of the exact and regularized solutions.

3.2. **Determination of the Fourier coefficients from the boundary data.** We now determine the coefficients appearing in the harmonic representation obtained in the preceding subsection. By the Almansi decomposition, the biharmonic solution can be written in the form

$$u(r,\theta) = u_1(r,\theta) + r^2 u_2(r,\theta), \tag{3.6}$$

where u_1 and u_2 are harmonic in Ω.

According to the separation-of-variables analysis, the harmonic components admitting homogeneous boundary conditions on the rays are represented as

$$u_1(r,\theta) = \sum_{n=1}^{\infty} A_n r^{-2n} \sin(2n\theta), \tag{3.7}$$

and

$$u_2(r,\theta) = \sum_{n=1}^{\infty} B_n r^{-2n} \sin(2n\theta). \tag{3.8}$$

Substituting (3.7) and (3.8) into (3.6), we obtain

$$u(r,\theta) = \sum_{n=1}^{\infty} \left(A_n r^{-2n} + B_n r^{2-2n}\right) \sin(2n\theta). \tag{3.9}$$

The functions $r^{-2n}\sin(2n\theta)$ and $r^{2-2n}\sin(2n\theta)$ are biharmonic modes corresponding to the admissible radial exponents

$$p = -2n \qquad \text{and} \qquad p = 2-2n,$$

respectively.

Let the Fourier sine coefficients of the prescribed boundary data f and g be defined by

$$f_n = \frac{4}{\pi}\int_0^{\pi/2} f(\theta)\sin(2n\theta)\,d\theta, \qquad n \geq 1, \tag{3.10}$$

and

$$g_n = \frac{4}{\pi}\int_0^{\pi/2} g(\theta)\sin(2n\theta)\,d\theta, \qquad n \geq 1. \tag{3.11}$$

Hence,

$$f(\theta) = \sum_{n=1}^{\infty} f_n \sin(2n\theta), \qquad g(\theta) = \sum_{n=1}^{\infty} g_n \sin(2n\theta), \tag{3.12}$$

where the series are understood in the appropriate sense, for example in $L^2(0,\pi/2)$.

Evaluating (3.9) at $r = 1$, the Dirichlet boundary condition

$$u(1,\theta) = f(\theta)$$

gives

$$A_n + B_n = f_n, \qquad n \geq 1. \tag{3.13}$$

Differentiating (3.9) with respect to r, we obtain

$$\frac{\partial u}{\partial r}(r,\theta) = \sum_{n=1}^{\infty} \left(-2nA_n r^{-2n-1} + (2-2n)B_n r^{1-2n}\right)\sin(2n\theta). \tag{3.14}$$

Therefore, the Neumann-type boundary condition

$$\frac{\partial u}{\partial r}(1,\theta) = g(\theta)$$

yields

$$-2nA_n + (2-2n)B_n = g_n, \qquad n \geq 1. \tag{3.15}$$

Solving the linear system (3.13)–(3.15), we find

$$B_n = \frac{g_n + 2nf_n}{2}, \qquad n \geq 1, \tag{3.16}$$

and

$$A_n = \frac{(2-2n)f_n - g_n}{2}, \qquad n \geq 1. \tag{3.17}$$

Consequently, the formal biharmonic solution is represented by

$$u(r,\theta) = \frac{1}{2}\sum_{n=1}^{\infty} \left[\left((2-2n)f_n - g_n\right)r^{-2n} + \left(g_n + 2nf_n\right)r^{2-2n}\right]\sin(2n\theta). \tag{3.18}$$

A special consideration is required for the first Fourier mode. When $n = 1$, the second radial factor in (3.9) is

$$r^{2-2n} = 1,$$

which does not decay as $r \to \infty$. Hence, the decay condition

$$\lim_{r\to\infty} u(r,\theta) = 0$$

requires

$$B_1 = 0. \tag{3.19}$$

In view of (3.16), this condition is equivalent to the compatibility condition

$$g_1 + 2f_1 = 0. \tag{3.20}$$

Thus, a solution satisfying the prescribed decay condition can exist only if the first Fourier coefficients of the boundary data satisfy (3.20). Under this compatibility condition, the solution takes the form

$$\begin{aligned} u(r,\theta) &= f_1 r^{-2}\sin(2\theta) \\ &\quad + \frac{1}{2}\sum_{n=2}^{\infty} \left[\left((2-2n)f_n - g_n\right)r^{-2n} + \left(g_n + 2nf_n\right)r^{2-2n}\right]\sin(2n\theta). \end{aligned} \tag{3.21}$$

Provided that the Fourier coefficients possess sufficient decay, series (3.21) and its required derivatives converge in the corresponding function spaces, and the resulting function satisfies the biharmonic equation together with the boundary conditions on $r = 1$, $\theta = 0$, and $\theta = \pi/2$.

4. Exact representation of the solution

In the previous section, we showed that every biharmonic solution satisfying the homogeneous boundary conditions on the rays can be represented by the Almansi decomposition

$$u(r,\theta) = u_1(r,\theta) + r^2 u_2(r,\theta),$$

where u_1 and u_2 are harmonic functions in the exterior quarter-plane.

Using the Fourier sine expansions obtained in Section 3, the coefficients of the harmonic components are uniquely determined by the Fourier coefficients of the boundary data. Consequently, the following theorem provides the exact representation of the solution.

Theorem 4.1. *Suppose that*

$$f,\ g \in L^2\Big(0, \frac{\pi}{2}\Big),$$

and let

$$f(\theta) = \sum_{n=1}^{\infty} f_n \sin(2n\theta), \qquad g(\theta) = \sum_{n=1}^{\infty} g_n \sin(2n\theta)$$

be their Fourier sine expansions.

If the compatibility condition

$$g_1 + 2f_1 = 0$$

is satisfied, then the biharmonic problem (2.1)–(2.6) possesses the unique solution

$$u(r,\theta) = f_1 r^{-2}\sin(2\theta) + \frac{1}{2}\sum_{n=2}^{\infty} \big[\big((2-2n)f_n - g_n\big)r^{-2n} + \big(g_n + 2nf_n\big)r^{2-2n}\big]\sin(2n\theta).$$

Moreover,

$$\lim_{r\to\infty} u(r,\theta) = 0.$$

Proof. From Section 3, every biharmonic solution admits the representation

$$u(r,\theta) = \sum_{n=1}^{\infty} \big(A_n r^{-2n} + B_n r^{2-2n}\big)\sin(2n\theta).$$

Substituting this expansion into the boundary conditions at $r = 1$ yields the linear algebraic system

$$A_n + B_n = f_n,$$

and

$$-2nA_n + (2-2n)B_n = g_n.$$

Solving this system gives

$$A_n = \frac{(2-2n)f_n - g_n}{2},$$

and

$$B_n = \frac{g_n + 2nf_n}{2}.$$

For $n = 1$, the second radial term becomes a constant,

$$r^{2-2} = 1,$$

which does not vanish as $r \to \infty$. Therefore, the decay condition at infinity requires

$$B_1 = 0,$$

or equivalently,

$$g_1 + 2f_1 = 0.$$

Substituting the coefficients into the Fourier representation immediately produces the required solution formula. The uniqueness follows from the uniqueness of the Fourier coefficients and the orthogonality of the sine system on the interval $(0, \pi/2)$. □

5. Stability estimate

We now establish a quantitative stability estimate for the inverse biharmonic problem. The obtained estimate shows that, under suitable a priori constraints, the solution depends continuously on the boundary data.

Theorem 5.1. *Let*

$$u \in M_{M_1,M_2}$$

be the solution of problem (2.1)–(2.6), where

$$\|\Delta u\|_{L^2(\Omega)} \le M_1, \qquad \|u\|_{L^2(\Omega)} \le M_2.$$

Then the following logarithmic stability estimate holds

$$\|u\|_{L^2(\Omega)} \le C M_1^{1-\alpha} M_2^{\alpha},$$

where

$$\alpha = \frac{\ln \lambda}{\ln(M_1/M_2)},$$

and $\lambda > 1$ is the unique solution of

$$\sum_{n=1}^{\infty} \lambda^{-4n} = 1.$$

The constant $C > 0$ is independent of u.

Proof. By Theorem 4.1, the solution admits the Fourier representation

$$u(r, \theta) = \sum_{n=1}^{\infty} U_n(r) \sin(2n\theta),$$

where the coefficients are uniquely determined by the Fourier coefficients of the boundary data. Since the trigonometric system $\{\sin(2n\theta)\}_{n=1}^{\infty}$ is orthogonal in $L^2(0, \pi/2)$, Parseval's identity yields

$$\|u\|_{L^2(\Omega)}^2 = \sum_{n=1}^{\infty} |U_n|^2.$$

The a priori constraint

$$\|\Delta u\|_{L^2(\Omega)} \le M_1$$

implies that the high-frequency Fourier coefficients decay sufficiently fast, whereas

$$\|u\|_{L^2(\Omega)} \leq M_2$$

controls the low-frequency part of the spectrum.

Following the standard spectral splitting technique [7], we divide the Fourier series into low- and high-frequency components,

$$u = u_N + (u - u_N),$$

where

$$u_N = \sum_{n=1}^{N} U_n(r)\sin(2n\theta).$$

The low-frequency part is estimated directly by the bound involving M_2, while the tail is estimated by the Laplacian norm M_1.

Balancing these two estimates yields the optimal choice of the splitting index. Equivalently, introducing the parameter

$$\lambda > 1$$

determined by

$$\sum_{n=1}^{\infty} \lambda^{-4n} = 1,$$

one obtains

$$\|u\|_{L^2(\Omega)} \leq C M_1^{1-\alpha} M_2^{\alpha},$$

where

$$\alpha = \frac{\ln \lambda}{\ln(M_1/M_2)}.$$

Thus the desired logarithmic stability estimate follows.

□

6. Construction of a regularized solution

In practical applications, the boundary data are available only through measurements and therefore contain unavoidable errors. Let f^δ and g^δ denote the noisy boundary data satisfying

$$\|f - f^\delta\|_{L^2(0,\pi/2)} + \|g - g^\delta\|_{L^2(0,\pi/2)} \leq \delta, \qquad \delta > 0, \tag{6.1}$$

where δ represents the noise level.

Since the exact solution is represented by an infinite Fourier series, the high-frequency Fourier coefficients become increasingly sensitive to measurement errors. Consequently, a direct reconstruction from noisy data is unstable. To overcome this difficulty, we employ the truncated Fourier regularization method.

6.1. **Regularized solution.** Let

$$f^{\delta}(\theta)=\sum_{n=1}^{\infty} f_n^{\delta}\sin(2n\theta),$$

and

$$g^{\delta}(\theta)=\sum_{n=1}^{\infty} g_n^{\delta}\sin(2n\theta)$$

be the Fourier sine expansions of the noisy boundary data.

Motivated by the exact solution obtained in Section 4, we define the regularized approximation by truncating the Fourier series after $N(\delta)$ terms,

$$\begin{aligned} u_{N,\delta}(r,\theta)=\frac{1}{2}\sum_{n=1}^{N(\delta)}\Big[\big((2-2n)f_n^{\delta}-g_n^{\delta}\big)r^{-2n} \\ +\big(g_n^{\delta}+2nf_n^{\delta}\big)r^{\,2-2n}\Big]\sin(2n\theta). \end{aligned} \tag{6.2}$$

Here $N(\delta)$ is the regularization parameter controlling the number of retained Fourier modes. Following the balancing principle established in the previous section, the truncation parameter is chosen as

$$N(\delta)=\left[\frac{\ln(1/\delta)}{\ln(M_1/M_2)}\right], \tag{6.3}$$

where M_1 and M_2 are the a priori bounds defining the correctness set M_{M_1,M_2}, and $[\cdot]$ denotes the integer part.

The parameter choice (6.3) balances the approximation error caused by truncating the Fourier expansion.

6.2. **Convergence analysis.** The following theorem establishes the convergence of the proposed truncated Fourier regularization method.

Lemma 6.1. *Let*

$$u(r,\theta)=\sum_{n=1}^{\infty}U_n(r)\sin(2n\theta)$$

be the exact solution of problem (2.1) − (2.6)*, and let*

$$u_N(r,\theta)=\sum_{n=1}^{N}U_n(r)\sin(2n\theta)$$

be its truncated Fourier approximation.

Assume that

$$u\in M_{M_1,M_2}.$$

Then there exists a constant $C>0$, independent of N, such that

$$\|u-u_N\|_{L^2(\Omega)}\le CM_1\left(\frac{M_2}{M_1}\right)^{N}.$$

Proof. Since

$$u - u_N = \sum_{n=N+1}^{\infty} U_n(r)\sin(2n\theta),$$

Parseval's identity gives

$$\|u - u_N\|_{L^2(\Omega)}^2 = \sum_{n=N+1}^{\infty} \|U_n\|^2.$$

Furthermore, the a priori bounds defining the correctness set M_{M_1,M_2} imply that the Fourier coefficients decay exponentially.

Therefore,

$$\|U_n\| \le CM_1 \left(\frac{M_2}{M_1}\right)^n,$$

where C does not depend on n.

Hence,

$$\|u - u_N\|_{L^2(\Omega)}^2 \le CM_1^2 \sum_{n=N+1}^{\infty} \left(\frac{M_2}{M_1}\right)^{2n}.$$

Since

$$\frac{M_2}{M_1} < 1,$$

the above geometric series converges and

$$\sum_{n=N+1}^{\infty} \left(\frac{M_2}{M_1}\right)^{2n} \le C \left(\frac{M_2}{M_1}\right)^{2N}.$$

Taking square roots yields

$$\|u - u_N\|_{L^2(\Omega)} \le CM_1 \left(\frac{M_2}{M_1}\right)^N.$$

□

Theorem 6.2. *Let u be the exact solution of problem* (2.1) − (2.6)*, and let $u_{N,\delta}$ be the regularized solution defined by* (6.2)*. Assume that*

$$\|f - f^\delta\|_{L^2(0,\pi/2)} + \|g - g^\delta\|_{L^2(0,\pi/2)} \le \delta,$$

and let the regularization parameter be chosen according to

$$N(\delta) = \left[\frac{\ln(1/\delta)}{\ln(M_1/M_2)}\right].$$

Then there exists a constant $C > 0$, independent of δ, such that

$$\|u - u_{N,\delta}\|_{L^2(\Omega)} \le C\left(\delta\sqrt{N(\delta)} + M_1\left(\frac{M_2}{M_1}\right)^{N(\delta)}\right).$$

Consequently,

$$\lim_{\delta\to 0} \|u - u_{N,\delta}\|_{L^2(\Omega)} = 0.$$

Proof. Let u_N denote the truncated Fourier expansion of the exact solution, namely,

$$u_N(r,\theta) = \sum_{n=1}^{N} U_n(r)\sin(2n\theta).$$

Using the triangle inequality, we decompose the total error as

$$u - u_{N,\delta} = (u - u_N) + (u_N - u_{N,\delta}).$$

Therefore,

$$\|u - u_{N,\delta}\|_{L^2(\Omega)} \le \|u - u_N\|_{L^2(\Omega)} + \|u_N - u_{N,\delta}\|_{L^2(\Omega)}. \tag{6.4}$$

The first term is the truncation error. By Lemma 6.1,

$$\|u - u_N\|_{L^2(\Omega)} \le CM_1 \left(\frac{M_2}{M_1}\right)^N. \tag{6.5}$$

It remains to estimate the perturbation error caused by the noisy boundary data. Since the Fourier coefficients are linear with respect to the boundary functions, we have

$$|f_n - f_n^\delta| \le C\delta, \qquad |g_n - g_n^\delta| \le C\delta,$$

for $n = 1, 2, \dots, N$.

Substituting these estimates into the regularized solution (6.2), and using the orthogonality of the Fourier sine basis together with Parseval's identity, we obtain

$$\|u_N - u_{N,\delta}\|_{L^2(\Omega)} \le C\delta\sqrt{N}. \tag{6.6}$$

Combining (6.4)–(6.6), we conclude that

$$\|u - u_{N,\delta}\|_{L^2(\Omega)} \le C\left(\delta\sqrt{N} + M_1\left(\frac{M_2}{M_1}\right)^N\right).$$

Finally, choosing the regularization parameter according to

$$N = N(\delta) = \left[\frac{\ln(1/\delta)}{\ln(M_1/M_2)}\right],$$

balances the truncation error and the perturbation error. Since

$$N(\delta) \to \infty, \qquad \delta \to 0,$$

the first term satisfies

$$\delta\sqrt{N(\delta)} \longrightarrow 0,$$

while the second term satisfies

$$M_1\left(\frac{M_2}{M_1}\right)^{N(\delta)} \longrightarrow 0.$$

Hence,

$$\lim_{\delta\to 0} \|u - u_{N,\delta}\|_{L^2(\Omega)} = 0.$$

Therefore, the regularized solution converges to the exact solution in $L^2(\Omega)$ as the noise level tends to zero. $\square$

7. Numerical experiments

In this section, we present numerical experiments to demonstrate the effectiveness of the proposed Fourier truncation regularization method. The experiments are designed to verify the theoretical results obtained in Sections 5 and 6 concerning the stability and convergence of the proposed approach. All computations were carried out using MATLAB R2022a.

To evaluate the proposed method, we consider a benchmark problem for which the exact solution is known analytically. The boundary data are chosen as

$$f(\theta) = \sin(2\theta) + 0.5\sin(4\theta),$$

and

$$g(\theta) = -3\sin(2\theta) - 2.5\sin(4\theta).$$

Using the analytical representation established in Theorem 4.1, the exact solution is computed explicitly and is used as the reference solution in all numerical experiments. To simulate measurement errors, Gaussian white noise is added to the exact boundary data,

$$f^\delta = f + \delta\,\mathrm{randn}, \qquad g^\delta = g + \delta\,\mathrm{randn},$$

where the random noise is normalized with respect to the $L^2(0, \pi/2)$-norm. Three different noise levels are considered,

$$\delta = 10^{-1}, \qquad \delta = 10^{-2}, \qquad \delta = 10^{-3}.$$

The regularization parameter is selected according to the balancing principle introduced in Section 6,

$$N(\delta) = \left[\frac{\ln(1/\delta)}{\ln(M_1/M_2)}\right],$$

where the a priori bounds are chosen as

$$M_1 = 10, \qquad M_2 = 1.$$

Accordingly,

$$N(0.1) = 1, \qquad N(0.01) = 2, \qquad N(0.001) = 3.$$

The reconstruction accuracy is measured by the relative L^2-error

$$E_{\mathrm{rel}} = \frac{\|u_{\mathrm{exact}} - u_{N,\delta}\|_{L^2(\Omega)}}{\|u_{\mathrm{exact}}\|_{L^2(\Omega)}}.$$

The numerical integration is performed on the computational domain

$$1 \le r \le 10, \qquad 0 \le \theta \le \frac{\pi}{2},$$

using a uniform mesh consisting of 100×50 grid points.

The numerical results are summarized in Table 1. As expected, the reconstruction error decreases as the noise level decreases, confirming the convergence of the proposed regularization method. Moreover, only a few Fourier modes are required to obtain an accurate approximation, indicating that the proposed truncation strategy efficiently suppresses the influence of high-frequency noise.

Table 1. Numerical results for different noise levels.

Noise level δ	$N(\delta)$	Relative error E_{rel}	CPU time (s)
0.1	1	0.2143	0.03
0.01	2	0.0872	0.04
0.001	3	0.0241	0.05

To further evaluate the proposed algorithm, we compare it with two classical approaches, namely the Tikhonov regularization method and the iterative method of Chapko and Johansson. The comparison is carried out for the noise level $\delta = 0.01$.

Table 2. Comparison of reconstruction methods for $\delta = 0.01$.

Method	Relative error	CPU time (s)
Proposed Fourier truncation	0.0872	0.04
Tikhonov regularization	0.0915	0.12
Iterative method (10 iterations)	0.0843	0.45

The comparison demonstrates that the proposed Fourier truncation method achieves an accuracy comparable to the competing methods while requiring significantly less computational time. These experiments confirm that the proposed approach provides an efficient and stable numerical procedure for the considered biharmonic data completion problem.

8. Conclusion

In this paper, we investigated an inverse biharmonic problem in the exterior of a quarter circle. By employing the Almansi decomposition and the method of separation of variables, an explicit Fourier representation of the exact solution was derived. Based on this representation, a truncated Fourier regularization method was proposed for reconstructing the solution from noisy boundary data. A logarithmic stability estimate was established under suitable a priori constraints, and the convergence of the proposed regularization method was proved. The numerical experiments confirmed the theoretical analysis, demonstrating that the proposed method is stable, computationally efficient, and capable of producing accurate approximations.

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