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CONTINUITY OF THE SPECTRUM AND WEYL TYPE THEOREMS FOR EXTENSION OF 2-ISOMETRIC OPERATORS

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Abstract. In this paper, we show that the spectrum, Weyl spectrum, and Browder spectrum are continuous on the set of all 2-quasi-2-isometric operators. Further, we show that k-quasi-2-isometric operator satisfies Bishop's property (β). Moreover, we prove Weyl type theorems for $f(d_{TS})$, where d_{TS} denote the generalized derivation or the elementary operator with k-quasi-2-isometric operator entries T and S^* and $f \in \mathcal{H}(\sigma(d_{TS}))$, the set of analytic functions which are defined on an open neighborhood of $\sigma(d_{TS})$.

1. Introduction

Let $B(\mathcal{H})$ denote the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ and let $\mathcal{N}(T)$, $\mathcal{R}(T)$, $\sigma(T)$, $\sigma_p(T)$ and $\sigma_a(T)$ denote the kernel, the range, the spectrum, point spectrum and the approximate point spectrum of $T \in B(\mathcal{H})$, respectively. Recall that an operator $T \in B(\mathcal{H})$ is called *2-isometry* if $T^{*2}T^2 - 2T^*T + I = 0$. This generalized isometries play an important role in the study of Dirichlet operators and other classes of operators [2, 10, 21]. An operator $T \in B(\mathcal{H})$ is called a *quasi-2-isometry* if $T^{*3}T^3 - 2T^{*2}T^2 + T^*T = 0$ [13]. An operator $T \in B(\mathcal{H})$ is said to be *k-quasi-2-isometric* if $T^{*k}(T^{*2}T^2 - 2T^*T + I)T^k = 0$ for some positive integer k [19]. See [1, 13, 14, 16, 19, 20] for more properties of these classes of operators. The following inclusions are well known:

2-isometry $\subseteq$ quasi-2-isometry $\subseteq$ k-quasi-2-isometry.

Let $\mathcal{G}$ denote the set of all compact subsets of $\mathbb{C}$ equipped with the Hausdorff metric. If $\mathcal{U}$ is a unital Banach algebra then the spectrum can be viewed as a function $\sigma : \mathcal{U} \to \mathcal{G}$, mapping each $T \in \mathcal{U}$ to its spectrum $\sigma(T)$. If $\{T_n\}$ is a sequence of elements of a unital Banach algebra $\mathcal{U}$, then $\liminf_n \sigma(T_n) = \{\lambda :$ there exists $\lambda_n \in \sigma(T_n)$ such that $\lambda = \lim_{n\to\infty} \lambda_n\}$ and $\limsup_n \sigma(T_n)$ is the set of all limit points of convergent sequences of the form $\{\lambda_{n_k}\}$, where $\lambda_{n_k} \in \sigma(T_{n_k})$. If $\liminf_n \sigma(T_n) = \limsup_n \sigma(T_n)$, we say $\lim_n \sigma(T_n) = \liminf_n \sigma(T_n) = \limsup_n \sigma(T_n)$. Since the set of invertible elements in $\mathcal{U}$ forms an open set, we can see that $\lim\ \inf_n \sigma(T_n) \subseteq \sigma(T)$ whenever the sequence of elements T_n converges

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to T in $\mathcal{U}$. We say that σ is continuous if

$$\sigma(T) = \lim_n \sigma(T_n)$$

or equivalently $\sigma(T) \subset \lim_n \sigma(T_n)$. It is well known that σ does have points of discontinuity in noncommutative algebras. The fundamental results on spectral continuity in general Banach algebras are contained in Newburgh's work [15]. Spectral continuity in the case where the Banach algebra is the C^* -algebra of all operators acting on a complex separable Hilbert space has been studied by Conway and Morrel[6]. There are many studies in the topic of spectral continuity in non-normal classes of operators. We refer the reader to [7, 8, 9, 11] and the references therein.

An operator $T \in B(\mathcal{H})$ is said to have the *single-valued extension property (SVEP)* if for every open subset U of $\mathbb{C}$ and any analytic function $f : U \to \mathcal{H}$ such that $(T - z)f(z) \equiv 0$ on U, we have $f(z) \equiv 0$ on U. A Hilbert space operator $T \in B(\mathcal{H})$ satisfies *Bishop's property* (β) if for every open subset U of $\mathbb{C}$ and every sequence $f_n : U \longrightarrow \mathcal{H}$ of analytic functions with $(T-z)f_n(z)$ converges uniformly to 0 in norm on compact subsets of U, $f_n(z)$ converges uniformly to 0 in norm on compact subsets of U. For more details see [3, 12, 17].

Let $\alpha(T)$ and $\beta(T)$ denote the nullity and the deficiency of $T \in \mathcal{B}(\mathcal{H})$, defined by $\alpha(T)$= dim($\mathcal{N}(T)$) and $\beta(T)$=dim($\mathcal{N}(T^*)$). An operator T is said to be *upper semi-Fredholm*(resp.,*lower semi- Fredholm*) if $\mathcal{R}(T)$ of $T \in B(\mathcal{H})$ is closed and $\alpha(T) < \infty$ (resp., $\beta(T) < \infty$). Let $SF_+(\mathcal{H})$ (resp., $SF_-(\mathcal{H})$) denote the semi-group of upper semi-Fredholm (resp., *lower semi-Fredholm*) operators on $\mathcal{H}$. An operator $T \in B(\mathcal{H})$ is said to be *semi-Fredhom* if $T \in SF_+(\mathcal{H}) \cup SF_-(\mathcal{H})$ and Fredholm if $T \in SF_+(\mathcal{H}) \cap SF_-(\mathcal{H})$. The index of semi-Fredholm operator T is defined by ind $(T) = \alpha(T) - \beta(T)$. Recall that the *ascent* of an operator $T \in B(\mathcal{H})$ is the smallest non negative integer $\mathtt{p}$:=$\mathtt{p}(T)$ such that $\mathcal{N}(T^{\mathtt{p}}) = \mathcal{N}(T^{(\mathtt{p}+1)})$. The *descent* of $T \in B(\mathcal{H})$ is defined as the smallest non negative integer $\mathtt{q} := \mathtt{q}(T)$ such that $\mathcal{R}(T^{\mathtt{q}}) = \mathcal{R}(T^{(\mathtt{q}+1)})$. An operator $T \in B(\mathcal{H})$ is *Weyl*, $T \in \mathcal{W}$, if it is Fredholm of index zero, and Browder, $T \in \mathcal{B}$, if T is Fredholm of finite ascent and descent. The *Weyl spectrum* $\sigma_w(T)$ and Browder spectrum $\sigma_b(T)$ of T are given by

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \mathcal{W}\}$$

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \mathcal{B}\}.$$

Let $E_0(T)$ denote the set of eigenvalues of T of finite geometric multiplicity. We say that $T \in B(\mathcal{H})$ satisfies *Weyl's theorem* if

$$\sigma(T) \setminus \sigma_w(T) = E_0(T).$$

We refer the readers to [3, 18], where Weyl type theorems are extensively treated.

The left multiplication operator L_T and the right multiplication operator R_T for $T \in B(\mathcal{H})$ on $B(\mathcal{H})$ are defined by $L_T(X) = TX$ and $R_T(X) = XT$ for every $X \in B(\mathcal{H})$. Let d_{TS} denote the *elementary operator* $\Delta_{TS} : B(\mathcal{H}) \to B(\mathcal{H})$ defined by $\Delta_{TS}(X) = (L_T R_S - I)(X)$ or the *generalized derivation* $\delta_{TS} : B(\mathcal{H}) \to B(\mathcal{H})$ defined by $\delta_{TS}(X) = L_T(X) - R_S(X)$.

J.Shen and F.Zuo[19] give a operator matrix representation of k-quasi- 2-isometric operator and studied some properties of this operator. In this paper, we prove that the spectrum, Weyl spectrum, and Browder spectrum are continuous on the set of all 2-quasi-2-isometric operators. We show that k-quasi-2-isometric operators have Bishop property(β) and study some variant of Weyl theorem for elementary operator with k-quasi-2-isometric operator entries.

2. Results

Theorem 2.1. *For a k-quasi-2-isometric operator T, $T^{*k+1}T^{k+1} \geq T^{*k}T^k$.*

Proof. Since T is a k-quasi-2-isometric operator, we have

$$T^{*k+2}T^{k+2} = 2T^{*k+1}T^{k+1} - T^{*k}T^k.$$

Claim: $T^{*n+k+1}T^{n+k+1} = (n+1)T^{*k+1}T^{k+1} - nT^{*k}T^k$ for all $n \geq 1$.
We use the method of mathematical induction to prove the claim. For $n = 1$, the result is trivial. It can be easily verified for $n = 2$. Suppose that the result is true for $n = m$. We have to prove the result for $n = m+1$. Consider,

$$\begin{aligned} T^{*(m+1)+k+1}T^{(m+1)+k+1} &= T^*(T^{*m+k+1}T^{m+k+1})T \\ &= T^*((m+1)T^{*k+1}T^{k+1} - mT^{*k}T^k)T \\ &= (m+1)T^{*k+2}T^{k+2} - mT^{*k+1}T^{k+1} \\ &= (m+1)(2T^{*k+1}T^{k+1} - T^{*k}T^k) - mT^{*k+1}T^{k+1} \\ &= (m+2)T^{*k+1}T^{k+1} - (m+1)T^{*k}T^k. \end{aligned}$$

Hence the result is true for $n = m+1$. Hence the claim is proved. Since $T^{*n+k+1}T^{n+k+1}$ for all $n \geq 1$ is a positive operator and by using the claim we have for all $n \geq 1$

$$\begin{aligned} (n+1)T^{*k+1}T^{k+1} - nT^{*k}T^k \geq 0 &\Rightarrow (n+1)T^{*k+1}T^{k+1} \geq nT^{*k}T^k \\ &\Rightarrow (1+\frac{1}{n})T^{*k+1}T^{k+1} \geq T^{*k}T^k. \end{aligned}$$

As $n \to \infty$, we have, $T^{*k+1}T^{k+1} \geq T^{*k}T^k$. □

Lemma 2.2. *Let T be a 2-quasi-2-isometric operator, $\lambda \in \sigma_p(T)$ and $T = \begin{pmatrix} \lambda I & T_{12} \\ 0 & T_{22} \end{pmatrix}$ on $\mathcal{H} = \mathcal{N}(T-\lambda I) \oplus \mathcal{N}(T-\lambda I)^\perp$. Then $\mathcal{N}(T_{22}-\lambda I) = \{0\}$.*

Proof. We have, $T = \begin{pmatrix} \lambda I & T_{12} \\ 0 & T_{22} \end{pmatrix}$ on $\mathcal{H} = \mathcal{N}(T-\lambda I) \oplus \mathcal{N}(T-\lambda I)^\perp$.
Since T is a 2-quasi-2-isometric operator, by Theorem 2.1 and [14, Theorem 2.2], we have

$$T^{*3}T^3 - T^{*2}T^2 = \begin{pmatrix} 0 & \overline{\lambda}^3 T_{12}T_{22}^2 \\ \lambda^3 T_{22}^{*2}T_{12}^* & A \end{pmatrix} \geq 0,$$

where $A = \overline{\lambda}^2 T_{12}^*T_{12}T_{22}^2 + \overline{\lambda}T_{22}^*T_{12}^*T_{12}T_{22}^2 + \lambda^2 T_{22}^{*2}T_{12}^*T_{12} + \lambda T_{22}^{*2}T_{12}^*T_{12}T_{22} + T_{22}^{*2}T_{12}^*T_{12}T_{22}^2 + T_{22}^{*3}T_{22}^3 - T_{22}^{*2}T_{22}^2$. Recall that $\begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix} \geq 0$ if and only if X ≥ 0, Z ≥ 0,$Y =$

$X^{\frac{1}{2}}WZ^{\frac{1}{2}}$ for some contraction W. So we have $T_{12}T_{22}^2 = 0$.
Let $(T_{22} - \lambda I)x = 0, \| x \| = 1$. If $\lambda = 0$, then for $y = 0 \oplus x$ we have,

$$T^3 y = \begin{pmatrix} T_{12}T_{22}^2 x \\ T_{22}^3 x \end{pmatrix} = 0.$$

Since T is a 2-quasi-2-isometric, $T^{*4}T^4 - 2T^{*3}T^3 + T^{*2}T^2 = 0$. Hence

$$\| T^4 y \|^2 + \| Ty \|^2 = 2 \| T^3 y \|^2 = 0.$$

Thus, $Ty = 0$. Therefore, $y \in \mathcal{N}(T - \lambda I) \cap \mathcal{N}(T - \lambda I)^\perp$ and so $x = 0$.
If $\lambda \neq 0$, then for $y = 0 \oplus x$ we have

$$(T - \lambda I)(0 \oplus x) = T_{12}x = \frac{1}{\lambda^2} T_{12}T_{22}^2 x = 0.$$

Thus, $y \in \mathcal{N}(T - \lambda I) \cap \mathcal{N}(T - \lambda I)^\perp$ and hence $x = 0$. This completes the proof. $\square$

Recall that an operator T is said to be *isoloid* if each isolate point in $\sigma(T)$ is an eigenvalue of T. An operator $T \in B(\mathcal{H})$ is said to be *polaroid* if each isolate point in $\sigma(T)$ is a pole of the resolvent of T. An operator $T \in B(\mathcal{H})$ is said to be algebraically k-quasi-2-isometric operator if $p(T)$ is k-quasi-2-isometric operator for some non-constant polynomial p.

Lemma 2.3. *Let T be a k-quasi-2-isometric operator. Then $\sigma(T) = \{\lambda\}$ implies that $Tx = \lambda x$ for some $x \in \mathcal{H}$ i.e. λ is an eigenvalue of T.*

Proof. By [19, Corollary 2.11], T is an isoloid and the required result follows. $\square$

Lemma 2.4. *Suppose that T is a quasinilpotent algebraically k-quasi-2-isometric operator. Then T is nilpotent.*

Proof. Suppose that $p(T)$ is k-quasi-2-isometric operator for some non-constant polynomial p. Since $\sigma(p(T)) = p(\sigma(T))$ and $\sigma(T) = 0$, the operator $p(T) - p(0)$ is quasinilpotent. By Lemma 2.3, $\sigma(p(T) - p(0)) = \{0\}$ implies that $p(T) - p(0) = 0$. Hence it follows that,

$$0 = p(T) - p(0) = cT^m(T - \lambda_1 I)(T - \lambda_2 I) \cdots (T - \lambda_n I)$$

where $m \geq 1$. Since $T - \lambda_i I$ is invertible for every $\lambda_i \neq 0$, we must have $T^m = 0$. This completes the proof. $\square$

Proposition 2.5. [5] *Let $\mathcal{H}$ be a complex Hilbert space. Then there exist a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and $\psi : B(\mathcal{H}) \to B(\mathcal{K})$ such that ψ is an $*$-isometric isomorphism preserving the order satisfying*
(i) $\psi(T^*) = \psi(T)^*$, $\psi(I_{\mathcal{H}}) = I_{\mathcal{K}}$, $\psi(\alpha T + \beta S) = \alpha\psi(T) + \beta\psi(S)$, $\psi(TS) = \psi(T)\psi(S)$, $\|\psi(T)\| = \|T\|$, $\psi(T) \leq \psi(S)$ *if* $T \leq S$ *for all* $T, S \in B(\mathcal{H})$ *and for all* $\alpha, \beta \in \mathbb{C}$.
(ii) $\sigma(T) = \sigma(\psi(T)), \sigma_a(T) = \sigma_a(\psi(T)) = \sigma_p(\psi(T))$, *where* $\sigma_p(T)$ *is the point spectrum of* T.

Definition 2.6. [8] The set $C(i)$ consists of the operators $T \in B(\mathcal{H})$ for which $\sigma(T) = \{0\}$ implies T is nilpotent and T° (the Berberian extension of T) satisfies the property;

$$T^\circ = \begin{pmatrix} \lambda I & T_{12} \\ 0 & T_{22} \end{pmatrix} \quad \text{on} \quad \mathcal{H} = \mathcal{N}(T^\circ - \lambda I) \oplus \mathcal{N}(T^\circ - \lambda I)^\perp$$

at every non-zero $\lambda \in \sigma_p(T^\circ)$ for some operators T_{12} and T_{22} such that $\lambda \notin \sigma_p(T_{22})$ and $\sigma(T^\circ) = \sigma(T_{22}) \cup \{\lambda\}$.

Theorem 2.7. *The spectrum σ is continuous on the set of all 2-quasi-2-isometric operators.*

Proof. Suppose that T is a 2-quasi-2-isometric operator. Let $\psi : B(\mathcal{H}) \to B(\mathcal{K})$ be Berberian's faithful $*$-representation of Proposition 2.5. Then,

$$\begin{aligned} &(\psi(T))^{*2}((\psi(T))^{*2}(\psi(T))^2 - 2(\psi(T))^*\psi(T) + I)(\psi(T))^2 \\ &= \psi(T^{*2})(\psi(T^{*2})\psi(T^2) - 2\psi(T^*)\psi(T) + \psi(I))\psi(T^2) \\ &= \psi(T^{*2}(T^{*2}T^2 - 2T^*T + I)T^2) = 0. \end{aligned}$$

Hence $\psi(T)$ is a 2-quasi-2-isometric operator. By Lemma 2.4 and Lemma 2.2 we have $T \in C(i)$. Therefore, we have that the spectrum σ is continuous on the set of 2-quasi-2-isometric operators by [8, Theorem 1.1]. □

Corollary 2.8. *The Weyl spectrum and the Browder spectrum are continuous in the class of all 2-quasi-2-isometric operators.*

Proof. By [19], 2-quasi-2-isometric operator satisfies Weyl's theorem. From Theorem 2.7, spectrum σ is continuous on the set of all 2-quasi-2-isometric operator. Applying Theorem 2.2 and Theorem 2.3 of [7] it follows that Weyl spectrum and Browder spectrum are continuous in the class of all 2-quasi-2-isometric operators. □

Theorem 2.9. *If $T \in B(\mathcal{H})$ is a k-quasi-2-isometric operator, then T satisfies Bishop's property (β).*

Proof. Suppose $T \in B(\mathcal{H})$ is a k-quasi-2-isometric operator. Let $\{f_n\}$ be a sequence of analytic functions from an open subset U of $\mathbb{C}$ to $\mathcal{H}$ such that $(T - \lambda I)f_n(\lambda) \to 0$ as $n \to \infty$ uniformly on compact subsets of U. Suppose T^k has a dense range in $\mathcal{H}$. Then T is a 2-isometric operator. Then by [20, Lemma 2.6] T has Bishop's property (β). Suppose T^k does not have a dense range in $\mathcal{H}$. Then by [19, Theorem 2.1]

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on} \quad \mathcal{H} = \overline{\mathcal{R}(T^k)} \oplus \mathcal{N}(T^{*k})$$

where T_1 is a 2-isometric operator and $T_3^k = 0$. Since $f_n(\lambda) \in \mathcal{H}$, $f_n(\lambda) = f_{n1}(\lambda) \oplus f_{n2}(\lambda)$ where $f_{n1}(\lambda) \in \overline{\mathcal{R}(T^k)}$ and $f_{n2}(\lambda) \in \mathcal{N}(T^{*k})$. Then,

$$\begin{aligned}(T-\lambda I)f_n(\lambda) &= \begin{pmatrix} T_1-\lambda I & T_2 \\ 0 & T_3-\lambda I\end{pmatrix}\begin{pmatrix} f_{n1}(\lambda) \\ f_{n2}(\lambda)\end{pmatrix} \\ &= \begin{pmatrix} (T_1-\lambda I)f_{n1}(\lambda)+T_2f_{n2}(\lambda) \\ (T_3-\lambda I)f_{n2}(\lambda)\end{pmatrix} \to 0\end{aligned}$$

as $n \to \infty$ uniformly on all compact subsets of U. Thus, $(T_3 - \lambda I)f_{n2}(\lambda) \to 0$ as $n \to \infty$ uniformly on all compact subsets of U. But T_3 is nilpotent and hence it has Bishop's property (β). Therefore, $f_{n2}(\lambda) \to 0$ as $n \to \infty$ uniformly on all compact subsets of U. Also, $(T_1-\lambda I)f_{n1}(\lambda)+T_2f_{n2}(\lambda) \to 0$ as $n \to \infty$ uniformly on all compact subsets of U. Since T_2 is bounded and $f_{n2}(\lambda) \to 0$ on all compact subsets of U, we have, $(T_1 - \lambda I)f_{n1}(\lambda) \to 0$ as $n \to \infty$ on all compact subsets of U. But T_1 is a 2-isometric operator and by [20, Lemma 2.6] every 2-isometric operator has Bishop's property (β). Hence, $f_{n1}(\lambda) \to 0$ as $n \to \infty$ uniformly on all compact subsets of U. Therefore, $f_n(\lambda) = f_{n1}(\lambda) \oplus f_{n2}(\lambda) \to 0$ as $n \to \infty$ uniformly on all compact subsets of U. Hence T has Bishop's property (β). □

Let $SF_+^-(\mathcal{H}) = \{T \in SF_+(\mathcal{H}) : \mathrm{ind(T)} \le 0\}$. The essential approximate point spectrum $\sigma_{SF_+^-}(T)$ of T is defined by $\sigma_{SF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin SF_+^-(\mathcal{H})\}$. Let $E_0^a(T) = \{\lambda \in \mathbb{C} : \lambda \in \mathrm{iso}\,\sigma_a(T) \text{ and } 0 < \alpha(T-\lambda) < \infty\}$. An operator $T \in B(\mathcal{H})$ holds *a-Weyl's theorem* if $\sigma_{SF_+^-}(T) = \sigma_a(T) \setminus E_0^a(T)$.

An operator $T \in B(\mathcal{H})$ is called *B-Fredholm*, if there exist a non negative integer n for which the induced operator $T_{[n]} : \mathcal{R}(T_{[n]}) \to \mathcal{R}(T_{[n]})$(in particular $T_{[0]} = T$) is Fredholm in the usual sense. Let $\mathcal{SBF}_+^-(\mathcal{H})$ denote the class of all upper B-Fredholm operators such that ind $(T) \le 0$. An operator $T \in B(\mathcal{H})$ is called *B-Weyl*, $T \in \mathcal{BW}$, if it is B-Fredholm with $\mathrm{ind}(T_{[n]}) = 0$. The upper B-Weyl spectrum $\sigma_{SBF_+^-}(T)$ and B-Weyl spectrum $\sigma_{BW}(T)$ of T are given by

$$\sigma_{SBF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \mathcal{SBF}_+^-(\mathcal{H})\}$$

$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \mathcal{BW}\}.$$

Let $E_a(T)$ be the set of all eigenvalues of T which are isolated in $\sigma_a(T)$. We say that T satisfies *generalized Weyl's theorem* if $\sigma_{BW}(T) = \sigma(T) \setminus E(T)$, *property* (w) if $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E^0(T)$, *property* (gw) if $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$, and *generalized a-Weyl's theorem* if $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E_a(T)$. We refer the readers to [3], where Weyl type theorems are extensively treated.

For $T \in \mathcal{B}(\mathcal{H})$ and $x \in \mathcal{H}$, the *local resolvent set* of T at x is the set $\rho_T(x)$ consist of elements $z_0 \in \mathbb{C}$ such that there exists an analytic function $f(z)$ defined in a neighborhood of z_0, with values in $\mathcal{H}$, which verifies $(T - z)f(z) = x$. The complement of the local resolvent set $\rho_T(x)$ of T at x in $\mathbb{C}$ is the *local spectrum* $\sigma_T(x)$ of T at x. For each subset U of $\mathbb{C}$, the set $\mathcal{H}_T(U) = \{x \in \mathcal{H} : \sigma_T(x) \subseteq U\}$ is called the local spectral subspace of T. An operator $T \in B(\mathcal{H})$ is said to have *Dunford's property* (C) if $\mathcal{H}_T(U)$ is closed for each closed subset U of $\mathbb{C}$. It is well known that, Bishop's property $(\beta) \Rightarrow$ Dunford's property $(C) \Rightarrow$ SVEP.

For more details see [3, 12].

In [16], the author studied Weyl type theorems for the elementary operator with quasi-2-isometric operator entries. Let $\mathcal{H}(\sigma(d_{TS}))$ denote the set of analytic functions which are defined on an open neighborhood of $\sigma(d_{TS})$. Now we show that if $T, S^* \in B(\mathcal{H})$ are k-quasi-2-isometric operator, then generalized Weyl's theorem, a-Weyl's theorem, property (w), property (gw) and generalized a-Weyl's theorem holds for $f(d_{TS})$ for every $f \in \mathcal{H}(\sigma(d_{TS}))$.

Lemma 2.10. *Let T and S^* be k-quasi-2-isometric operators. Then d_{TS} satisfies single-valued extension property (SVEP).*

Proof. By Theorem 2.9, T and S^* satisfies Bishop's property (β). Since Bishop's property implies Dunford's property (C) and by [12, Corollary 3.6.11], it follows that L_T satisfies Dunford's property(C). By [12, Theorem 2.5.5] and [12, Corollary 3.6.11], it follows that R_S satisfies Dunford's property (C). Hence both TX and XS satisfies Dunford's property (C) for all $X \in B(\mathcal{H})$. Since L_T and R_S commutes, it follows that SVEP holds for both $TX - XS$ and $TXS - X$ by [12, Theorem 3.6.3]. Then d_{TS} has SVEP. □

Lemma 2.11. *Let T and S^* be k-quasi-2-isometric operators. Then d_{TS} is a polaroid.*

Proof. Since T and S^* are k-quasi-2-isometric operators, T and S^* are polaroids by [19, Theorem 2.10]. Also, if S^* is a polaroid, then S is a polaroid. Therefore, d_{TS} is a polaroid by [23, Lemma 4.1]. □

Theorem 2.12. *Let $T, S^* \in B(\mathcal{H})$ be k-quasi-2-isometric operators. Then generalized Weyl's theorem holds for d_{TS}.*

Proof. Since T and S^* are *k-quasi-2-isometric* operators, Lemma 2.10 implies d_{TS} has SVEP. By Lemma 2.11, d_{TS} is a polaroid. Then by [4, Theorem 3.10], it follows that generalized Weyl's theorem holds for d_{TS}. □

Theorem 2.13. *Let $T, S^* \in B(\mathcal{H})$ be k-quasi-2-isometric operators. Then $f(d_{TS})$ satisfies generalized Weyl's theorem, a-Weyl's theorem, property (w), property (gw) and generalized a-Weyl's theorem hold for every $f \in \mathcal{H}(d_{TS})$.*

Proof. Since T and S^* are k-quasi-2-isometric operators, Lemma 2.10 implies d_{TS} has SVEP. By Lemma 2.11, d_{TS} is a polaroid and so d_{TS} is an isoloid. By [22, Theorem 2.2], it follows that generalized Weyl's theorem holds for $f(d_{TS})$ for every $f \in \mathcal{H}(\sigma(d_{TS}))$. Applying [4, Theorem 3.12], $f(d_{TS})$ satisfies a-Weyl's theorem, property (w), property (gw) and generalized a-Weyl's theorem for every $f \in \mathcal{H}(\sigma(d_{TS}))$. □

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