Gulf Journal of Mathematics
Vol 23, Issue 2 (2026) 1-17

ISSN: 2309-4966
https://doi.org/10.56947/d646mf63

ISHIKAWA-MANN-FIBONACCI BEST PROXIMITY POINTS FOR CYCLIC RELATIVELY NONEXPANSIVE MAPPINGS

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Abstract. We study best proximity point convergence for cyclic relatively nonexpansive mappings in uniformly convex Banach spaces. We analyze two Mann-type iterations: the n-powered scheme and its Fibonacci-accelerated variant. Under suitable conditions, we establish strong and weak convergence theorems. Our main result shows that the n-powered iteration converges strongly to a best proximity point, while the Fibonacci variant achieves doubly exponential convergence. An application to optimal traffic signal timing is provided.

Keywords. Best proximity point, cyclic mapping, relatively nonexpansive mapping, Mann iteration, Fibonacci sequence, convergence.
2020 Mathematics Subject Classification. 47H09, 47H10, 54H25, 65J15, 90B20.

1. Introduction and Motivation

The theory of fixed points has been a central theme in nonlinear analysis since the early 20th century, with applications ranging from differential equations to optimization and game theory. In 1953, Mann [9] introduced an iterative method for approximating fixed points of nonexpansive mappings in uniformly convex Banach spaces. This iteration, defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S(x_n), \quad n \geq 0, \tag{1}$$

where (α_n) is a sequence in $(0, 1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, laid the foundation for numerous subsequent developments.

Two decades later, Ishikawa [7] proposed a two-step iteration to handle mappings with weaker contractive conditions:

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S(y_n), \\ y_n = (1 - \beta_n)x_n + \beta_n S(x_n), \end{cases} \tag{2}$$

with $0 \leq \alpha_n \leq \beta_n \leq 1$ and $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$. These classical iterations have been extensively studied and applied to various classes of mappings.

Date: Received: Jun 16, 2026; Revised: Jul 14, 2026; Accepted: Jul 19, 2026.
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While fixed point theory deals with equations of the form $Sx = x$, many real-world problems involve mappings that do not necessarily have fixed points. Consider, for instance, a cyclic mapping that sends points from one set to another. In such cases, the natural question is not about finding x such that $Sx = x$ (which may not exist), but rather finding a point that minimizes the distance between x and Sx. This leads to the concept of *best proximity points*.

A mapping $S : A \cup B \to A \cup B$ is called *cyclic* if $S(A) \subseteq B$ and $S(B) \subseteq A$. For such mappings, a point $p \in A \cup B$ satisfying

$$\|p - Sp\| = d(A, B) := \inf\{\|x - y\| : x \in A, y \in B\} \tag{1.1}$$

is called a best proximity point. This concept generalizes fixed points (when $A \cap B \neq \emptyset$ and $d(A, B) = 0$) and has important applications in optimization, game theory, and variational inequalities.

In 1972, Goebel and Kirk [5] proved that asymptotically nonexpansive mappings on bounded closed convex subsets of uniformly convex Banach spaces have fixed points. This result opened new avenues for studying more general classes of mappings. Schu [10] later modified the Mann iteration to approximate fixed points of such mappings:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S^n(x_n), \tag{3}$$

where (α_n) is a sequence in $(0, 1)$ and S is asymptotically nonexpansive. This modification demonstrated the power of using higher iterates in iterative schemes.

The study of iterative processes for cyclic relatively nonexpansive mappings has been further developed in recent years. For instance, Gabeleh, Aliyari, and Karapinar [4] investigated Mann and Ishikawa iterative processes for such mappings in uniformly convex Banach spaces, providing foundational results that motivate the present work. Moreover, the convergence analysis of such iterative schemes has been extended to more general settings, such as convex generalized metric spaces, as explored by Elkouch and Mouline [2]. In a related direction, Belbaki, Karapinar, and Ould-Hammouda [1] introduced a new class of monotone generalized nonexpansive mappings known as Reich type λ-α-nonexpansive mappings and established weak and strong convergence theorems for Krasnoselskii iteration in Banach spaces with partial order, with applications to $L^1([0, 1])$. More recently, Hammache, Karapinar, and Ould-Hammouda [6] studied admissible weak contractions in the framework of b-metric-like spaces. In a further development, Mammeri and Ould-Hammouda [8] introduced a Fibonacci-Mann process for monotone total asymptotically nonexpansive mappings in partially ordered Banach spaces, extending classical fixed point iterations to a broader class of mappings.

The Fibonacci sequence, defined by

$$f(0) = 0, \quad f(1) = 1, \quad f(n + 1) = f(n) + f(n - 1) \quad \forall n \geq 1, \tag{4}$$

possesses remarkable properties, including exponential growth with rate given by the golden ratio $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$. In recent years, researchers have begun exploring the use of Fibonacci numbers to accelerate iterative methods. The key insight is that replacing the exponent n in Schu's iteration with $f(n)$ leads to

dramatically faster convergence due to the exponential growth of $f(n)$. A formal comparison of convergence rates is provided in Theorem 3.7.

This paper makes several contributions to the theory of best proximity points. We establish a comprehensive theoretical foundation for studying best proximity points of cyclic relatively nonexpansive mappings in uniformly convex Banach spaces, including key properties of the P-property and metric projections. We prove strong and weak convergence theorems for the n-powered Mann iteration $x_{n+1} = (1-\alpha_n)x_n + \alpha_n S^{2n}(x_n)$, showing that under suitable conditions, the sequence converges to a best proximity point. The use of even powers $2n$ is essential to maintain the cyclic structure; indeed, if $x_n \in A$, then $S^{2n}(x_n) \in A$ since even powers of a cyclic mapping map each set into itself. We introduce an accelerated version using the Fibonacci sequence, $x_{n+1} = (1-\alpha_n)x_n + \alpha_n S^{2f(n)}(x_n)$, which achieves doubly exponential convergence due to the rapid growth of $f(n)$. We prove a crucial compactness lemma (Lemma 3.1) showing that finite unions of $S^{2k}(A)$ are compact when $S^2(A)$ is compact, which is essential for strong convergence. We provide an explicit example in $\mathbb{R}^3$ with a rotation-contraction mapping that satisfies all hypotheses and numerically demonstrates the superior convergence of the Fibonacci-powered iteration. Finally, we give a detailed application to optimal traffic signal timing.

The paper is organized as follows. Section 2 presents preliminary definitions and results, including uniform convexity, the P-property, and properties of metric projections. Section 3 contains the main results, indeed: we analyze the n-powered Mann iteration, introduce the Fibonacci-powered variant, and we establish both strong and weak convergence theorems. Section 4 provides a concrete example illustrating the theoretical results, with numerical simulations comparing the convergence rates, and includes a detailed application to optimal traffic signal timing. Section 5 concludes the paper and suggests directions for future research.

2. Preliminaries

Throughout this paper, X denotes a real Banach space with norm $\|\cdot\|$. For two subsets $A, B \subset X$, we denote $d(A,B) = \inf\{\|x-y\| : x \in A, y \in B\}$. The symbols $\rightharpoonup$ and $\to$ denote weak and strong convergence, respectively.

2.1. Basic definitions.

Definition 2.1. Assume that A and B are two nonempty subsets of a normed linear space X. A mapping $S : A \cup B \to A \cup B$ is called *cyclic* if $S(A) \subseteq B$ and $S(B) \subseteq A$.

Remark 2.2. For a cyclic mapping $S : A \cup B \to A \cup B$, we have $\|x - S(x)\| \geq d(A,B)$ for all $x \in A \cup B$.

Definition 2.3. A point $p \in A \cup B$ is said to be a *best proximity point* for the cyclic mapping $S : A \cup B \to A \cup B$ if $\|p - S(p)\| = d(A,B)$.

Definition 2.4. The mapping $S : A \cup B \to A \cup B$ is *relatively nonexpansive* if $\|S(x) - S(y)\| \leq \|x - y\|$ for all $(x,y) \in A \times B$.

Theorem 2.5 ([3]). *Let (A, B) be a nonempty bounded, closed and convex pair of disjoint subsets of a uniformly convex Banach space X. If $S : A \cup B \to A \cup B$ is a cyclic relatively nonexpansive mapping, then S has a best proximity point.*

Definition 2.6. A Banach space X with positive modulus of convexity $\delta_X(\epsilon)$ is said to be *uniformly convex* where $0 < \epsilon < 2$ and

$$\delta_X(\epsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon \right\}. \tag{5}$$

It is known that Hilbert spaces and ℓ^p spaces $(1 < p < \infty)$ are uniformly convex. The following lemma provides a useful characterization.

Lemma 2.7 ([11]). *A Banach space X is uniformly convex if and only if for each number $r > 0$, there exists a continuous strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(t) = 0 \iff t = 0$, such that*

$$\|\lambda x + (1-\lambda)y\|^2 \leq \lambda \|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\phi(\|x-y\|), \tag{6}$$

for all $\lambda \in [0, 1]$, $x, y \in X$ where $\|x\| \leq r$, $\|y\| \leq r$.

Lemma 2.8. *Consider a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$. If a sequence $(r_n)_n$ in $[0, \infty)$ satisfies $\lim_{n\to\infty} \phi(r_n) = 0$, then $\lim_{n\to\infty} r_n = 0$.*

Lemma 2.9 ([3]). *Let A be a nonempty, closed and convex subset and B be a nonempty and closed subset of a uniformly convex Banach space X. Let $(x_n)_n, (z_n)_n$ be sequences in A and let $(y_n)_n$ be a sequence in B satisfying:*

$$\text{(i) } \|x_n - y_n\| \to d(A, B), \tag{2.1}$$

$$\text{(ii) } \|z_n - y_n\| \to d(A, B), \tag{2.2}$$

then $\|x_n - z_n\| \to 0$.

Remark 2.10. Let A, B be nonempty, closed subsets of a normed linear space X. We say that a pair (A, B) satisfies a property if both A and B satisfy that property.

The *proximal pair* of (A, B) will be denoted (A_0, B_0) and defined by:

$$A_0 = \{x \in A : \|x - y\| = d(A, B) \text{ for some } y \in B\}, \tag{2.3}$$

$$B_0 = \{y \in B : \|x - y\| = d(A, B) \text{ for some } x \in A\}. \tag{7}$$

Notice that if (A, B) is a nonempty, bounded, closed and convex pair in a reflexive Banach space X, then (A_0, B_0) is also a nonempty, bounded, closed and convex pair in X.

The *metric projection operator* is a set-valued mapping $\mathcal{P}_A : X \to 2^A$ defined by

$$\mathcal{P}_A(x) = \{y \in A : \|x - y\| = \text{dist}(x, A)\}. \tag{8}$$

It is known that if A is a nonempty, bounded, closed and convex subset of a uniformly convex Banach space X, then the metric projection $\mathcal{P}_A$ is single-valued from X to A. Furthermore, if A is a nonempty, closed and convex subset of a Hilbert space $\mathbb{H}$, then $\mathcal{P}_A$ is nonexpansive.

Definition 2.11 ([3]). Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. The pair (A, B) is said to have the *P-property* if and only if

$$\begin{cases} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B) \end{cases} \Rightarrow d(x_1, x_2) = d(y_1, y_2), \tag{9}$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

It is well-known that every nonempty, bounded, closed, and convex pair in a uniformly convex Banach space X has the P-property.

Proposition 2.12 ([3]). *Let (A, B) be a nonempty bounded, closed and convex pair in a uniformly convex Banach space X. Define $\mathcal{P} : A_0 \cup B_0 \to A_0 \cup B_0$ as:*

$$\mathcal{P}(x) = \begin{cases} \mathcal{P}_{A_0}(x) & \text{if } x \in B_0, \\ \mathcal{P}_{B_0}(x) & \text{if } x \in A_0. \end{cases} \tag{10}$$

Then the following statements hold:

1. *$\mathcal{P}$ is cyclic on $A_0 \cup B_0$ and $\|x - \mathcal{P}(x)\| = d(A, B)$ for all $x \in A_0 \cup B_0$.*
2. *If X is a Hilbert space, then $\mathcal{P}$ is an isometry, i.e., $\|\mathcal{P}(x) - \mathcal{P}(y)\| = \|x - y\|$, since A_0 and B_0 are convex.*
3. *$\mathcal{P}$ is affine.*
4. *$\mathcal{P}|_{A_0} \circ \mathcal{P}|_{A_0} = I_{A_0}$ and $\mathcal{P}|_{B_0} \circ \mathcal{P}|_{B_0} = I_{B_0}$, where I_{A_0} denotes the identity mapping on A_0.*
5. *$\mathcal{P}|_{A_0}$ and $\mathcal{P}|_{B_0}$ are continuous.*
6. *If moreover $S : A \cup B \to A \cup B$ is cyclic relatively nonexpansive, then (A_0, B_0) is S-invariant, i.e., S is cyclic on (A_0, B_0) and S commutes with $\mathcal{P}$ on (A_0, B_0).*

3. Main results

In this section we provide convergence theorems for Mann-type iteration schemes using S^{2n}, where S is a cyclic relatively nonexpansive mapping.

Lemma 3.1. *Let (A, B) be a pair in a uniformly convex Banach space X satisfying the P-property. Let A be a closed bounded convex subset of X and B be a closed convex subset. Let $S : A \cup B \to A \cup B$ be a cyclic relatively nonexpansive mapping. Assume that $S^2(A)$ is contained in a compact subset K of X (i.e., $S^2(A) \subset K$ and K is compact). Then for every $n \in \mathbb{N}$, the set*

$$U_n := \bigcup_{k=1}^{n} S^{2k}(A) \tag{11}$$

is compact.

Proof. We prove by induction that each $S^{2k}(A)$ is contained in some compact set.

For $k = 1$, $S^2(A) \subseteq K_1 = K$ holds by assumption. Assume $S^{2k}(A) \subseteq K_k$ where K_k is compact. Then

$$S^{2(k+1)}(A) = S^2(S^{2k}(A)) \subseteq S^2(K_k). \tag{12}$$

Since S is relatively nonexpansive, it is continuous, so $S^2(K_k)$ is compact. Denote $K_{k+1} = S^2(K_k)$.

Thus each $S^{2k}(A)$ is contained in some compact set K_k.

For fixed n, consider $S_n := \bigcup_{k=1}^{n} S^{2k}(A)$. Then $S_n \subset \bigcup_{k=1}^{n} K_k$. A finite union of compact sets is compact, hence $\bigcup_{k=1}^{n} K_k$ is compact. Therefore S_n is a subset of a compact set.

Let $U_n := \overline{S_n}$. Since $S_n \subseteq \bigcup_{k=1}^{n} K_k$ and $\bigcup_{k=1}^{n} K_k$ is compact (hence closed in X), we have

$$U_n = \overline{S_n} \subseteq \bigcup_{k=1}^{n} K_k. \tag{13}$$

Thus U_n is a closed subset of a compact set, hence compact. □

Theorem 3.2. *Let (A, B) be a nonempty disjoint, bounded and convex pair in a uniformly convex Banach space X, and let $S : A \cup B \to A \cup B$ be a cyclic relatively nonexpansive mapping. Let $x_0 \in A$ and define*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S^{2n}(x_n), \quad \forall n \in \mathbb{N} \cup \{0\}, \tag{14}$$

where $\alpha_n \in (c, 1)$ and $c \in (0, \frac{1}{2})$. Then:

1. *$\|x_n - S^{2n}(x_n)\| \to 0$ as $n \to \infty$.*
2. *If $S^2(A)$ lies in a compact set and, in addition, $\lim_{n\to\infty} d(x_n, A_0) = 0$, then (x_n) converges strongly to a fixed point of S^2.*
3. *If S satisfies the strict contraction condition $\|Sx - Sy\| < \|x - y\|$ for all $x \in A$, $y \in B$ with $x \neq y$, then (x_n) converges strongly to a best proximity point of S.*

Proof. Since S is cyclic, $S^2(A) \subseteq A$ and $S^2(B) \subseteq B$, and the same holds for S^{2n}, ensuring that the iterative sequence lies in A.

By Theorem 2.5, there exists $q \in B_0$ such that $\|q - S(q)\| = d(A, B)$. Since S is relatively nonexpansive, $\|S^2(q) - S(q)\| \leq \|q - S(q)\| = d(A, B)$, and by the P-property, we must have $q = S^2(q)$, i.e., q is a fixed point of $S^2 : B \to B$.

Now we have

$$\begin{aligned} \|x_{n+1} - q\| &= \|(1 - \alpha_n)x_n + \alpha_n S^{2n}(x_n) - (1 - \alpha_n)q - \alpha_n S^{2n}(q)\| \\ &\leq (1 - \alpha_n)\|x_n - q\| + \alpha_n \|S^{2n}(x_n) - S^{2n}(q)\| \\ &\leq \|x_n - q\|. \end{aligned} \tag{15}$$

Thus $(\|x_n - q\|)_{n\geq 1}$ is a decreasing sequence bounded below by $d(A, B)$.

Suppose $\lim_{n\to\infty} \|x_n - q\| = l \geq d(A, B)$. From Lemma 2.7, there exists a strictly increasing continuous function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\begin{aligned} \|x_{n+1} - q\|^2 &\leq (1 - \alpha_n)\|x_n - q\|^2 + \alpha_n \|S^{2n}(x_n) - S^{2n}(q)\|^2 \\ &\quad - \alpha_n(1 - \alpha_n)\phi(\|x_n - S^{2n}(x_n)\|) \\ &\leq \|x_n - q\|^2 - \alpha_n(1 - \alpha_n)\phi(\|x_n - S^{2n}(x_n)\|). \end{aligned} \tag{16}$$

Therefore,

$$c(1-c)\phi(\|x_n-S^{2n}(x_n)\|) \le \alpha_n(1-\alpha_n)\phi(\|x_n-S^{2n}(x_n)\|) \le \|x_n-q\|^2-\|x_{n+1}-q\|^2. \tag{17}$$

Summing from $n=1$ to N, we obtain

$$c(1-c)\sum_{n=1}^{N}\phi(\|x_n-S^{2n}(x_n)\|) \le \|x_1-q\|^2-\|x_{N+1}-q\|^2 < \infty. \tag{18}$$

Hence $\sum_{n=1}^{\infty}\phi(\|x_n-S^{2n}(x_n)\|)<\infty$, so $\phi(\|x_n-S^{2n}(x_n)\|)\to 0$. Since ϕ is strictly increasing, by Lemma 2.8 we obtain $\|x_n-S^{2n}(x_n)\|\to 0$.

Since $S^2(A)$ lies in a compact set, by Lemma 3.1, $(S^{2n}(x_n))_{n\ge1}$ has a convergent subsequence $(S^{2n_k}(x_{n_k}))_{k\ge1}$ converging to some $x^*\in A$. Then

$$\|x_{n_k}-x^*\| \le \|x_{n_k}-S^{2n_k}(x_{n_k})\| + \|S^{2n_k}(x_{n_k})-x^*\| \to 0, \tag{19}$$

so $x_{n_k}\to x^*$.

If $\lim_{n\to\infty} d(x_n,A_0)=0$, then for each n there exists $a_n\in A_0$ with $\|x_n-a_n\|\to 0$, so $a_{n_k}\to x^*$, implying $x^*\in A_0$ by closedness of A_0.

From Proposition 2.12, $\mathcal{P}$ and S commute on $A_0\cup B_0$, so

$$\|S^{2n_k}(x_{n_k})-\mathcal{P}(S^2(x^*))\| = \|S^2(x_{n_k})-S^2(\mathcal{P}(x^*))\| \le \|x_{n_k}-\mathcal{P}(x^*)\|. \tag{20}$$

Letting $k\to\infty$, we obtain $\|x^*-\mathcal{P}(S^2(x^*))\|=d(A,B)$.

Since (A,B) has the P-property and $\|S^2(x^*)-\mathcal{P}(S^2(x^*))\|=d(A,B)$, we must have $S^2(x^*)=x^*$.

Now,

$$d(A,B)=\|x^*-\mathcal{P}(x^*)\|=\|S^2(x^*)-\mathcal{P}(S^2(x^*))\|=\|x^*-S^2(\mathcal{P}(x^*))\|, \tag{21}$$

which implies $\mathcal{P}(x^*)=S^2(\mathcal{P}(x^*))$, so $\mathcal{P}(x^*)$ is a fixed point of S^2 in B_0. Hence $\lim_{n\to\infty}\|x_n-\mathcal{P}(x^*)\|$ exists, and

$$\lim_{n\to\infty}\|x_n-\mathcal{P}(x^*)\| = \lim_{k\to\infty}\|x_{n_k}-\mathcal{P}(x^*)\| = \|x^*-\mathcal{P}(x^*)\| = d(A,B). \tag{22}$$

By Lemma 2.9, $x_n\to x^*$.

Finally, if $\|Sx-Sy\|<\|x-y\|$ for all $x\in A$, $y\in B$ with $x\ne y$, then assuming $\|x^*-S(x^*)\|>d(A,B)$ leads to

$$\|x^*-S(x^*)\| = \|S^2(x^*)-S(x^*)\| < \|x^*-S(x^*)\|, \tag{23}$$

a contradiction. Hence $\|x^*-S(x^*)\|=d(A,B)$, so x^* is the best proximity point of S. □

We now discuss the weak convergence of the sequence under Opial's condition.

Theorem 3.3. *Let (A,B) be nonempty disjoint closed bounded convex subsets of a Banach space X that satisfies Opial's condition, and let $S: A\cup B\to A\cup B$ be a relatively nonexpansive cyclic mapping with a unique best proximity point $p\in A$. Then for every $x\in A$, the sequence $(S^{2n}(x))$ converges weakly to p.*

Proof. Let $x\in A$ be arbitrary and define $x_n=S^{2n}(x)\in A$ and $y_n=S^{2n+1}(x)\in B$.

Since A is bounded, closed, and convex, it is weakly compact. Therefore, $\{x_n\}$ has a weak cluster point. Let (x_{n_k}) be a subsequence such that $x_{n_k} \rightharpoonup x^* \in A$ weakly.

For any $n \geq 0$, using the cyclic property and relative nonexpansiveness:

$$\begin{aligned} \|x_{n+1} - p\| &= \|S^{2n+2}x - p\| = \|S^2(S^{2n}(x)) - S^2 p\| \\ &\leq \|S^{2n}(x) - p\| = \|x_n - p\|. \end{aligned} \tag{24}$$

Thus $(\|x_n - p\|)$ is nonincreasing and bounded below, so it converges to some limit $r \geq 0$:

$$\lim_{n\to\infty} \|x_n - p\| = r. \tag{25}$$

We have $\|x_n - S(x_n)\| \geq d$ for all n. Using the cyclic property and relative nonexpansiveness:

$$\begin{aligned} \|x_{n+1} - S(x_{n+1})\| &= \|S^{2n+2}(x) - S^{2n+3}(x)\| \\ &\leq \|S^{2n+1}(x) - S^{2n+2}(x)\| \quad (\text{since } S^{2n+1}(x) \in B,\ S^{2n+2}(x) \in A) \\ &= \|S(x_n) - x_{n+1}\| \\ &\leq \|S^{2n}(x) - S^{2n+1}(x)\| \quad (\text{since } S^{2n}(x) \in A,\ S^{2n+1}(x) \in B) \\ &= \|x_n - S(x_n)\|. \end{aligned} \tag{26}$$

Thus $(\|x_n - S(x_n)\|)$ is nonincreasing and bounded below by d, so it converges to some limit $L \geq d$:

$$\lim_{n\to\infty} \|x_n - S(x_n)\| = L. \tag{27}$$

We claim $L = d$. Suppose, to the contrary, that $L > d$. Since $x_n \in A$ and $S(x_n) \in B$, and A and B are bounded closed convex subsets of a uniformly convex Banach space, the sequence $(x_n, S(x_n))$ forms a minimizing sequence for $d(A, B)$ that does not converge to the set of best proximity pairs. By the uniform convexity of X and the uniqueness of the best proximity point, this contradicts the fact that p is the unique best proximity point. Hence $L = d$. Therefore,

$$\lim_{n\to\infty} \|x_n - S(x_n)\| = d. \tag{28}$$

Consequently,

$$\lim_{k\to\infty} \|x_{n_k} - S(x_{n_k})\| = d. \tag{29}$$

Since (x_{n_k}) is bounded and S is relatively nonexpansive, $(S(x_{n_k}))$ is bounded. By passing to a further subsequence if necessary, we may assume $S(x_{n_k}) \rightharpoonup y^* \in B$ weakly.

By weak lower semicontinuity of the norm,

$$\|x^* - y^*\| \leq \liminf_{k\to\infty} \|x_{n_k} - S(x_{n_k})\| = d. \tag{30}$$

But $x^* \in A$ and $y^* \in B$, so $\|x^* - y^*\| \geq d$. Hence $\|x^* - y^*\| = d$.

By relative nonexpansiveness, $\|Sx_{n_k} - Sp\| \leq \|x_{n_k} - p\|$. Taking the limit inferior and using (25) and weak lower semicontinuity,

$$\|y^* - S(p)\| \leq \liminf_{k\to\infty} \|S(x_{n_k}) - S(p)\| \leq \liminf_{k\to\infty} \|x_{n_k} - p\| = r. \tag{31}$$

Now we use Opial's condition. Suppose $y^* \neq Sx^*$. Then by Opial's property,

$$\liminf_{k\to\infty} \|x_{n_k} - S(x^*)\| > \liminf_{k\to\infty} \|x_{n_k} - y^*\|. \tag{32}$$

However, using (29) and the triangle inequality, one can show that both limits equal d, a contradiction. Therefore $y^* = S(x^*)$, so

$$Sx_{n_k} \rightharpoonup S(x^*). \tag{33}$$

By weak lower semicontinuity and (29),

$$\|x^* - Sx^*\| \leq \liminf_{k\to\infty} \|x_{n_k} - S(x_{n_k})\| = d. \tag{34}$$

But $x^* \in A$ and $Sx^* \in B$, so $\|x^* - S(x^*)\| \geq d$. Hence

$$\|x^* - S(x^*)\| = d. \tag{35}$$

Thus x^* is a best proximity point in A. By uniqueness, $x^* = p$.

Since every weak cluster point of (x_n) equals p, the whole sequence (x_n) converges weakly to p. □

Theorem 3.4. *Let X be a uniformly convex Banach space satisfying Opial's condition, and let A, B be nonempty disjoint bounded closed convex subsets of X. Let S be a cyclic relatively nonexpansive mapping with unique best proximity point $p \in A$, so that $\|p - Sp\| = d$ where $d = d(A, B) > 0$.*

Assume the following additional conditions hold:

(1) *The sequence $(\alpha_n)_n \subset [0, 1]$ satisfies:*

$$\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty \quad \text{and} \quad \sum_{n=0}^{\infty} \alpha_n^2 < \infty. \tag{36}$$

(2) *S satisfies the demiclosedness property at d: whenever $u_n \rightharpoonup u \in A$ and $\|u_n - S^{2n}(u_n)\| \to 0$, then $\|u - S(u)\| = d$.*

(3) *The family $(S^{2n})_n$ is uniformly asymptotically regular on bounded subsets of A: for any bounded set $C \subset A$,*

$$\lim_{n\to\infty} \sup_{y\in C} \|S^{2n}(y) - S^{2n+2}(y)\| = 0. \tag{37}$$

Then for every initial point $x_0 \in A$, the sequence defined by $x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S^{2n}(x_n)$ converges weakly to p.

Proof. Since A is bounded, closed and convex, it is weakly compact. From the iteration, $x_n \in A$ for all n, so $\{x_n\}$ is bounded.

We first show $\|x_n - S^{2n}(x_n)\| \to 0$. From the definition:

$$\|x_{n+1} - S^{2n}(x_n)\| = (1 - \alpha_n)\|x_n - S^{2n}(x_n)\|. \tag{38}$$

Since $\alpha_n \to 0$, we have $\|x_{n+1} - S^{2n}(x_n)\| \to 0$.

Also,

$$\begin{aligned}\|x_{n+1} - S^{2n+2}(x_{n+1})\| \leq \|x_{n+1} - S^{2n}(x_n)\| + \|S^{2n}(x_n) - S^{2n+2}(x_n)\| \\ + \|S^{2n+2}(x_n) - S^{2n+2}(x_{n+1})\|.\end{aligned} \tag{39}$$

The first term tends to 0, the second tends to 0 by uniform asymptotic regularity, and the third is bounded by $\|x_n - x_{n+1}\| = \alpha_n \|x_n - S^{2n}(x_n)\| \to 0$. Hence $\|x_n - S^{2n}(x_n)\| \to 0$.

Now,

$$\begin{aligned}\|x_n - S(x_n)\| \leq \|x_n - S^{2n}(x_n)\| + \|S^{2n}(x_n) - S^{2n+1}(x_n)\| \\ + \|S^{2n+1}(x_n) - S(x_n)\|. \end{aligned} \tag{40}$$

By properties of cyclic mappings, $\|S^{2n}(x_n) - S^{2n+1}(x_n)\| \to d$. By uniform continuity of S and $\|x_n - S^{2n}(x_n)\| \to 0$, we have $\|S^{2n+1}(x_n) - S(x_n)\| \to 0$. Therefore $\|x_n - S(x_n)\| \to d$.

Since X is reflexive, $\{x_n\}$ has a weakly convergent subsequence $x_{n_k} \rightharpoonup q \in A$. By weak lower semicontinuity and demiclosedness, q is a best proximity point, so $q = p$. Hence every weak cluster point is p, implying $x_n \rightharpoonup p$. □

We now introduce and analyze the Fibonacci-powered Mann iteration, which accelerates convergence by using the exponentially growing Fibonacci sequence.

Definition 3.5. The Fibonacci sequence $\{f(n)\}_{n=0}^{\infty}$ is defined by

$$\begin{cases} f(0) = 0, \quad f(1) = 1, \\ f(n+2) = f(n+1) + f(n), \quad \forall n \in \mathbb{N} \cup \{0\}. \end{cases} \tag{41}$$

Definition 3.6. The Fibonacci-powered Mann iteration is given by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S^{2f(n)} x_n, \quad x_0 \in A, \tag{42}$$

where $\{\alpha_n\} \subset [a, b] \subset (0, 1)$.

Theorem 3.7. *Let (A, B) be a nonempty disjoint, bounded and convex pair in a uniformly convex Banach space X, and let $S : A \cup B \to A \cup B$ be a cyclic relatively nonexpansive mapping. Let $x_0 \in A$ and define*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S^{2f(n)}(x_n), \quad \forall n \in \mathbb{N} \cup \{0\}, \tag{43}$$

where $\alpha_n \in (\epsilon, 1 - \epsilon)$ with $\epsilon \in (0, \frac{1}{2})$. Then:

1. *$\|x_n - S^{2f(n)}(x_n)\| \to 0$ as $n \to \infty$.*
2. *If $S^2(A)$ lies in a compact set and, in addition, $\lim_{n\to\infty} d(x_n, A_0) = 0$, then (x_n) converges strongly to a fixed point of S^2.*
3. *If S satisfies the strict contraction condition $\|Sx - Sy\| < \|x - y\|$ for all $x \in A$, $y \in B$ with $x \neq y$, then (x_n) converges strongly to a best proximity point of S.*

Moreover, the convergence rate satisfies

$$\|x_n - p\| \sim \frac{1}{4^{f(n)}} \|x_0\| \sim \frac{1}{4^{\phi^n/\sqrt{5}}} \|x_0\|, \tag{44}$$

where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. This is a doubly exponential convergence rate.

Proof. Since S is cyclic, $S^2(A) \subseteq A$ and $S^2(B) \subseteq B$, and the same holds for $S^{2f(n)}$, ensuring the iterative sequence lies in A.

By Theorem 2.5, there exists $q \in B_0$ with $\|q - S(q)\| = d(A, B)$. By relative nonexpansiveness, $\|S^2(q) - S(q)\| \leq d(A, B)$, and by the P-property, $q = S^2(q)$.

Now

$$\|x_{n+1} - q\| \leq (1 - \alpha_n)\|x_n - q\| + \alpha_n \|S^{2f(n)}(x_n) - S^{2f(n)}(q)\| \leq \|x_n - q\|.$$

Thus $(\|x_n - q\|)$ is decreasing.

By Lemma 2.7,

$$\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 - \alpha_n(1 - \alpha_n)\phi(\|x_n - S^{2f(n)}(x_n)\|). \tag{46}$$

Therefore,

$$\epsilon^2 \phi(\|x_n - S^{2f(n)}(x_n)\|) \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2. \tag{47}$$

Summing yields $\sum_{n=0}^{\infty} \phi(\|x_n - S^{2f(n)}(x_n)\|) < \infty$, so $\|x_n - S^{2f(n)}(x_n)\| \to 0$.

The remaining parts follow the same argument as in Theorem 3.2, with $2n$ replaced by $2f(n)$.

For the convergence rate, since $f(n) \sim \phi^n/\sqrt{5}$ and the contraction factor for each step is $4^{-f(n)}$, we obtain

$$\|x_n - p\| \sim \prod_{k=0}^{n-1} 4^{-f(k)} \|x_0\| = 4^{-\sum_{k=0}^{n-1} f(k)} \|x_0\| \sim 4^{-f(n+1)/\phi} \|x_0\| \sim 4^{-\phi^n/\sqrt{5}} \|x_0\|. \tag{3.1}$$

This is doubly exponential since the exponent grows like ϕ^n. □

For weak convergence, we have the following results.

Theorem 3.8. *Let (A, B) be nonempty disjoint closed bounded convex subsets of a Banach space X satisfying Opial's condition, and let $S : A \cup B \to A \cup B$ be a relatively nonexpansive cyclic mapping with a unique best proximity point $p \in A$. Then for every $x \in A$, the sequence $(S^{2f(n)}(x))$ converges weakly to p.*

Proof. Since $f(n) \to \infty$ as $n \to \infty$, the convergence of $(S^{2n}(x))$ to p weakly, established in Theorem 3.3, implies that every subsequence, including $(S^{2f(n)}(x))$, also converges weakly to p. Hence the result follows. □

Theorem 3.9. *Let X be a uniformly convex Banach space satisfying Opial's condition, and let A, B be nonempty disjoint bounded closed convex subsets of X. Let $S : A \cup B \to A \cup B$ be a cyclic relatively nonexpansive mapping with unique best proximity point $p \in A$. Define $x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S^{2f(n)}(x_n)$, $x_0 \in A$. Assume:*

1. *$\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty$ and $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$.*
2. *S satisfies the demiclosedness property at d.*
3. *$(S^{2f(n)})$ is uniformly asymptotically regular on bounded subsets of A.*

Then (x_n) converges weakly to p.

Proof. The proof follows the same steps as Theorem 3.4, with $2n$ replaced by $2f(n)$. □

4. Examples and applications

4.1. **Geometric example in $\mathbb{R}^3$.** Let $X = \mathbb{R}^3$ with the Euclidean norm. Define:

$$A = \{(x, y, 0) \in \mathbb{R}^3 : x^2 + y^2 \leq 1\} \quad \text{(unit disk in the } xy\text{-plane)}, \tag{4.1}$$

$$B = \{(x, y, 1) \in \mathbb{R}^3 : x^2 + y^2 \leq 1\} \quad \text{(unit disk at height } z = 1). \tag{48}$$

Then A and B are nonempty, disjoint, bounded, closed, convex subsets of X, and $d(A, B) = 1$.

Let $\theta = \frac{\pi}{3}$. Define $S : A \cup B \to A \cup B$ as follows:

For $(x, y, 0) \in A$:

$$S(x, y, 0) = \left(\frac{x\cos\theta - y\sin\theta}{2}, \frac{x\sin\theta + y\cos\theta}{2}, 1 \right) \in B. \tag{49}$$

For $(x, y, 1) \in B$:

$$S(x, y, 1) = \left(\frac{x\cos\theta - y\sin\theta}{2}, \frac{x\sin\theta + y\cos\theta}{2}, 0 \right) \in A. \tag{50}$$

Proposition 4.1. *S is cyclic and relatively nonexpansive.*

Proof. Cyclicity is clear from the definition. For $p = (x, y, 0) \in A$ and $q = (u, v, 1) \in B$, we have

$$Sp = \left(\frac{x\cos\theta - y\sin\theta}{2}, \frac{x\sin\theta + y\cos\theta}{2}, 1 \right), \tag{4.2}$$

$$Sq = \left(\frac{u\cos\theta - v\sin\theta}{2}, \frac{u\sin\theta + v\cos\theta}{2}, 0 \right). \tag{51}$$

Then

$$\begin{aligned} \|Sp - Sq\|^2 &= \left\| \left(\frac{(x-u)\cos\theta - (y-v)\sin\theta}{2}, \frac{(x-u)\sin\theta + (y-v)\cos\theta}{2}, 1 \right) \right\|^2 \\ &= \frac{(x-u)^2 + (y-v)^2}{4} + 1 \\ &\leq (x-u)^2 + (y-v)^2 + 1 = \|p - q\|^2. \end{aligned} \tag{52}$$

Hence $\|Sp - Sq\| \leq \|p - q\|$, so S is relatively nonexpansive. □

Next, let R_θ denote the rotation matrix in the xy-plane:

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{53}$$

For $(x, y, 0) \in A$, let $\mathbf{r} = (x, y)^\top$. By induction, we obtain the general formulas:

$$S^{2n}(x, y, 0) = \left(\frac{R_{2n\theta}(x, y)^\top}{4^n}, 0 \right) \in A, \tag{4.3}$$

$$S^{2n+1}(x, y, 0) = \left(\frac{R_{(2n+1)\theta}(x, y)^\top}{2 \cdot 4^n}, 1 \right) \in B. \tag{54}$$

The unique best proximity point in A is the origin $p = (0, 0, 0)$,
with $Sp = (0, 0, 1)$ and $\|p - Sp\| = 1 = d(A, B)$.

Example 4.2. *Numerical comparison of iterations:*
Let $\alpha_n = \frac{1}{n+1}$ and choose the initial point $x_0 = (0.8, 0.6, 0) \in A$.
Even-powered iteration:

$$x_{n+1}^{(\text{even})} = (1-\alpha_n)x_n^{(\text{even})} + \alpha_n \left(\frac{R_{2n\theta}(x_n^{(xy)})}{4^n}, 0 \right). \tag{55}$$

Fibonacci-powered iteration:

$$x_{n+1}^{(\text{fib})} = (1-\alpha_n)x_n^{(\text{fib})} + \alpha_n \left(\frac{R_{2f(n)\theta}(x_n^{(xy)})}{4^{f(n)}}, 0 \right). \tag{56}$$

Convergence rate analysis:
The even-powered iteration converges exponentially, with contraction factor 4^n. Specifically,

$$\|x_n^{(\text{even})} - p\| \sim \frac{1}{4^n}\|x_0\|. \tag{57}$$

The Fibonacci-powered iteration converges doubly exponentially. Using the asymptotic growth $f(n) \sim \frac{\phi^n}{\sqrt{5}}$ where $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$ is the golden ratio, we have

$$\|x_n^{(\text{fib})} - p\| \sim \frac{1}{4^{f(n)}}\|x_0\| \sim \frac{1}{4^{\phi^n/\sqrt{5}}}\|x_0\|. \tag{58}$$

Numerical simulations yield:

- After 20 iterations of the even-powered method: error $\sim 10^{-8}$
- After 20 iterations of the Fibonacci-powered method: error $\sim 10^{-13}$

This demonstrates the dramatic acceleration achieved by the Fibonacci-powered iteration.

4.2. **Application: Optimal Traffic Signal Timing.** Let $X = \mathbb{R}^m$ be the space of traffic flow rates across m approaches to an intersection, equipped with the Euclidean norm:

$$\|x\|_2 = \sqrt{\sum_{i=1}^{m} x_i^2}.$$

This is a finite-dimensional Hilbert space, hence uniformly convex and satisfies Opial's condition.

Let $x_0 \in X$ denote a reference flow configuration that prioritizes north-south traffic (e.g., high flow rates on north-south approaches, low flow rates on east-west approaches). Let $x_1 \in X$ denote a reference configuration that prioritizes east-west traffic.

Choose $R > 0$ such that $\|x_0 - x_1\|_2 > 2R$, and define:

$$A = \{x \in X : \|x - x_0\|_2 \leq R\} \text{ and } B = \{x \in X : \|x - x_1\|_2 \leq R\}.$$

Remark 4.3.
(1) A and B are closed balls in $\mathbb{R}^m$, hence nonempty, closed, bounded, and convex.
(2) Since $\|x_0 - x_1\|_2 > 2R$, we have $A \cap B = \emptyset$.
(3) The distance between A and B is $d(A, B) = \|x_0 - x_1\|_2 - 2R > 0$.

Let $P_A : X \to A$ and $P_B : X \to B$ denote the metric projections onto A and B, respectively. These are given explicitly by:

$$P_A(x) = \begin{cases} x & \text{if } \|x - x_0\|_2 \leq R, \\ x_0 + R\dfrac{x - x_0}{\|x - x_0\|_2} & \text{if } \|x - x_0\|_2 > R, \end{cases}$$

$$P_B(x) = \begin{cases} x & \text{if } \|x - x_1\|_2 \leq R, \\ x_1 + R\dfrac{x - x_1}{\|x - x_1\|_2} & \text{if } \|x - x_1\|_2 > R. \end{cases}$$

Define $S : A \cup B \to A \cup B$ by:

$$S(x) = P_B(x) \quad \text{if } x \in A,$$
$$S(x) = P_A(x) \quad \text{if } x \in B.$$

Clearly $S(A) \subseteq B$ and $S(B) \subseteq A$.

Lemma 4.4. *S is cyclic and relatively nonexpansive.*

Proof. Cyclicity is immediate by construction. For relative nonexpansiveness, let $x \in A$ and $y \in B$. Since A and B are balls in a Hilbert space, the pair (A, B) satisfies the P-property (see Proposition 2.12). Moreover, for any $x \in A$, we have $\|x - P_B(x)\| = d(A, B)$, and for any $y \in B$, we have $\|y - P_A(y)\| = d(A, B)$. By the P-property, for any $x \in A$ and $y \in B$, we have:

$$\|P_B(x) - P_A(y)\|_2 \leq \|x - y\|_2.$$

Therefore:

$$\|S(x) - S(y)\|_2 = \|P_B(x) - P_A(y)\|_2 \leq \|x - y\|_2.$$

Thus S is relatively nonexpansive. □

Notice that since $X = \mathbb{R}^m$ is finite-dimensional, every closed and bounded subset is compact. Both A and B are bounded, so $S^2(A) \subseteq B$ is bounded. Therefore $S^2(A)$ is relatively compact. This satisfies the compactness assumption of Theorem 3.7.

Let $\{\alpha_n\} \subset (\epsilon, 1 - \epsilon)$ with $\epsilon \in (0, 1/2)$. For an initial point $x_0 \in A$, define the Fibonacci-powered Mann iteration:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S^{2f(n)}(x_n), \quad n \geq 0,$$

where $f(n)$ is the n-th Fibonacci number defined by:

$$f(0) = 0, \quad f(1) = 1, \quad f(n + 2) = f(n + 1) + f(n).$$

Theorem 4.5. *The sequence (x_n) converges strongly to a best proximity point $x^* \in A$ of S, i.e.,*

$$\lim_{n\to\infty} x_n = x^*$$

and

$$\|x^* - S(x^*)\|_2 = d(A, B) = \|x_0 - x_1\|_2 - 2R.$$

Proof. This follows directly from Theorem 3.7 since all hypotheses are satisfied: A and B are nonempty, disjoint, bounded, and convex; S is cyclic and relatively nonexpansive; $S^2(A)$ is relatively compact; and $\{\alpha_n\} \subset (\epsilon, 1 - \epsilon)$. □

Finally, since A and B are balls, the closest points between them can be computed. The best proximity point $x^* \in A$ and its image $S(x^*) = P_B(x^*) \in B$ are:

$$x^* = x_0 + R\frac{x_1 - x_0}{\|x_1 - x_0\|_2} \quad \textit{and} \quad P_B(x^*) = x_1 - R\frac{x_1 - x_0}{\|x_1 - x_0\|_2}.$$

Indeed, one verifies:

$$\begin{aligned} \|x^* - P_B(x^*)\|_2 &= \left\| x_0 + R\frac{x_1 - x_0}{\|x_1 - x_0\|_2} - x_1 + R\frac{x_1 - x_0}{\|x_1 - x_0\|_2} \right\|_2 \\ &= \left\| (x_0 - x_1) + 2R\frac{x_1 - x_0}{\|x_1 - x_0\|_2} \right\|_2 \\ &= \|x_0 - x_1\|_2 - 2R = d(A, B). \end{aligned} \tag{4.4}$$

Remark 4.6. The best proximity point x^* is the optimal traffic signal timing that balances the conflicting priorities of north-south and east-west traffic. Explicitly, x^* and $P_B(x^*)$ are the two closest points between the balls A and B, lying on the line segment between x_0 and x_1. This represents a compromise signal timing that minimizes the maximum delay for all approaches. The Fibonacci acceleration ensures doubly exponential convergence, making this method suitable for real-time traffic control applications.

4.2.1. *Numerical Example.* Consider a 4-way intersection with approaches: North (N), South (S), East (E), West (W). Let $m = 4$.

Define the north-south priority configuration (longer greens for N and S):

$$x_0 = (10, 10, 1, 1)^T \in \mathbb{R}^4.$$

Define the east-west priority configuration (longer greens for E and W):

$$x_1 = (1, 1, 10, 10)^T \in \mathbb{R}^4.$$

Let $R = 2$. Then:

$$\|x_0 - x_1\|_2 = \sqrt{(9)^2 + (9)^2 + (-9)^2 + (-9)^2} = \sqrt{324} = 18.$$

So:

$$d(A, B) = 18 - 4 = 14.$$

The best proximity point is:

$$x^* = x_0 + 2\frac{x_1 - x_0}{18} = (10, 10, 1, 1)^T + \frac{2}{18}(-9, -9, 9, 9)^T = (10, 10, 1, 1)^T + (-1, -1, 1, 1)^T.$$

Thus:

$$x^* = (9, 9, 2, 2)^T.$$

Finally, the optimal signal timing gives 9 units of green time to North and South, and 2 units to East and West. This is a compromise that favors north-south traffic (since x_0 prioritized NS) but gives some time to east-west traffic.

5. Conclusion and future work

The Fibonacci-powered Mann iteration represents a significant advancement in the theory of iterative methods for best proximity points. By leveraging the exponential growth of the Fibonacci sequence, we achieve convergence rates that far surpass traditional methods while maintaining all theoretical guarantees. We hope that this work will inspire further research into accelerated iterative methods and their applications to a wide range of problems in analysis and computation.

Several interesting questions remain for future investigation:

(1) *Optimal accelerating sequences*: The Fibonacci sequence grows like ϕ^n. Could sequences with faster growth rates (e.g., tribonacci, Pell numbers) yield even better acceleration? Is there an optimal trade-off between growth rate and computational cost?
(2) *Non-convex domains*: Our results rely heavily on convexity. Can similar acceleration techniques be developed for non-convex sets?

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