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GORENSTEIN FI-FLAT (PRE)COVERS

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ABSTRACT. In this paper, we give a characterization of left GFI_F-closed ring and we prove that over a GFI_F-closed ring every R-module has Gorenstein FI-flat (pre)cover.

1. Introduction

The notions of (pre)covers and (pre)envelopes of modules were introduced by Enochs [4] in 1981. Since then the existence and the properties of (pre)covers and (pre)envelopes relative to certain submodule categories have been studied widely. The theory of (pre)covers and (pre)envelopes, which plays an important role in homological algebra and representation theory of algebras, becomes now one of the main research topics in relative homological algebra.

Bican et al. [2] proved that the class of flat R-modules is a precovering class. On the other hand, Enochs et al. [8] proved that the class of Gorenstein flat R-modules is a precovering class over a right coherent ring. Furthermore, it was shown in [10] that the result holds over a left GF-closed ring. Also, Selvaraj et al. proved in [16] over n-coherent ring, the class of Gorenstein n-flat R-modules is a precovering class. Selvaraj et al. [15] introduced the concept of Gorenstein FI-flat modules. Motivated by these works, it is natural to arise a question that for any ring R, do all modules have Gorenstein FI-flat covers? In this paper, we prove that the class of Gorenstein FI-flat R-modules is a precovering class over a left GFI_F-closed ring.

This paper is organized as follows. Section 2, recalls some known notions which will be needed in the subsequent sections. In the third section, we give a characterization of left GFI_F-closed ring (See Theorem 3.10) and in the final section we prove that over a left GFI_F-closed ring, all modules have Gorenstein FI-flat covers.

Throughout this paper, R is an associative ring with identity and all modules are, if not specified otherwise, left R-modules. Let M and N be two R-modules. $M^+ = Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ denotes the character module of M. $Hom(M, N)$ (resp.

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$Ext^n(M,N)$) means $Hom_R(M,N)$ (resp. $Ext^n_R(M,N)$), and similarly $M \otimes N$ (resp. $Tor_n(M,N)$) denotes $M \otimes_R N$ (resp. $Tor^R_n(M,N)$) for an integer $n \geq 1$.

2. Preliminaries

Let $\mathcal{C}$ be a class of R-modules and X an R-module. Following [4], we say that a homomorphism $f : C \to X$ is a *$\mathcal{C}$-precover* if $C \in \mathcal{C}$ and the abelian group homomorphism $Hom(C', f) : Hom(C', C) \to Hom(C', X)$ is surjective for each $C' \in \mathcal{C}$. A $\mathcal{C}$-precover $f : C \to X$ is called a *$\mathcal{C}$-cover* if every endomorphism $g : C \to C$ such that $fg = f$ is an isomorphism. Dually, we have the definitions of a $\mathcal{C}$-(pre) envelope. $\mathcal{C}$-covers ($\mathcal{C}$-envelopes) may not exist in general, but if they exist, they are unique up to isomorphism.

Given a class $\mathcal{C}$ of R-modules, we will denote the class of R-modules X satisfying $Ext^1(C,X) = 0$ (resp. $Ext^1(X,C) = 0$) for every $C \in \mathcal{C}$ by $\mathcal{C}^{\perp}$ (resp. ${}^{\perp}\mathcal{C}$), which is called the right (resp. left) orthogonal class of $\mathcal{C}$. The notion of a cotorsion pair was first introduced by Salce in [14]. Its importance in homological algebra has been shown by its use in the proof of the existence of flat covers of modules over any ring [2]. Following [5], a pair of classes of R-modules $(\mathcal{A},\mathcal{B})$ is said to be a *cotorsion pair* if $\mathcal{A}^{\perp} = \mathcal{B}$ and ${}^{\perp}\mathcal{B} = \mathcal{A}$.

A cotorsion pair $(\mathcal{A},\mathcal{B})$ is said to be *complete* if for any R-module X there are exact sequences $0 \to X \to B \to A \to 0$ and $0 \to B' \to A' \to X \to 0$ with $B, B' \in \mathcal{B}$ and $A, A' \in \mathcal{A}$. A cotorsion pair $(\mathcal{A},\mathcal{B})$ is said to be *perfect* if every R-module has an $\mathcal{A}$-cover and a $\mathcal{B}$-envelope. A cotorsion pair $(\mathcal{A},\mathcal{B})$ is said to be *hereditary* if whenever $0 \to A' \to A \to A'' \to 0$ is exact with $A, A'' \in \mathcal{A}$ then A' is also in $\mathcal{A}$, or equivalently, if $0 \to B' \to B \to B'' \to 0$ is exact with $B', B \in \mathcal{B}$ then B'' is also in $\mathcal{B}$.

3. Gorenstein FI-flat modules

First, we recall the following definition.

Definition 3.1. [7] A left R-module M is said to be Gorenstein flat, if there exists an exact sequence of flat left R-modules,

$$\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

such that $M \cong \mathrm{Im}(F_0 \to F^0)$ and such that $B \otimes -$ leaves the sequence exact whenever B is an injective right R-module. $\mathcal{GF}(R)$ denotes class of all Gorenstein flat R-modules.

Definition 3.2. [12] A left R-module M is called FI-injective if $Ext^1(G,M) = 0$ for any FP-injective left R-module G. A right R-module N is said to be FI-flat if $Tor_1(N,G) = 0$ for any FP-injective left R-module G.

Definition 3.3. [15] A left R-module M is said to be Gorenstein FI-flat if there is a $\mathscr{A} \otimes -$ exact exact sequence $\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$ of

FI-flat left R-modules with $M = \mathrm{Ker}(F^0 \longrightarrow F^1)$, where $\mathscr{A}$ denotes class of all FI-injective right R-modules.

Example 3.4. Since every flat R-module is FI-flat and injective R-module is FI-injective, we get Gorenstein flat modules are Gorenstein FI-flat R-modules. However, converse need not be true. But it is true by [12, Proposition 2.3].

Definition 3.5. [1] A ring R is said to be left GF-closed if $\mathcal{GF}(R)$ is closed under extensions.

Definition 3.6. [15] A ring R is said to be left GFI_F-closed if $\mathcal{GFI_F}(R)$ is closed under extensions. i.e., for every short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ of left R-modules if M_1 and M_3 are in $\mathcal{GFI_F}(R)$, then M_2 is in $\mathcal{GFI_F}(R)$, where $\mathcal{GFI_F}(R)$ is the class of all Gorenstein FI-flat R-modules.

Example 3.7. Since Gorenstein flat modules are Gorenstein FI-flat modules, clearly we get GF-closed rings are GFI_F-closed rings.

We begin with the following result.

Lemma 3.8. *The following conditions are equivalent for a left R-module M:*

1. *M is Gorenstein FI-flat;*
2. *M satisfies the two following conditions:*
   (i) *$Tor_i(I, M) = 0$ for all $i > 0$ and all FI-injective right R-modules I; and*
   (ii) *There exists an exact sequence of left R-modules $0 \longrightarrow M \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$ where each F^i is FI-flat, such that $I \otimes -$ leaves the sequence exact whenever I is an FI-injective right R-module;*
3. *There exists a short exact sequence of left R-modules $0 \longrightarrow M \longrightarrow F \longrightarrow G \longrightarrow 0$, where F is FI-flat and G is Gorenstein FI-flat.*

Proof. The equivalent conditions $(1) \Longleftrightarrow (2)$ and $(1) \Longleftrightarrow (3)$ hold by the definition of Gorenstein FI-flat modules. It is enough to prove that $(3) \Longrightarrow (2)$. Suppose that there exists a short exact sequence of left R-modules:

$$0 \longrightarrow M \longrightarrow F \longrightarrow G \longrightarrow 0, \tag{3.1}$$

where F is FI-flat and G is Gorenstein FI-flat. Let I be an FI-injective right R-module. Since G is Gorenstein FI-flat and by the equivalence $(1) \Longleftrightarrow (2)$, $Tor_{i+1}(I, G) = 0$ for all $i \geq 0$. Then, use the long exact sequence,

$$Tor_{i+1}(I, G) \longrightarrow Tor_i(I, M) \longrightarrow Tor_i(I, F),$$

to get $Tor_i(I, M)$ for all $i > 0$.

On the other hand, G is Gorenstein FI-flat, there exists an exact sequence of left R-modules:

$$0 \longrightarrow G \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots, \tag{3.2}$$

where each F^i is FI-flat, such that $I \otimes -$ leaves the sequence exact whenever I is an FI-injective right R-module. Assembling the sequences (3.1) and (3.2), we get the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] & F \arrow[r] \arrow[dr] & {} \arrow[r] & F^0 \arrow[r] & F^1 \arrow[r] & \cdots \\
& & & G \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

such that $I \otimes -$ leaves the upper sequence exact whenever I is an FI-injective right R-module, as desired. $\square$

Lemma 3.9. *Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be a short exact sequence of left R-modules. If A is Gorenstein FI-flat and C is FI-flat, then B is Gorenstein FI-flat.*

Proof. Since A is Gorenstein FI-flat, there exists a short exact sequence of left R-modules $0 \longrightarrow A \longrightarrow F \longrightarrow G \longrightarrow 0$, where F is FI-flat and G is Gorenstein FI-flat. Consider the following pushout diagram

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, dashed] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & F \arrow[r, dashed] \arrow[d] & F^{'} \arrow[r] \arrow[d] & C \arrow[r] & 0 \\
& G \arrow[r, equal] \arrow[d] & G \arrow[d] & & \\
& 0 & 0. & &
\end{tikzcd}$$

In the sequence $0 \longrightarrow F \longrightarrow F^{'} \longrightarrow C \longrightarrow 0$, both F and C are FI-flat, hence so is $F^{'}$. Then by the middle vertical sequence and from Lemma 3.8, B is Gorenstein FI-flat. $\square$

We now prove the main result of this section, which is a generalization of [11, Theorem 3.7]

Theorem 3.10. *The following conditions are equivalent for a ring R:*

(i) *R is left GFI_F-closed;*
(ii) *The class $\mathcal{GFI}_{\mathcal{F}}(R)$ is projectively resolving;*
(iii) *For every short exact sequence of left R-modules $0 \longrightarrow G_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0$, where G_0 and G_1 are Gorenstein FI-flat. If $Tor_1(I, M) = 0$ for all FI-injective right R-modules I, then M is Gorenstein FI-flat.*

Proof. $(i) \Longrightarrow (ii)$. To claim that the class $\mathcal{GFI}_{\mathcal{F}}(R)$ is projectively resolving, it suffices to prove that it is closed under kernals of epimorphisms. Then, consider a short exact sequence of left R-modules $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, where B and C are Gorenstein FI-flat. We prove that A is Gorenstein FI-flat. Since

B is Gorenstein FI-flat, there exists a short exact sequence of left R-modules $0 \longrightarrow B \longrightarrow F \longrightarrow G \longrightarrow 0$, where F is FI-flat and G is Gorenstein FI-flat. Consider the following pushout diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & A \arrow[r] & F \arrow[r, dashed] \arrow[d] & D \arrow[r] \arrow[d] & 0 \\
 & & G \arrow[r, equal] \arrow[d] & G \arrow[d] & \\
 & & 0 & 0. &
\end{tikzcd}$$

By the right vertical sequence and since R is left GFI_F-closed, the R-module D is Gorenstein FI-flat. Therefore, by the middle horizontal sequence and Lemma 3.8, A is Gorenstein FI-flat.

$(i) \Longrightarrow (iii)$. Since G_1 is Gorenstein FI-flat, there exists a short exact sequence of left R-modules $0 \longrightarrow G_1 \longrightarrow F_1 \longrightarrow H \longrightarrow 0$, where F_1 is FI-flat and H is Gorenstein FI-flat. Consider the following pushout diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & G_1 \arrow[r] \arrow[d] & G_0 \arrow[r] \arrow[d, dashed] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & F_1 \arrow[r, dashed] \arrow[d] & D \arrow[r] \arrow[d] & M \arrow[r] & 0 \\
 & H \arrow[r, equal] \arrow[d] & H \arrow[d] & & \\
 & 0 & 0. & &
\end{tikzcd}$$

In the short exact sequence $0 \longrightarrow G_0 \longrightarrow D \longrightarrow H \longrightarrow 0$, both G_0 and H are Gorenstein FI-flat, then so is D (since R is left GFI_F-closed). Then, there exists a short exact sequence of left R-modules $0 \longrightarrow D \longrightarrow F \longrightarrow G \longrightarrow 0$, where F is FI-flat and G is Gorenstein FI-flat. Consider the following pushout diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & F_1 \arrow[r] \arrow[d, equal] & D \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & F_1 \arrow[r] & F \arrow[r, dashed] \arrow[d] & F' \arrow[r] \arrow[d] & 0 \\
 & & G \arrow[r, equal] \arrow[d] & G \arrow[d] & \\
 & & 0 & 0. &
\end{tikzcd}$$

We prove that $F^{'}$ is FI-flat. Consider the sequence $0 \longrightarrow M \longrightarrow F^{'} \longrightarrow G \longrightarrow 0$. Let I be an FI-injective right R-module. By the following exact sequence,

$$0 = Tor_1(I, M) \longrightarrow Tor_1(I, F^{'}) \longrightarrow Tor_1(I, G) = 0,$$

we get

$$Tor_1(I, F^{'}) = 0. \tag{3.3}$$

On the other hand, consider the sequence $0 \longrightarrow F_1 \longrightarrow F \longrightarrow F^{'} \longrightarrow 0$. Then, we get the short exact sequence of character modules:

$$0 \longrightarrow (F^{'})^+ \longrightarrow (F)^+ \longrightarrow (F_1)^+ \longrightarrow 0. \tag{3.4}$$

now, by [13, Theorem 3.52] F^+ and $(F_1)^+$ are FI-injective right R-modules. Then by (3.3) and [3, Proposition 5.1] we get,

$$Ext^1((F_1)^+, (F^{'})^+) \cong (Tor_1((F_1)^+, (F^{'})^+))^+ = 0.$$

Then, the sequence (3.4) splits, and so $(F^{'})^+$ is FI-injective being a direct summand of the FI-injective right R-module F^+. Therefore, $F^{'}$ is a FI-flat left R-module [13, Theorem 3.52].

Finally, by Lemma 3.8 and the short exact sequence $0 \longrightarrow M \longrightarrow F^{'} \longrightarrow G \longrightarrow 0$, M is Gorenstein FI-flat.

$(iii) \Longrightarrow (i)$. Consider a short exact sequence of left R-modules $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, where A and C are Gorenstein FI-flat. We show that B is Gorenstein FI-flat. Let I be an FI-injective right R-module. Applying the functor $I \otimes -$ to the short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, we get the long exact sequence,

$$Tor_i(I, A) \longrightarrow Tor_i(I, B) \longrightarrow Tor_i(I, C).$$

Then, $Tor_i(I, B) = 0$ for all $i > 0$ since A and C are Gorenstein FI-flat and by Lemma 3.8.

On the other hand, since C is Gorenstein FI-flat, there exists a short exact sequence of left R-modules $0 \longrightarrow G \longrightarrow F \longrightarrow C \longrightarrow 0$, where F is FI-flat and

C is Gorenstein FI-flat. Consider the following pullback diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & G \arrow[r, equal] \arrow[d] & G \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & D \arrow[r, dashed] \arrow[d, dashed] & F \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0. &
\end{tikzcd}$$

Also, since A is Gorenstein FI-flat, there exists a short exact sequence of left R-modules $0 \longrightarrow A \longrightarrow F^{'} \longrightarrow G^{'} \longrightarrow 0$, where $F^{'}$ is FI-flat and $G^{'}$ is Gorenstein FI-flat. Consider the following pushout diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & D \arrow[r] \arrow[d, dashed] & F \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & F^{'} \arrow[r, dashed] \arrow[d] & D^{'} \arrow[r] \arrow[d] & F \arrow[r] & 0 \\
 & G^{'} \arrow[r, equal] \arrow[d] & G^{'} \arrow[d] & & \\
 & 0 & 0. & &
\end{tikzcd}$$

In the short exact sequence $0 \longrightarrow F^{'} \longrightarrow D^{'} \longrightarrow F \longrightarrow 0$ both $F^{'}$ and F are FI-flat, then so is $D^{'}$. Then, by the short exact sequence $0 \longrightarrow D \longrightarrow D^{'} \longrightarrow G^{'} \longrightarrow 0$ and from Lemma 3.8, D is Gorenstein FI-flat. Finally, consider the short exact sequence $0 \longrightarrow G \longrightarrow D \longrightarrow B \longrightarrow 0$. We get G and D are Gorenstein FI-flat, and $Tor_i(I, B) = 0$ for all $i > 0$ and all FI-injective right modules I. Therefore, by (iii), B is Gorenstein FI-flat. $\square$

4. Gorenstein FI-flat covers

In this section, we investigate the existence of Gorenstein FI-flat covers for every module over a GFI_F-closed ring.

Proposition 4.1. ([6, Proposition 1.4]) *Let $\mathfrak{X}$ be a class of modules. In addition, suppose $\mathfrak{X}$ is closed under direct limits. Then for a left R-module M, the existence of an $\mathfrak{X}$-precover of M implies the existence of an $\mathfrak{X}$-cover.*

The following lemma is due to Schanuel's Lemma.

Lemma 4.2. *Let*

$$0 \to K \to F \to M \to 0$$

$$0 \to L \to G \to M \to 0$$

be exact, where $f : F \to M$ and $k : G \to M$ are Gorenstein FI-flat precover of a module M. Then $G \oplus K \cong F \oplus L$.

Proof. Consider the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "g"] \arrow[d, "m"] & F \arrow[r, "f"] \arrow[d, "p"] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & L \arrow[r, "h"] & G \arrow[r, "k"] & M \arrow[r] & 0. \end{tikzcd}$$

Here the linear map p is available such that $f = kp$ since $k : G \to M$ is Gorenstein FI-flat precover. This means that we have above completed commutative diagram with exact rows.
Similarly, we have the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "h"] \arrow[d, "l"] & G \arrow[r, "k"] \arrow[d, "q"] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & K \arrow[r, "q"] & F \arrow[r, "f"] & M \arrow[r] & 0. \end{tikzcd}$$

We now can define two linear maps σ and ψ as follows:

$$\sigma : G \oplus K \to F \oplus L$$

$$(x', y) \mapsto (q(x') - g(y), -m(y) - h^{-1}(1 - pq)x')$$

$$\psi : F \oplus L \to G \oplus K$$

$$(x, y') \mapsto (p(x) - h(y'), -l(y') - g^{-1}(1 - qp)x)$$

Note that $f = kp$ and $k = fq$, we have that $f = fqp$ and $k = kpq$, or equivalently, $f(1 - qp) = 0$ and $k(1 - pq) = 0$. Therefore, we have the following

$$(1 - qp)x \in Ker(f) = g(K), x \in F$$

$$(1 - pq)x' \in Ker(k) = h(L), x' \in G.$$

Therefore the maps above are well defined. It is easy to verify that $\sigma\psi = 1$ and $\psi\sigma = 1$ and then that $G \oplus K \cong F \oplus L$. □

Next, we introduce the notion of Gorenstein FI-cotorsion right R-module as follows.

Definition 4.3. A left R-module N is said to be Gorenstein FI-cotorsion if $Ext^1(G, N) = 0$ for all Gorenstein FI-flat left R-modules G.

Theorem 4.4. *Let $0 \to N_1 \to N_2 \to N_3 \to 0$ be a short exact sequence with N_1 and N_3 are Gorenstein FI-cotorsion then N_2 also Gorenstein FI-cotorsion.*

Proof. It is straightforward. □

Lemma 4.5. *Suppose $0 \to A \to B \to C \to 0$ is exact and both A and C have Gorenstein FI-flat covers. Also, suppose $A \in \mathcal{GFI_F}^{\perp}$. Then there is a commutative diagram with exact rows and columns*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

such that $D = F \oplus G$ and such that $F \to A$ and $G \to C$ are Gorenstein FI-flat covers. Moreover, $D \to B$ is a Gorenstein FI-flat cover and $N \in \mathcal{GFI_F}^{\perp}$.

Proof. Since $g : G \to C$ is a Gorenstein FI-flat cover, there exists a linear map $u : G \to B$ such that $g = pu$, here $p : B \to C$. Define $h : F \oplus G \to B$ with $(x, y) \longmapsto gf(x) + u(y)$, $x \in F$, $y \in G$ and here $f : F \to A$ is the Gorenstein FI-flat cover. By a standard diagram chasing argument, we can easily verify that the diagram can be completed with the standard properties. In particular, $D \to B$ is Gorenstein FI-flat precover and $N \in \mathcal{GFI_F}^{\perp}$. $\square$

Now, we give an important result about direct limits.

Lemma 4.6. *If R is a left GFI_F-closed ring, then the class of Gorenstein FI-flat modules is closed under direct limits.*

Proof. Let us take that $(M_\alpha)_{\alpha<\lambda}$ is a well ordered direct system of Gorenstein FI-flat modules. If $\lambda = n < \omega$, then $\varinjlim M_\alpha = M_{n-1}$ is Gorenstein FI-flat, and so we have finished. Then let $\lambda = \omega$, and let us start by showing that $\varinjlim M_n (n < \omega)$ is Gorenstein FI-flat. Since M_0 is Gorenstein FI-flat, there exists an exact sequence

$$A(0) =: 0 \longrightarrow M_0 \longrightarrow F_0^0 \longrightarrow F_0^1 \longrightarrow F_0^2 \longrightarrow \cdots$$

with each F_0^i is FI-flat for $i \geq 0$ such that $I \otimes -$ leaves it exact whenever I is an FI-injective right R-module. Simultaneously, we have that each module $K_0^i = \mathrm{Ker}(F_0^i \longrightarrow F_0^{i+1})$ is Gorenstein FI-flat for $i \geq 0$ (where $K_0^0 = M_0$). Now, consider the push out diagram of morphisms $M_0 \longrightarrow F_0^0$ and $M_0 \longrightarrow M_1$

$$\begin{tikzcd}
0 \arrow[r] & M_0 \arrow[r] \arrow[d] & F_0^0 \arrow[r] \arrow[d] & K_0^1 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M_1 \arrow[r] & U \arrow[r] & K_0^1 \arrow[r] & 0
\end{tikzcd}$$

Then, we get that U is Gorenstein FI-flat since M_1 and K_0^1 are so, and since R is left GFI_F-closed, there exists a short exact sequence $0 \longrightarrow U \longrightarrow F_1^0 \longrightarrow L \longrightarrow 0$

with F_1^0 is FI-flat such that L is Gorenstein FI-flat. Again consider the following pushout diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M_1 \arrow[r] \arrow[d, equal] & U \arrow[r] \arrow[d] & K_0^1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_1 \arrow[r] & F_1^0 \arrow[r] \arrow[d] & K_1^1 \arrow[r] \arrow[d] & 0 \\
 & & L \arrow[r, equal] \arrow[d] & L \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

since L and K_0^1 are Gorenstein FI-flat, and R is left GFI_F-closed, we get that K_1^1 is Gorenstein FI-flat. We now get the following morphism of exact sequences induced by the morphism $M_0 \longrightarrow M_1$:

$$\begin{tikzcd}
0 \arrow[r] & M_0 \arrow[r] \arrow[d] & F_0^0 \arrow[r] \arrow[d] & K_0^1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_1 \arrow[r] & F_1^0 \arrow[r] & K_1^1 \arrow[r] & 0.
\end{tikzcd}$$

Using the same method above, we will get a morphism of exact sequences induced by the morphism $K_0^1 \longrightarrow K_1^1$

$$\begin{tikzcd}
0 \arrow[r] & K_0^1 \arrow[r] \arrow[d] & F_0^1 \arrow[r] \arrow[d] & K_0^2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_1^1 \arrow[r] & F_1^1 \arrow[r] & K_1^2 \arrow[r] & 0,
\end{tikzcd}$$

where F_1^1 is FI-falt and K_1^2 is Gorenstein FI-flat. Thus, continuously using the arguments above, we can get an exact sequence

$$A(1) =: 0 \longrightarrow M_1 \longrightarrow F_1^0 \longrightarrow F_1^1 \longrightarrow F_1^2 \longrightarrow \cdots$$

with each F_1^i is FI-flat and each $K_1^i = \mathrm{Ker}(F_1^i \longrightarrow F_1^{i+1})$ is Gorenstein FI-flat for all $i \geq 0$ (where $K_1^0 = M_1$), and a morphism $A(0) \longrightarrow A(1)$ induced by $M_0 \longrightarrow M_1$. If we continue with this process, then we get a commutative diagram

$$\begin{tikzcd}
A(0) \arrow[r, equal] \arrow[d] & 0 \arrow[r] & M_0 \arrow[r] \arrow[d] & F_0^0 \arrow[r] \arrow[d] & F_0^1 \arrow[r] \arrow[d] & F_0^2 \arrow[r] \arrow[d] & \cdots \\
A(1) \arrow[r, equal] \arrow[d] & 0 \arrow[r] & M_1 \arrow[r] \arrow[d] & F_1^0 \arrow[r] \arrow[d] & F_1^1 \arrow[r] \arrow[d] & F_1^2 \arrow[r] \arrow[d] & \cdots \\
A(2) \arrow[r, equal] \arrow[d] & 0 \arrow[r] & M_2 \arrow[r] \arrow[d] & F_2^0 \arrow[r] \arrow[d] & F_2^1 \arrow[r] \arrow[d] & F_2^2 \arrow[r] \arrow[d] & \cdots \\
\vdots & & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

with exact rows and each F_j^i is FI flat, and each $K_j^i = \mathrm{Ker}(F_j^i \longrightarrow F_j^{i+1})$ Gorenstein FI-flat for all $i \geq 0$ and $j \geq 0$ (where $K_j^0 = M_j$). Thus, it is easy to see that each exact sequence $A(n), n = 0, 1, 2, \cdots$ is still exact after applying the functor $I \otimes -$ for any FI-injective right R-module I. Applying the exact functor $\varinjlim$ to the above commutative diagram, we then obtain an exact sequence

$$\varinjlim A(n) =: 0 \to \varinjlim M_n \to \varinjlim F_n^0 \to \varinjlim F_n^1 \to \varinjlim F_n^2 \to \cdots,$$

where each module $\varinjlim F_n^i$, $i = 0, 1, 2, \cdots$ is FI-flat module. Note that I is an FI-injective right R-module, then $I \otimes \varinjlim A(n) \cong \varinjlim (I \otimes A(n))$ is exact since $\varinjlim$ commutes with the tensor product functor. Since each $M_n, n = 0, 1, 2, \cdots$ Gorenstein FI-flat, it follows that $Tor_i(I, \varinjlim M_n) \cong \varinjlim Tor_i(I, M_n) = 0$ for all $i > 0$, and all FI-injective right R-modules I. Hence, $\varinjlim M_n$ is Gorenstein FI-flat by Lemma 3.8.

Now, we reindex the modules $M_0, M_1, \cdots, M_\omega, M_{\omega+1}, \cdots$ such that $M_\omega = \varinjlim M_n$ and $M_{\omega+1}$ is the old of M_ω and so forth. Therefore, we may assume that the system $(M_\alpha)_{\alpha<\lambda}$ is continuous, i.e., $M_\beta = \varinjlim M_\alpha (\alpha < \beta)$ if β is a limit ordinal with $\beta < \lambda$. Then using transfinite induction, we can prove that $\varinjlim M_\alpha (\alpha < \lambda)$ is Gorenstein FI-flat, and the desired result follows. □

Therefore, by Proposition 4.1 and Lemma 4.6 we only need to find Gorenstein FI-flat precover of M. We recall from [9] the definition of Kaplansky classes. A class $\mathcal{K}$ of R-modules is a Kaplansky class if there exists a cardinal $\mathcal{N}$ such that for every $M \in \mathcal{K}$ and for each $x \in M$, there exists a submodule F of M such that $x \in F \subseteq M, F, M/F \in \mathcal{K}$ and $Card(F) \leq \mathcal{N}$.

Lemma 4.7. [9, Theorem 2.9] *Let $\mathcal{K}$ be a Kaplansky class. If $\mathcal{K}$ contains the projective modules and it is closed under extensions and direct limits,then $(\mathcal{K}, \mathcal{K}^\perp)$ is a perfect cotorsion pair in R-Mod.*

The following lemma is due to [9, Proposition 2.10].

Lemma 4.8. *The class of Gorenstein FI-flat modules over any ring is a Kaplansky class.*

Now, we are in a position to give the main result of this paper, which extends [8, Theorem 2.12]

Theorem 4.9. *Let R be a left GFI_F-closed ring. Then, $(\mathcal{GFI}_{\mathcal{F}}, \mathcal{GFI}_{\mathcal{F}}{}^{\perp})$ is hereditary perfect cotorsion pair.*

Proof. By Lemma 4.6, 4.7, and 4.8, we get that $(\mathcal{GFI}_{\mathcal{F}}, \mathcal{GFI}_{\mathcal{F}}{}^{\perp})$ is a perfect cotorsion pair. Since the class of Gorenstein FI-flat modules is projectively resolving by Theorem 3.10, then we get that $(\mathcal{GFI}_{\mathcal{F}}, \mathcal{GFI}_{\mathcal{F}}{}^{\perp})$ is hereditary. □

By the above theorem, we immediately have the following result.

Corollary 4.10. *All left modules over a left GFI_F-closed ring have Gorenstein FI-flat covers.*

Corollary 4.11. *If R is left GFI_F-closed ring, then every module has a Gorenstein FI-cotorsion envelope.*

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References

1. D. Bennis, *Rings over which the class of Gorenstein flat modules is closed under extensions,* Comm. Algebra, **37(3)** (2009), 855-868.
2. L. Bican, E. Bashir, and E. E. Enochs, *All modules have flat covers,* Bull. London Math. Soc., **33** (2001), 385-390.
3. H. Carten and S. Eilenberg, Homological Algebra, Princeton University Press, Princeton, 1956.
4. E. E. Enochs, *Injective and flat covers, envelopes and resolvents,* Israel J. Math., **39(3)** (1981), 189-209.
5. E. E. Enochs and O. M. G. Jenda, Relative Homological Algebra, De Gruyter Expositions in Mathematics, **30**, New York, 2000.
6. E. E. Enochs and J. Xu, *Gorenstein flat covers of modules over Gorenstein rings,* J. Algebra, **181** (1996), 288-313.
7. E. Enochs, O. M. G. Jenda and B. Torrecillas, *Gorenstein flat modules,* Nanjing Daxue Shuxue Bannian Kan, **10** (1993), 1-9.
8. E. Enochs, O. M. G. Jenda and J. A. López - Ramos, *The existence of Gorenstein flat covers,* Math. Scand., **94** (2004), 46-62.
9. E. Enochs and J. A. Lopez - Ramos, *Kapalansky classes,* Rend. Semin. Mat. Univ. Padova, **107** (2002), 67-79.
10. Y. Gang and L. Zhongkui, *Gorenstein flat covers over GF-closed rings,* Comm. Algebra, **40(5)** (2012), 1632-1640.
11. H. Holm, *Gorenstein Homological Dimensions,* Journal of Pure and Applied Algebra, **189** (2004), 167-193.
12. L. Mao and N. Ding, *FI-injective and FI-flat modules,* J. Algebra, **209** (2007), 367-385.
13. J. J. Rotman, An Introduction to Homological Algebra, Academic Press, 1979.
14. L. Salce, *Cotorsion theories for abelian groups,* Sympos Math., Vol XIII, (1979), 11-32.
15. C. Selvaraj, V.Biju and R. Udhayakumar, *Stability of Gorenstein FI-flat modules,* Far East J. Math., **95(2)** (2014), 159-168.
16. C. Selvaraj, R. Udhayakumar and A. Umamaheswaran, *Gorenstein n-flat modules and their covers,* Asian Eur. J. Math., **7(3)** (2014) 1450051 (13 pages).

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