

WOLFF POTENTIAL AND BIRKHOFF'S ERGODIC THEOREM IN GRAND VARIABLE HERZ SPACES

ROSALIO G. ARTES JR.¹, WAQAR AFZAL^{3,4}, SAMRA ASJAD⁵, MUHAMMAD TARIQ⁶,
and HIJAZ AHMAD^{7,8,9,10}

ABSTRACT. In this article, we establish the boundedness of the Wolff potential operator and Birkhoff's ergodic theorem in grand variable Herz spaces. These concepts were previously studied in the setting of Lebesgue spaces. Our results extend and refine the existing literature by considering a more general functional framework, thereby providing broader applicability and sharper estimates than the previously known results.

Keywords. Wolff potential; grand variable Herz spaces; grand Lebesgue spaces; variable exponent; boundedness of fractional operators; ergodic theorem.

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1. INTRODUCTION

Classical operators of harmonic analysis, among them the Hardy–Littlewood maximal operator, the Riesz potential, the Bessel–Riesz potential, and the Bochner–Riesz means, form the backbone of modern function space theory and provide the principal analytical tools for the study of partial differential equations (PDEs). Each of these operators is linear, and their boundedness theory in weighted Lebesgue, Morrey, and Herz spaces is by now well understood. In this direction, Izuki [19] proved the boundedness of fractional potential on Herz spaces and derived a wavelet characterisation providing a concrete function-space framework for multi-resolution analysis, while Almeida and Drihem [3] developed a comprehensive theory of fractional potential and singular-type operators on Herz spaces, establishing sharp boundedness criteria under log-Hölder continuity. The mapping properties of classical operators in spaces with grand integrability structure were further advanced by Sultan, Azmi, Sultan, Mehmood, and Mlaiki [25], who established the boundedness of Riesz potential operators in grand Herz–Morrey spaces, and by AlNemer, Basendwah, Sultan, and Popa [5], who derived Sobolev-type theorems for the rough fractional operator in grand Herz spaces, highlighting the role of the grand integrability parameter in treating borderline

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* Corresponding author

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cases. The foundational framework of grand variable Herz spaces itself was introduced by Nafis, Rafeiro, and Zaighum [22], who verified the boundedness of sublinear operators on these spaces via discrete grand Lebesgue sequence spaces; this framework forms the direct basis for the present investigation.

When attention shifts to quasilinear or fully nonlinear PDEs, the linear Riesz potential is no longer adequate, and a genuinely nonlinear replacement is needed. This role is played by the Wolff potential, which has established itself as a fundamental tool in nonlinear analysis and classical operator theory; unlike Riesz potentials it is tailor-made for the nonlinear regime, rendering it essential in the study of capacities, Sobolev-type embeddings, and sharp pointwise regularity. Almeida, Harjulehto, Hästö, and Lukkari [4] provided one of the earliest systematic studies of the Riesz and Wolff-type operators within variable exponent spaces, together with regularity results for nonlinear PDEs. Building on this, Borowski, Chlebicka, and Miasojedow [10] established important boundedness properties for Wolff-type potentials in the analysis of nonlinear PDEs. In a related direction, Chlebicka, Song, Youn, and Zatorska-Goldstein [14] obtained Riesz potential estimates for mixed local-nonlocal problems with measure data, extending potential-theoretic techniques to settings where local and nonlocal contributions interact. In a parallel development on exponentially modulated potential operators, Afzal, Abbas, Macías-Díaz, Gallegos, and Almalki [1] introduced exponentially damped fractional operators extending the classical Riesz potential and discussed applications to elliptic equations in nondivergence form.

The analysis of pseudo-differential and singular integral operators on non-Euclidean structures has attracted considerable recent attention. Cardona, Delgado, and Ruzhansky [13] established L^p -bounds for pseudo-differential operators on graded Lie groups, opening new directions for operator theory beyond the flat Euclidean setting. In a geometric function-space direction, Afzal, Abbas, Tariq, Hincal, and Abdelfattah [11] proved sharp boundedness criteria for the Wolff potential on complete Riemannian manifolds, demonstrating that the nonlinear potential framework extends naturally to curved geometries. Nonlinear elliptic problems in weighted and anisotropic spaces have been studied by Alla, Azraïbi, and El Haji [6], who investigated measurable boundary conditions in anisotropic weighted Sobolev spaces on irregular domains, providing a bridge between abstract function-space theory and boundary-value problems with non-standard growth.

Recent contributions have further broadened the scope of operator theory in non-standard function spaces. Afzal and Abbas [24] established the regularity of parabolic Ornstein–Uhlenbeck equations via boundedness of fractional Muckenhoupt-type weighted singular operators in variable Herz spaces, revealing that parabolic regularity can be captured within the variable-Herz framework. Afzal, Abbas, Tariq, Macías-Díaz, and Ahmad [2] established the boundedness of the multidimensional Katugampola operator in Campanato spaces, revealing new connections between fractional integral operators and classical function-space embeddings that complement the Wolff potential framework. Gürdal and Stojiljkovic [17] derived Berezin radius inequalities for finite sums of functional Hilbert space operators, highlighting the relevance of spectral-theoretic methods

in the analysis of operators on abstract spaces. Al-Shutnawi [7] investigated generalized Marcinkiewicz integrals associated with surfaces of revolution, extending the boundedness theory of singular integral operators to geometric settings with variable curvature. Taken together, these works illustrate the broad contemporary interest in pushing classical harmonic analysis tools—boundedness of integral and singular operators, Sobolev-type embeddings, and spectral estimates—into function spaces with variable or grand integrability structure, and onto non-Euclidean geometric settings.

The Wolff potential $\mathcal{W}_{\beta, \nu}$ [4], acting on locally integrable functions $\psi \in L^1_{\text{loc}}(\mathbb{R}^n)$, takes the form

$$\mathcal{W}_{\beta, \nu} \psi(\varepsilon) := \int_0^\infty \left(r^{\beta \nu - n} \int_{\mathbb{B}(\varepsilon, r)} |\psi(\tau)| \, d\tau \right)^{\frac{1}{\nu-1}} \frac{dr}{r}, \quad 0 < \beta < \frac{n}{\nu}. \quad (1.1)$$

When $\nu = 2$ this expression reduces to the classical Riesz potential, while for $\nu \neq 2$ it produces a genuinely nonlinear object ideally suited to quasilinear and fully nonlinear PDEs.

Novelty and motivating gaps. Boundedness of the Wolff potential is essential for regularity in evolution equations, yet prior investigations have been confined to local settings such as classical Lebesgue spaces [4] and standard variable Herz spaces [3]; the grand variable Herz space setting has remained unexplored for this operator. Unlike classical Riesz potentials, the Wolff potential is genuinely nonlinear, which yields stronger regularising effects and greater adaptability for non-standard growth. Fixed-exponent results are inadequate for non-homogeneous media whose local integrability varies from point to point. The present paper closes these gaps by establishing, for the first time, the full boundedness of the Wolff potential in grand variable Herz spaces with completely variable parameters. The central novelty is the interplay between the grand integrability exponent, which handles borderline Sobolev-type embeddings as studied in [25, 5, 22], and the variable exponent structure, which adapts to local integrability as in [19, 3]. We further establish Birkhoff’s ergodic theorem in this setting, capturing both fine local and global behaviour of dynamical systems. Our findings unify and improve upon multiple earlier theorems [19, 3, 25, 22, 1, 24] as limiting cases, providing a versatile tool for PDE analysis in irregular domains where constant-exponent spaces fall short.

The article is structured as follows. Section 2 collects preliminary material: definitions, key lemmas, and auxiliary results organised in the natural hierarchy, covering variable Lebesgue spaces, grand Lebesgue sequence spaces, variable Herz spaces, and grand variable Herz spaces. Section 3 presents and proves the main results, namely the mapping properties of the Wolff potential and the ergodic theorem in the grand variable Herz framework. Section 4 provides a brief summary and directions for future work.

2. PRELIMINARIES

We work throughout in the n -dimensional Euclidean space $\mathbb{R}^n$. The notational conventions below are fixed for the remainder of the paper.

Notation. For a centre $\mathbf{x} \in \mathcal{R}^n$ and radius $\rho > 0$, the open ball $\{\mathbf{y} \in \mathcal{R}^n : |\mathbf{x} - \mathbf{y}| < \rho\}$ is written $\mathcal{B}(\mathbf{x}, \rho)$. The Lebesgue measure of a measurable set $E \subset \mathcal{R}^n$ is $|E|$, and χ_E denotes its indicator function. For a measurable function f , the notation $\text{supp } f$ refers to its support. We write $\eta \lesssim \zeta$ if there exists a positive constant C such that $\eta \leq C \zeta$. In addition, the notation $\eta \approx \zeta$ indicates that both $\eta \lesssim \zeta$ and $\zeta \lesssim \eta$ hold simultaneously. For every $\mathfrak{l} \in \mathbb{Z}$, the dyadic annuli of $\mathcal{R}^n$ are defined by

$$\mathcal{B}_{\mathfrak{l}} := \mathcal{B}(0, 2^{\mathfrak{l}}), \quad \mathcal{R}_{\mathfrak{l}} := \mathcal{B}_{\mathfrak{l}} \setminus \mathcal{B}_{\mathfrak{l}-1}, \quad \chi_{\mathfrak{l}} := \chi_{\mathcal{R}_{\mathfrak{l}}}.$$

For a measurable exponent function $\mathbf{p} : \mathcal{R}^n \rightarrow [1, \infty)$, we write $\mathbf{p}^+ := \text{ess sup}_{\mathbf{y} \in \mathcal{R}^n} \mathbf{p}(\mathbf{y})$, $\mathbf{p}^- := \text{ess inf}_{\mathbf{y} \in \mathcal{R}^n} \mathbf{p}(\mathbf{y})$, and $\mathbf{p}'(\cdot)$ for the pointwise conjugate exponent given by $1/\mathbf{p}(\mathbf{y}) + 1/\mathbf{p}'(\mathbf{y}) = 1$.

2.1. Variable Lebesgue spaces. Denote by $\mathcal{P}(\mathcal{R}^n)$ the collection of all measurable mappings $\mathbf{p} : \mathcal{R}^n \rightarrow [1, \infty)$, which are called variable exponents. For any measurable subset $\Omega \subset \mathcal{R}^n$, define

$$\mathbf{p}_{\Omega}^- := \text{ess inf}_{\mathbf{y} \in \Omega} \mathbf{p}(\mathbf{y}), \quad \mathbf{p}_{\Omega}^+ := \text{ess sup}_{\mathbf{y} \in \Omega} \mathbf{p}(\mathbf{y}).$$

Definition 2.1. For a variable exponent $\mathbf{p} \in \mathcal{P}(\mathcal{R}^n)$, the generalized Lebesgue space $L^{\mathbf{p}(\cdot)}(\mathcal{R}^n)$ is defined as the set of all measurable functions $h : \mathcal{R}^n \rightarrow \mathbb{R}$ satisfying

$$\int_{\mathcal{R}^n} |h(\mathbf{y})|^{\mathbf{p}(\mathbf{y})} d\mathbf{y} < \infty.$$

The corresponding modular functional and Luxemburg norm are given by

$$\rho_{\mathbf{p}(\cdot)}(h) := \int_{\mathcal{R}^n} |h(\mathbf{y})|^{\mathbf{p}(\mathbf{y})} d\mathbf{y}, \quad \|h\|_{\mathbf{p}(\cdot)} := \inf \left\{ \lambda > 0 : \rho_{\mathbf{p}(\cdot)}\left(\frac{h}{\lambda}\right) \leq 1 \right\}.$$

Equipped with $\|\cdot\|_{\mathbf{p}(\cdot)}$, the space $L^{\mathbf{p}(\cdot)}(\mathcal{R}^n)$ is a Banach space. The modular and the norm satisfy the two-sided inequality

$$\min(\|h\|_{\mathbf{p}(\cdot)}^{\mathbf{p}^-}, \|h\|_{\mathbf{p}(\cdot)}^{\mathbf{p}^+}) \leq \rho_{\mathbf{p}(\cdot)}(h) \leq \max(\|h\|_{\mathbf{p}(\cdot)}^{\mathbf{p}^-}, \|h\|_{\mathbf{p}(\cdot)}^{\mathbf{p}^+}).$$

Whenever $1/\mathfrak{s}(\mathbf{y}) = 1/\mathbf{p}(\mathbf{y}) + 1/\mathbf{q}(\mathbf{y})$ holds pointwise, the generalised Hölder inequality reads

$$\|hg\|_{\mathfrak{s}(\cdot)} \leq 2\|h\|_{\mathbf{p}(\cdot)}\|g\|_{\mathbf{q}(\cdot)}.$$

The exponent $\mathbf{p}$ is *locally log-Hölder continuous* if there exists $c_{\log} > 0$ with

$$|\mathbf{p}(\mathbf{y}) - \mathbf{p}(\mathbf{x})| \leq \frac{c_{\log}}{\log(e + 1/|\mathbf{y} - \mathbf{x}|)} \quad \text{for all } \mathbf{y}, \mathbf{x} \in \mathcal{R}^n.$$

The log-Hölder decay condition at infinity holds when there exist $\mathbf{p}_{\infty} \in \mathbb{R}$ and $c_{\log} > 0$ such that

$$|\mathbf{p}(\mathbf{y}) - \mathbf{p}_{\infty}| \leq \frac{c_{\log}}{\log(e + |\mathbf{y}|)} \quad \text{for all } \mathbf{y} \in \mathcal{R}^n.$$

The symbol $\mathcal{P}_0^{\log}$ (resp. $\mathcal{P}_{\infty}^{\log}$) denotes the class of variable exponents satisfying the log-Hölder condition near the origin (resp. at infinity), and $\mathcal{P}^{\log} = \mathcal{P}_0^{\log} \cap \mathcal{P}_{\infty}^{\log}$. The principal exponent classes used in this paper are: $\mathcal{P}(\mathcal{R}^n)$, consisting of all measurable $\mathbf{p} : \mathcal{R}^n \rightarrow [1, \infty)$ with $\mathbf{p}^- > 1$ and $\mathbf{p}^+ < \infty$; $\mathcal{P}^{\log}(\mathcal{R}^n)$, the subset

satisfying both the local log-Hölder condition and the decay condition at infinity; and $\mathcal{P}_0(\mathcal{R}^n)$, the subset satisfying the log-Hölder decay condition near the origin.

2.2. Grand Lebesgue sequence spaces. The grand Lebesgue sequence spaces were constructed in [23] and constitute the discrete building blocks of the grand variable Herz spaces used throughout this work.

Definition 2.2 ([23]). Let $1 \leq p < \infty$ and $\theta > 0$. The grand Lebesgue sequence space $\ell^{p),\theta}$ is defined via the norm

$$\|\{x_{\mathfrak{k}}\}_{\mathfrak{k} \in \mathbb{Z}}\|_{\ell^{p),\theta}} := \sup_{\epsilon > 0} \epsilon^\theta \left(\sum_{\mathfrak{k} \in \mathbb{Z}} |x_{\mathfrak{k}}|^{p(1+\epsilon)} \right)^{\frac{1}{p(1+\epsilon)}} = \sup_{\epsilon > 0} \epsilon^{\frac{\theta}{p(1+\epsilon)}} \|\{x_{\mathfrak{k}}\}\|_{\ell^{p(1+\epsilon)}}.$$

The following chain of continuous embeddings holds: for $0 < \epsilon_0 < 1/p$, $\delta > 0$, and $0 < \theta_1 \leq \theta_2$,

$$\ell^{p(1-\epsilon_0)} \hookrightarrow \ell^p \hookrightarrow \ell^{p),\theta_1} \hookrightarrow \ell^{p),\theta_2} \hookrightarrow \ell^{p(1+\delta)}.$$

In particular, $\ell^p \subsetneq \ell^{p),\theta}$ for every $\theta > 0$.

2.3. Variable Herz spaces.

Definition 2.3 ([22]). Let $p, q \in \mathcal{P}(\mathcal{R}^n)$ and assume that $\alpha \in L^\infty(\mathcal{R}^n)$. The homogeneous variable exponent Herz space

$$\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\mathcal{R}^n)$$

is defined as the set of all functions

$$\psi \in L_{\text{loc}}^{p(\cdot)}(\mathcal{R}^n \setminus \{0\})$$

such that the associated Herz-type norm remains finite

$$\sum_{l \in \mathbb{Z}} \|2^{l\alpha(\cdot)} \psi \chi_l\|_{p(\cdot)}^{q(\cdot)} < \infty.$$

The corresponding quasi-norm is

$$\|\psi\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} = \left(\sum_{l \in \mathbb{Z}} \|2^{l\alpha(\cdot)} \psi \chi_l\|_{p(\cdot)}^{q(\cdot)} \right)^{1/q(\cdot)}.$$

When α , p , and q are constants, this space coincides with the classical homogeneous Herz space introduced in [18].

Proposition 2.4. *Let $\beta \in L^\infty(\mathcal{R}^n)$ be log-Hölder continuous both in a neighbourhood of the origin and at infinity. Then every function $\psi \in \dot{K}_{p(\cdot),q(\cdot)}^{\beta(\cdot)}(\mathcal{R}^n)$ satisfies the norm equivalence*

$$\|\psi\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\beta(\cdot)}} \approx \left\| \{2^{l\beta(0)} \psi \chi_l\} \right\|_{\ell_{<}^{q(\cdot)}(L^{p(\cdot)})} + \left\| \{2^{l\beta_\infty} \psi \chi_l\} \right\|_{\ell_{>}^{q(\cdot)}(L^{p(\cdot)})}.$$

2.4. Grand variable Herz spaces. The grand variable Herz spaces, introduced in [22], are obtained by substituting the classical ℓ^p summability in the Herz norm with the grand Lebesgue sequence space $\ell^{p(\cdot),\theta}$. This replacement strictly enlarges the space and unlocks the treatment of borderline embedding phenomena that lie outside the reach of the standard framework.

Definition 2.5 ([22]). Let $\alpha \in \mathcal{R}$, $1 \leq p < \infty$, $q(\cdot) \in \mathcal{P}(\mathcal{R}^n)$, and $\theta > 0$. The grand homogeneous variable Herz space $\dot{K}_{q(\cdot)}^{\alpha,p,\theta}(\mathcal{R}^n)$ is the set of all $\psi \in L_{\text{loc}}^{q(\cdot)}(\mathcal{R}^n \setminus \{0\})$ for which

$$\|\psi\|_{\dot{K}_{q(\cdot)}^{\alpha,p,\theta}(\mathcal{R}^n)} := \sup_{\epsilon > 0} \epsilon^\theta \left(\sum_{l \in \mathbb{Z}} 2^{l\alpha p(1+\epsilon)} \|\psi \chi_l\|_{L^{q(\cdot)}(\mathcal{R}^n)}^{p(1+\epsilon)} \right)^{\frac{1}{p(1+\epsilon)}} < \infty.$$

An equivalent formulation is

$$\|\psi\|_{\dot{K}_{q(\cdot)}^{\alpha,p,\theta}} = \sup_{\epsilon > 0} \epsilon^{\frac{\theta}{p(1+\epsilon)}} \|\psi\|_{\dot{K}_{q(\cdot)}^{\alpha,p(1+\epsilon)}},$$

where $\dot{K}_{q(\cdot)}^{\alpha,p(1+\epsilon)}$ is the homogeneous variable Herz space with fixed exponent $p(1+\epsilon)$.

Remark 2.6. The nesting structure of grand Lebesgue sequence spaces immediately yields

$$\dot{K}_{q(\cdot)}^{\alpha,p}(\mathcal{R}^n) \subsetneq \dot{K}_{q(\cdot)}^{\alpha,p,\theta}(\mathcal{R}^n) \subsetneq \dot{K}_{q(\cdot)}^{\alpha,p(1+\delta)}(\mathcal{R}^n) \quad \text{for every } \delta > 0.$$

Hence the grand variable Herz space is strictly larger than the standard variable Herz space, affording the treatment of borderline embedding cases inaccessible in the classical framework. Furthermore, for $0 < \theta_1 \leq \theta_2$ there is a continuous embedding $\dot{K}_{q(\cdot)}^{\alpha,p,\theta_1} \hookrightarrow \dot{K}_{q(\cdot)}^{\alpha,p,\theta_2}$.

Definition 2.7 ([22]). The non-homogeneous grand variable Herz space $K_{q(\cdot)}^{\alpha,p,\theta}(\mathcal{R}^n)$ is the collection of all $\psi \in L_{\text{loc}}^{q(\cdot)}(\mathcal{R}^n)$ such that

$$\|\psi\|_{K_{q(\cdot)}^{\alpha,p,\theta}(\mathcal{R}^n)} := \sup_{\epsilon > 0} \epsilon^\theta \left(\|\psi \chi_{\mathcal{B}_0}\|_{L^{q(\cdot)}(\mathcal{R}^n)}^{p(1+\epsilon)} + \sum_{l \geq 1} 2^{l\alpha p(1+\epsilon)} \|\psi \chi_l\|_{L^{q(\cdot)}(\mathcal{R}^n)}^{p(1+\epsilon)} \right)^{\frac{1}{p(1+\epsilon)}} < \infty.$$

Definition 2.8. Let (Ω, μ) be a probability space equipped with a non-decreasing sequence of measurable sets $\{\mathcal{B}_k\}_{k \geq 0}$ satisfying $\mu(\mathcal{B}_0) > 0$, $\mu(\mathcal{B}_k) \rightarrow 1$, with annuli $\mathcal{R}_0 := \mathcal{B}_0$ and $\mathcal{R}_k := \mathcal{B}_k \setminus \mathcal{B}_{k-1}$ for $k \geq 1$. Let $q(\cdot)$ be a measurable variable exponent on Ω with $1 \leq q^- \leq q^+ < \infty$, let $\alpha \in \mathbb{R}$, $1 \leq p < \infty$, and $\theta > 0$.

The non-homogeneous grand variable Herz space $K_{q(\cdot)}^{\alpha,p,\theta}(\Omega)$ is the collection of all μ -measurable functions $\psi : \Omega \rightarrow \mathbb{R}$ for which

$$\|\psi\|_{K_{q(\cdot)}^{\alpha,p,\theta}(\Omega)} := \sup_{\epsilon > 0} \epsilon^\theta \left(\|\psi \chi_{\mathcal{R}_0}\|_{L^{q(\cdot)}(\Omega, \mu)}^{p(1+\epsilon)} + \sum_{k \geq 1} 2^{k\alpha p(1+\epsilon)} \|\psi \chi_{\mathcal{R}_k}\|_{L^{q(\cdot)}(\Omega, \mu)}^{p(1+\epsilon)} \right)^{\frac{1}{p(1+\epsilon)}} < \infty,$$

where $L^{q(\cdot)}(\Omega, \mu)$ denotes the variable-exponent Lebesgue space on (Ω, μ) with Luxemburg norm. When $\Omega = \mathbb{R}^n$, μ is Lebesgue measure, and $\mathcal{B}_k = \mathcal{B}(0, 2^k)$, this definition recovers Definition 2.7.

Theorem 2.9 (Hölder's inequality, [22]). *Let $0 < p_i \leq \infty$, $1 \leq q^- \leq q^+ < \infty$, $-\infty < \alpha_i < \infty$ for $i = 1, 2$, and suppose*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q(\cdot)} + \frac{1}{q'(\cdot)} = 1, \quad \alpha = \alpha_1 + \alpha_2.$$

Then

$$\|\Psi\Phi\|_{\dot{K}_1^{\alpha, p, \theta}(\mathbb{R}^n)} \leq \|\Psi\|_{\dot{K}_{q(\cdot)}^{\alpha_1, p_1, \theta}(\mathbb{R}^n)} \cdot \|\Phi\|_{\dot{K}_{q'(\cdot)}^{\alpha_2, p_2, \theta}(\mathbb{R}^n)}.$$

2.5. Technical lemmas.

Lemma 2.10 ([3]). *Suppose that*

$$\beta \in L^\infty(\mathbb{R}^n)$$

satisfies the log-Hölder continuity condition. Let

$$\varepsilon \in \mathcal{B}(0, s_1) \setminus \mathcal{B}(0, s_1/2)$$

and

$$\tau \in \mathcal{B}(0, s_2) \setminus \mathcal{B}(0, s_2/2),$$

where $s_1 > 0$. Then the following estimate holds:

$$s_1^{\beta(\varepsilon)} \lesssim s_2^{\beta(\tau)} \begin{cases} \left(\frac{s_1}{s_2}\right)^{\beta^+}, & 0 < s_2 \leq \frac{s_1}{2}, \\ 1, & \frac{s_1}{2} < s_2 \leq 2s_1, \\ \left(\frac{s_1}{s_2}\right)^{\beta^-}, & s_2 > 2s_1. \end{cases}$$

If

$$p \in \mathcal{P}^{\log},$$

then the characteristic function associated with a ball $\mathcal{B}$ satisfies the asymptotic relation

$$\|\chi_{\mathcal{B}}\|_{p(\cdot)} \approx |\mathcal{B}|^{1/p(\tau)}$$

whenever the radius of $\mathcal{B}$ is sufficiently small, whereas for large balls one has

$$\|\chi_{\mathcal{B}}\|_{p(\cdot)} \approx |\mathcal{B}|^{1/p_\infty}.$$

Lemma 2.11 ([15]). *Let*

$$p \in \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$$

and define the annular region

$$\mathcal{R} := \mathcal{B}(0, r) \setminus \mathcal{B}(0, r/2).$$

If

$$|\mathcal{R}| \geq 2^{-n},$$

then for every

$$\tau \in \mathcal{R},$$

the norm of the characteristic function obeys

$$\|\chi_{\mathcal{R}}\|_{p(\cdot)} \approx |\mathcal{R}|^{1/p(\tau)} \approx |\mathcal{R}|^{1/p_\infty}.$$

Lemma 2.12 ([15]). *Let $0 < \kappa < 1$, $0 < q \leq \infty$, and suppose $\{\omega_j\} \in \ell^q$. Define*

$$\delta_l := \sum_{j \leq l} \kappa^{l-j} \omega_j, \quad \eta_l := \sum_{j \leq l} \kappa^{j-l} \omega_j.$$

Then $\{\delta_l\}, \{\eta_l\} \in \ell^q$ and

$$\|\{\delta_l\}\|_{\ell^q} + \|\{\eta_l\}\|_{\ell^q} \leq c \|\{\omega_j\}\|_{\ell^q}.$$

3. MAIN RESULTS

3.1. Boundedness of the Wolff Potential in Grand Variable Herz Spaces.

We now state and prove the principal boundedness results for the Wolff potential within the grand variable Herz framework. Each proof rests on a dyadic partition of $\mathbb{R}^n$ into three carefully chosen regions: an inner ball, a middle enlarged annulus, and a far-field tail, with the grand norm recovered by assembling the three resulting pointwise estimates via the quasi-triangle inequality and the discrete convolution control furnished by Lemma 2.12.

Theorem 3.1. *The following assumptions are imposed: $n \in \mathbb{N}$, $0 < q \leq \infty$, and $\theta > 0$; $p(\cdot) \in \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$; $a(\cdot) \in L^\infty(\mathbb{R}^n)$ is log-Hölder continuous at infinity with limit a_∞ satisfying*

$$-\frac{n}{p_\infty} < a_\infty < \frac{n}{p'_\infty};$$

and the Wolff potential operator $\mathcal{W}_{\beta, \nu}$ is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ into $L^{q(\cdot)}(\mathbb{R}^n)$, where

$$\frac{1}{q(\varepsilon)} = \frac{1}{p(\varepsilon)} - \frac{\beta}{n}.$$

Under these conditions, the operator $\mathcal{W}_{\beta, \nu}$ extends to a bounded map

$$\mathcal{W}_{\beta, \nu} : \dot{K}_{p(\cdot)}^{a(\cdot), p(\cdot), \theta}(\mathbb{R}^n) \longrightarrow \dot{K}_{q(\cdot)}^{a(\cdot), p(\cdot), \theta}(\mathbb{R}^n).$$

More precisely, there exists a constant $\mathcal{M} > 0$ such that

$$\|\mathcal{W}_{\beta, \nu}(\psi)\|_{\dot{K}_{q(\cdot)}^{a(\cdot), p(\cdot), \theta}} \leq \mathcal{M} \|\psi\|_{\dot{K}_{p(\cdot)}^{a(\cdot), p(\cdot), \theta}}, \quad (3.1)$$

for all $\psi \in \dot{K}_{p(\cdot)}^{a(\cdot), p(\cdot), \theta}(\mathbb{R}^n) \cap L^{p(\cdot)}(\mathbb{R}^n)$.

Proof. Let $\varepsilon > 0$ be fixed throughout. For each $l \in \mathbb{Z}$, set $\mathcal{B}_l := \mathcal{B}(0, 2^l)$ and $\mathcal{R}_l := \mathcal{B}_l \setminus \mathcal{B}_{l-1}$, and decompose

$$\psi = \underbrace{\psi \chi_{\mathcal{B}_{l-1}}}_{\psi_1^{(l)}} + \underbrace{\psi \chi_{\mathcal{B}_{l+1} \setminus \mathcal{B}_{l-1}}}_{\psi_2^{(l)}} + \underbrace{\psi \chi_{\mathbb{R}^n \setminus \mathcal{B}_{l+1}}}_{\psi_3^{(l)}}. \quad (3.2)$$

Sub-additivity of $\mathcal{W}_{\beta, \nu}$ gives, for every $\varepsilon \in \mathcal{R}_l$,

$$|\mathcal{W}_{\beta, \nu} \psi(\varepsilon)| \leq \sum_{i=1}^3 \left| \mathcal{W}_{\beta, \nu} \psi_i^{(l)}(\varepsilon) \right|. \quad (3.3)$$

It suffices to bound each summand in the grand Herz quasi-norm.

Inner region. For $\varepsilon \in \mathcal{R}_l$ and $\tau \in \mathcal{B}_{l-1}$,

$$|\varepsilon - \tau| \geq |\varepsilon| - |\tau| \geq 2^{l-1} - 2^{l-2} = 2^{l-2},$$

so $\mathcal{B}(\varepsilon, r) \cap \mathcal{B}_{l-1} = \emptyset$ for all $r < c 2^l$. Consequently,

$$\begin{aligned} \left| \mathcal{W}_{\beta, \nu}(\psi_1^{(l)})(\varepsilon) \right| &\lesssim \left(\int_{c 2^l}^{\infty} \left(r^{\beta \nu - n} \int_{\mathcal{B}(\varepsilon, r)} |\psi_1^{(l)}(\tau)| d\tau \right)^{\nu-1} \frac{dr}{r} \right)^{\frac{1}{\nu-1}} \\ &\leq \left(\int_{c 2^l}^{\infty} r^{(\beta \nu - n)(\nu-1)} \frac{dr}{r} \right)^{\frac{1}{\nu-1}} \left(\int_{\mathcal{B}_{l-1}} |\psi(\tau)| d\tau \right), \end{aligned}$$

where the radial integral converges since $(\beta \nu - n)(\nu - 1) - 1 < -1$. Evaluating it explicitly,

$$\left| \mathcal{W}_{\beta, \nu}(\psi_1^{(l)})(\varepsilon) \right| \lesssim 2^{-l \frac{n - \beta \nu}{\nu-1}} \left(\int_{\mathcal{B}_{l-1}} |\psi(\tau)| d\tau \right)^{\frac{1}{\nu-1}}. \quad (3.4)$$

Since the right-hand side of (3.4) is constant on $\mathcal{R}_l$, taking the $L^{q(\cdot)}$ -norm over $\mathcal{R}_l$ and applying Lemma 2.11 gives

$$\left\| \mathcal{W}_{\beta, \nu}(\psi_1^{(l)}) \chi_l \right\|_{L^{q(\cdot)}} \lesssim 2^{ln/q_\infty} \cdot 2^{-l \frac{n - \beta \nu}{\nu-1}} \left(\int_{\mathcal{B}_{l-1}} |\psi(\tau)| d\tau \right)^{\frac{1}{\nu-1}}.$$

Applying the variable-exponent Hölder inequality and Lemma 2.11 to the remaining integral,

$$\int_{\mathcal{B}_{l-1}} |\psi(\tau)| d\tau \leq \sum_{j \leq l-1} \|\psi \chi_j\|_{L^{p(\cdot)}} \|\chi_j\|_{L^{p'(\cdot)}} \lesssim \sum_{j \leq l-1} 2^{jn/p'_\infty} \|\psi \chi_j\|_{L^{p(\cdot)}},$$

and inserting back we obtain

$$\left\| \mathcal{W}_{\beta, \nu}(\psi_1^{(l)}) \chi_l \right\|_{L^{q(\cdot)}} \lesssim 2^{ln/q_\infty} \cdot 2^{-l \frac{n - \beta \nu}{\nu-1}} \left(\sum_{j \leq l-1} 2^{jn/p'_\infty} \|\psi \chi_j\|_{L^{p(\cdot)}} \right)^{\frac{1}{\nu-1}}.$$

Multiplying by 2^{la_∞} , raising to the power $p(1+\epsilon)$, and summing over $l \in \mathbb{Z}$, the right-hand side takes the form of a discrete convolution of the sequence $\{2^{ja_\infty} \|\psi \chi_j\|_{L^{p(\cdot)}}\}_j$ with the kernel $\Phi(l-j) := 2^{(l-j)\delta}$, where

$$\delta := a_\infty + \frac{n}{q_\infty} - \frac{n - \beta \nu}{\nu - 1} + \frac{n}{(\nu - 1)p'_\infty} < 0$$

by the assumption $a_\infty < n/p'_\infty$. Since $\delta < 0$, the kernel $\{\Phi(k)\}_{k \leq 0}$ is summable in ℓ^1 , and Lemma 2.12 applied to the convolution sum yields

$$\sup_{\epsilon > 0} \epsilon^\theta \left(\sum_{l \in \mathbb{Z}} 2^{la_\infty p(1+\epsilon)} \left\| \mathcal{W}_{\beta, \nu}(\psi_1^{(l)}) \chi_l \right\|_{L^{q(\cdot)}}^{p(1+\epsilon)} \right)^{\frac{1}{p(1+\epsilon)}} \lesssim \|\psi\|_{\dot{\mathbf{K}}_{p(\cdot)}^{a(\cdot), p, \theta}}. \quad (3.5)$$

Middle region. Since $\text{supp } \psi_2^{(\mathfrak{l})} \subseteq \mathcal{B}_{\mathfrak{l}+1} \setminus \mathcal{B}_{\mathfrak{l}-1} = \mathcal{R}_{\mathfrak{l}-1} \cup \mathcal{R}_{\mathfrak{l}} \cup \mathcal{R}_{\mathfrak{l}+1}$, the assumed $L^{\mathfrak{p}(\cdot)} \rightarrow L^{\mathfrak{q}(\cdot)}$ boundedness of $\mathcal{W}_{\beta, \nu}$ together with the triangle inequality gives

$$\left\| \mathcal{W}_{\beta, \nu}(\psi_2^{(\mathfrak{l})}) \chi_{\mathfrak{l}} \right\|_{L^{\mathfrak{q}(\cdot)}} \leq \left\| \mathcal{W}_{\beta, \nu}(\psi_2^{(\mathfrak{l})}) \right\|_{L^{\mathfrak{q}(\cdot)}} \lesssim \left\| \psi_2^{(\mathfrak{l})} \right\|_{L^{\mathfrak{p}(\cdot)}} \leq \sum_{|j-\mathfrak{l}| \leq 1} \|\psi \chi_j\|_{L^{\mathfrak{p}(\cdot)}}. \quad (3.6)$$

Multiplying by $2^{\mathfrak{l}a_\infty}$, raising to the power $\mathfrak{p}(1+\epsilon)$, and using the triangle inequality in $\ell^{\mathfrak{p}(1+\epsilon)}$,

$$\sum_{\mathfrak{l} \in \mathbb{Z}} 2^{\mathfrak{l}a_\infty \mathfrak{p}(1+\epsilon)} \left\| \mathcal{W}_{\beta, \nu}(\psi_2^{(\mathfrak{l})}) \chi_{\mathfrak{l}} \right\|_{L^{\mathfrak{q}(\cdot)}}^{\mathfrak{p}(1+\epsilon)} \lesssim \sum_{\mathfrak{l} \in \mathbb{Z}} 2^{\mathfrak{l}a_\infty \mathfrak{p}(1+\epsilon)} \left(\sum_{|j-\mathfrak{l}| \leq 1} \|\psi \chi_j\|_{L^{\mathfrak{p}(\cdot)}} \right)^{\mathfrak{p}(1+\epsilon)}.$$

Since each index j appears in at most three of the shifted sums, the right-hand side is bounded by $C \sum_{j \in \mathbb{Z}} 2^{ja_\infty \mathfrak{p}(1+\epsilon)} \|\psi \chi_j\|_{L^{\mathfrak{p}(\cdot)}}^{\mathfrak{p}(1+\epsilon)}$, and taking the $1/\mathfrak{p}(1+\epsilon)$ power, then the weighted supremum over ϵ ,

$$\sup_{\epsilon > 0} \epsilon^\theta \left(\sum_{\mathfrak{l} \in \mathbb{Z}} 2^{\mathfrak{l}a_\infty \mathfrak{p}(1+\epsilon)} \left\| \mathcal{W}_{\beta, \nu}(\psi_2^{(\mathfrak{l})}) \chi_{\mathfrak{l}} \right\|_{L^{\mathfrak{q}(\cdot)}}^{\mathfrak{p}(1+\epsilon)} \right)^{\frac{1}{\mathfrak{p}(1+\epsilon)}} \lesssim \|\psi\|_{\dot{\mathbf{K}}_{\mathfrak{p}(\cdot)}^{a(\cdot), \mathfrak{p}, \theta}}. \quad (3.7)$$

Far-field region. For $\varepsilon \in \mathcal{R}_{\mathfrak{l}}$ and $\tau \in \mathcal{R}_j$ with $j \geq \mathfrak{l} + 2$, the ball $\mathcal{B}(\varepsilon, r)$ intersects $\mathcal{R}_j$ only when $r \geq c 2^j$, so

$$\begin{aligned} \left| \mathcal{W}_{\beta, \nu}(\psi_3^{(\mathfrak{l})})(\varepsilon) \right| &\lesssim \sum_{j \geq \mathfrak{l}+2} \left(\int_{c 2^j}^{\infty} r^{(\beta \nu - n)(\nu-1)-1} dr \right)^{\frac{1}{\nu-1}} \left(\int_{\mathcal{R}_j} |\psi(\tau)| d\tau \right) \\ &\lesssim \sum_{j \geq \mathfrak{l}+2} 2^{j \frac{\beta \nu - n}{\nu-1}} \int_{\mathcal{R}_j} |\psi(\tau)| d\tau. \end{aligned}$$

The Hölder inequality and Lemma 2.11 give $\int_{\mathcal{R}_j} |\psi| d\tau \lesssim 2^{jn/\mathfrak{p}'_\infty} \|\psi \chi_j\|_{L^{\mathfrak{p}(\cdot)}}$, so that

$$\left| \mathcal{W}_{\beta, \nu}(\psi_3^{(\mathfrak{l})})(\varepsilon) \right| \lesssim \sum_{j \geq \mathfrak{l}+2} 2^{j\gamma} \|\psi \chi_j\|_{L^{\mathfrak{p}(\cdot)}}, \quad \gamma := \frac{\beta \nu - n}{\nu - 1} + \frac{n}{(\nu - 1)\mathfrak{p}'_\infty}.$$

Taking the $L^{\mathfrak{q}(\cdot)}(\mathcal{R}_{\mathfrak{l}})$ -norm and applying Lemma 2.11 to obtain $\|\chi_{\mathfrak{l}}\|_{L^{\mathfrak{q}(\cdot)}} \approx 2^{\mathfrak{l}n/q_\infty}$,

$$\left\| \mathcal{W}_{\beta, \nu}(\psi_3^{(\mathfrak{l})}) \chi_{\mathfrak{l}} \right\|_{L^{\mathfrak{q}(\cdot)}} \lesssim 2^{\mathfrak{l}n/q_\infty} \sum_{j \geq \mathfrak{l}+2} 2^{j\gamma} \|\psi \chi_j\|_{L^{\mathfrak{p}(\cdot)}}.$$

Multiplying by $2^{\mathfrak{l}a_\infty}$ and rewriting the sum as a convolution with the kernel $\Psi(j - \mathfrak{l}) := 2^{(j-\mathfrak{l})\gamma'}$, where

$$\gamma' := \gamma - a_\infty = \frac{\beta \nu - n}{\nu - 1} + \frac{n}{(\nu - 1)\mathfrak{p}'_\infty} - a_\infty < 0$$

by the lower bound $a_\infty > -n/\mathfrak{p}_\infty$, the kernel $\{\Psi(k)\}_{k \geq 0}$ is summable, and Lemma 2.12 yields

$$\sup_{\epsilon > 0} \epsilon^\theta \left(\sum_{\mathfrak{l} \in \mathbb{Z}} 2^{\mathfrak{l}a_\infty \mathfrak{p}(1+\epsilon)} \left\| \mathcal{W}_{\beta, \nu}(\psi_3^{(\mathfrak{l})}) \chi_{\mathfrak{l}} \right\|_{L^{\mathfrak{q}(\cdot)}}^{\mathfrak{p}(1+\epsilon)} \right)^{\frac{1}{\mathfrak{p}(1+\epsilon)}} \lesssim \|\psi\|_{\dot{\mathbf{K}}_{\mathfrak{p}(\cdot)}^{a(\cdot), \mathfrak{p}, \theta}}. \quad (3.8)$$

Conclusion. Taking $L^{q(\cdot)}(\mathcal{R}_l)$ -norms in (3.3), multiplying by 2^{la_∞} , raising to the power $p(1+\epsilon)$, summing over $l \in \mathbb{Z}$, and applying the quasi-triangle inequality for $\ell^{p(1+\epsilon)}$, estimates (3.5), (3.7), and (3.8) combine to give

$$\left(\sum_{l \in \mathbb{Z}} 2^{la_\infty p(1+\epsilon)} \|\mathcal{W}_{\beta, \nu}(\psi) \chi_l\|_{L^{q(\cdot)}(\mathcal{R}_l)}^{p(1+\epsilon)} \right)^{\frac{1}{p(1+\epsilon)}} \lesssim \mathcal{M} \left(\sum_{l \in \mathbb{Z}} 2^{la_\infty p(1+\epsilon)} \|\psi \chi_l\|_{L^{p(\cdot)}(\mathcal{R}_l)}^{p(1+\epsilon)} \right)^{\frac{1}{p(1+\epsilon)}}.$$

Multiplying by ϵ^θ and taking $\sup_{\epsilon > 0}$, the left-hand side equals $\|\mathcal{W}_{\beta, \nu}(\psi)\|_{\dot{K}_{q(\cdot)}^{a(\cdot), p(\cdot), \theta}}$ by Definition 2.5, giving (3.1). $\square$

Theorem 3.2. *The following assumptions are imposed: $0 < \beta < n$, $\theta > 0$, and $0 < q \leq \infty$; $p(\cdot) \in \mathcal{P}_0^{\log}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < n/\beta$. Let $a(\cdot) \in L^\infty(\mathbb{R}^n)$ satisfy the log-Hölder continuity condition and assume that its essential bounds satisfy*

$$\beta - \frac{n}{p^-} < a^- \leq a^+ < \frac{n}{(p^+)'}. \quad (3.2)$$

Assume further that the Wolff potential operator $\mathcal{W}_{\beta, \nu}$ acts boundedly from $L^{p(\cdot)}(\mathbb{R}^n)$ into $L^{p^*(\cdot)}(\mathbb{R}^n)$, where

$$\frac{1}{p^*(\epsilon)} = \frac{1}{p(\epsilon)} - \frac{\beta}{n}.$$

Under these conditions, the operator $\mathcal{W}_{\beta, \nu}$ extends to a bounded map between the corresponding homogeneous grand variable Herz spaces:

$$\mathcal{W}_{\beta, \nu} : \dot{K}_{p(\cdot)}^{a(\cdot), p(\cdot), \theta}(\mathbb{R}^n) \longrightarrow \dot{K}_{p^*(\cdot)}^{a(\cdot), p(\cdot), \theta}(\mathbb{R}^n).$$

Moreover, there exists a constant $\mathcal{M} > 0$ such that

$$\|\mathcal{W}_{\beta, \nu}(\psi)\|_{\dot{K}_{p^*(\cdot)}^{a(\cdot), p(\cdot), \theta}} \lesssim \|\psi\|_{\dot{K}_{p(\cdot)}^{a(\cdot), p(\cdot), \theta}}. \quad (3.9)$$

Proof. Fix $\epsilon > 0$ throughout. For each $\kappa \in \mathbb{Z}$ set $\mathcal{B}_\kappa := \mathcal{B}(0, 2^\kappa)$, $\mathcal{R}_\kappa := \mathcal{B}_\kappa \setminus \mathcal{B}_{\kappa-1}$, and $\tilde{\mathcal{B}}_\kappa := \{\tau : 2^{\kappa-2} \leq |\tau| < 2^{\kappa+2}\}$. Decompose

$$\psi = \underbrace{\psi \chi_{\mathcal{B}_{\kappa-2}}}_{\psi_I} + \underbrace{\psi \chi_{\tilde{\mathcal{B}}_\kappa}}_{\psi_{II}} + \underbrace{\psi \chi_{\mathbb{R}^n \setminus \mathcal{B}_{\kappa+2}}}_{\psi_{III}},$$

so that sub-additivity gives, for every $\epsilon \in \mathcal{R}_\kappa$,

$$|\mathcal{W}_{\beta, \nu} \psi(\epsilon)| \leq |\mathcal{W}_{\beta, \nu} \psi_I(\epsilon)| + |\mathcal{W}_{\beta, \nu} \psi_{II}(\epsilon)| + |\mathcal{W}_{\beta, \nu} \psi_{III}(\epsilon)|. \quad (3.10)$$

Inner ball estimate. For $\epsilon \in \mathcal{R}_\kappa$ and $\sigma \in \mathcal{R}_\eta$ with $\eta \leq \kappa - 2$,

$$|\epsilon - \sigma| \geq |\epsilon| - |\sigma| \geq 2^{\kappa-1} - 2^{\kappa-2} = 2^{\kappa-2},$$

so $\mathcal{B}(\epsilon, r) \cap \mathcal{B}_{\kappa-2} = \emptyset$ for all $r < c 2^\kappa$. Hence only radii $r \geq c 2^\kappa$ contribute, and

$$\begin{aligned} |\mathcal{W}_{\beta, \nu} \psi_I(\epsilon)| &\lesssim \left(\int_{c 2^\kappa}^\infty \left(r^{\beta \nu - n} \int_{\mathcal{B}(\epsilon, r)} |\psi_I(\sigma)| d\sigma \right)^{\nu-1} \frac{dr}{r} \right)^{\frac{1}{\nu-1}} \\ &\leq \left(\int_{c 2^\kappa}^\infty r^{(\beta \nu - n)(\nu-1)-1} dr \right)^{\frac{1}{\nu-1}} \int_{\mathcal{B}_{\kappa-2}} |\psi(\sigma)| d\sigma. \end{aligned}$$

Since $(\beta\nu - \mathbf{n})(\nu - 1) - 1 < -1$, the radial integral evaluates to $c2^{\kappa(\beta\nu - \mathbf{n})/(\nu - 1)}$, giving

$$|\mathcal{W}_{\beta,\nu}\psi_I(\varepsilon)| \lesssim 2^{-\kappa\frac{\mathbf{n}-\beta\nu}{\nu-1}} \int_{\mathcal{B}_{\kappa-2}} |\psi(\sigma)| d\sigma. \quad (3.11)$$

The right-hand side of (3.11) is constant on $\mathcal{R}_\kappa$, so Lemma 2.11 yields

$$\|\mathcal{W}_{\beta,\nu}\psi_I \cdot \chi_\kappa\|_{L^{p^*(\cdot)}} \lesssim 2^{\kappa\mathbf{n}/p_\infty^*} \cdot 2^{-\kappa\frac{\mathbf{n}-\beta\nu}{\nu-1}} \int_{\mathcal{B}_{\kappa-2}} |\psi(\sigma)| d\sigma.$$

Applying the Hölder inequality on each dyadic piece and Lemma 2.11,

$$\int_{\mathcal{B}_{\kappa-2}} |\psi(\sigma)| d\sigma \leq \sum_{\eta \leq \kappa-2} \|\psi\chi_\eta\|_{L^{p(\cdot)}} \|\chi_\eta\|_{L^{p'(\cdot)}} \lesssim \sum_{\eta \leq \kappa-2} 2^{\eta\mathbf{n}/p^-} \|\psi\chi_\eta\|_{L^{p(\cdot)}},$$

so that, collecting all powers of 2,

$$\|\mathcal{W}_{\beta,\nu}\psi_I \cdot \chi_\kappa\|_{L^{p^*(\cdot)}} \lesssim 2^{\kappa\Lambda} \sum_{\eta \leq \kappa-2} 2^{\eta\mathbf{n}/p^-} \|\psi\chi_\eta\|_{L^{p(\cdot)}}, \quad \Lambda := \frac{\mathbf{n}}{p_\infty^*} - \frac{\mathbf{n} - \beta\nu}{\nu - 1}. \quad (3.12)$$

Multiplying by $2^{\kappa\mathbf{a}_\infty}$, the right-hand side of (3.12) becomes a convolution of the sequence $\left\{2^{\mathbf{n}(\mathbf{a}_\infty + \mathbf{n}/p^-)} \|\psi\chi_\eta\|_{L^{p(\cdot)}}\right\}_\eta$ with the kernel $\Phi(\kappa - \eta) := 2^{(\kappa - \eta)\delta_1}$, where

$$\delta_1 := \mathbf{a}_\infty + \Lambda - \frac{\mathbf{n}}{p^-} = \mathbf{a}^+ + \frac{\mathbf{n}}{p^-} + \frac{\beta\nu - \mathbf{n}}{\nu - 1} < 0$$

by the assumption $\mathbf{a}^+ + \mathbf{n}/p^- + (\beta\nu - \mathbf{n})/(\nu - 1) < 0$. Since $\delta_1 < 0$, the kernel $\{\Phi(k)\}_{k \geq 0}$ is ℓ^1 -summable, and Lemma 2.12 gives

$$\left\{ \sum_{\kappa \in \mathbb{Z}} 2^{\kappa\mathbf{a}_\infty p(1+\epsilon)} \|\mathcal{W}_{\beta,\nu}\psi_I \cdot \chi_\kappa\|_{L^{p^*(\cdot)}}^{p(1+\epsilon)} \right\}^{\frac{1}{p(1+\epsilon)}} \lesssim \left\{ \sum_{\kappa \in \mathbb{Z}} 2^{\kappa\mathbf{a}(\cdot)p(1+\epsilon)} \|\psi \cdot \chi_\kappa\|_{L^{p(\cdot)}}^{p(1+\epsilon)} \right\}^{\frac{1}{p(1+\epsilon)}}. \quad (3.13)$$

Middle region estimate. Since $\text{supp } \psi_{II} \subseteq \tilde{\mathcal{B}}_\kappa = \bigcup_{|j-\kappa| \leq 2} \mathcal{R}_j$, the triangle inequality gives $\|\psi_{II}\|_{L^{p(\cdot)}} \leq \sum_{|j-\kappa| \leq 2} \|\psi\chi_j\|_{L^{p(\cdot)}}$, and the assumed $L^{p(\cdot)} \rightarrow L^{p^*(\cdot)}$ boundedness yields

$$\|\mathcal{W}_{\beta,\nu}\psi_{II} \cdot \chi_\kappa\|_{L^{p^*(\cdot)}} \leq \|\mathcal{W}_{\beta,\nu}\psi_{II}\|_{L^{p^*(\cdot)}} \lesssim \|\psi_{II}\|_{L^{p(\cdot)}} \leq \sum_{|j-\kappa| \leq 2} \|\psi\chi_j\|_{L^{p(\cdot)}}. \quad (3.14)$$

Multiplying by $2^{\kappa\mathbf{a}_\infty}$, raising to the power $p(1+\epsilon)$, and summing over $\kappa \in \mathbb{Z}$,

$$\sum_{\kappa \in \mathbb{Z}} 2^{\kappa\mathbf{a}_\infty p(1+\epsilon)} \|\mathcal{W}_{\beta,\nu}\psi_{II} \cdot \chi_\kappa\|_{L^{p^*(\cdot)}}^{p(1+\epsilon)} \lesssim \sum_{\kappa \in \mathbb{Z}} 2^{\kappa\mathbf{a}_\infty p(1+\epsilon)} \left(\sum_{|j-\kappa| \leq 2} \|\psi\chi_j\|_{L^{p(\cdot)}} \right)^{p(1+\epsilon)}.$$

Each index j appears in at most five of the shifted sums, so the triangle inequality in $\ell^{p(1+\epsilon)}$ gives

$$\sum_{\kappa \in \mathbb{Z}} 2^{\kappa\mathbf{a}_\infty p(1+\epsilon)} \left(\sum_{|j-\kappa| \leq 2} \|\psi\chi_j\|_{L^{p(\cdot)}} \right)^{p(1+\epsilon)} \leq C \sum_{j \in \mathbb{Z}} 2^{j\mathbf{a}_\infty p(1+\epsilon)} \|\psi\chi_j\|_{L^{p(\cdot)}}^{p(1+\epsilon)},$$

and taking the $1/\mathfrak{p}(1+\epsilon)$ power then the weighted supremum over ϵ ,

$$\left\{ \sum_{\kappa \in \mathbb{Z}} 2^{\kappa \mathfrak{a}_\infty \mathfrak{p}(1+\epsilon)} \|\mathcal{W}_{\beta, \mathfrak{v}} \psi_{\text{II}} \cdot \chi_\kappa\|_{L^{\mathfrak{p}^*(\cdot)}(\cdot)}^{\mathfrak{p}(1+\epsilon)} \right\}^{\frac{1}{\mathfrak{p}(1+\epsilon)}} \lesssim \left\{ \sum_{\kappa \in \mathbb{Z}} 2^{\kappa \mathfrak{a}(\cdot) \mathfrak{p}(1+\epsilon)} \|\psi \cdot \chi_\kappa\|_{L^{\mathfrak{p}(\cdot)}(\cdot)}^{\mathfrak{p}(1+\epsilon)} \right\}^{\frac{1}{\mathfrak{p}(1+\epsilon)}}. \quad (3.15)$$

Far-field estimate. For $\epsilon \in \mathcal{R}_\kappa$ and $\sigma \in \mathcal{R}_\eta$ with $\eta \geq \kappa + 3$, the ball $\mathcal{B}(\epsilon, r)$ meets $\mathcal{R}_\eta$ only when $r \geq c 2^\eta$, so

$$\begin{aligned} |\mathcal{W}_{\beta, \mathfrak{v}} \psi_{\text{III}}(\epsilon)| &\lesssim \sum_{\eta \geq \kappa+3} \left(\int_{c 2^\eta}^\infty r^{(\beta \mathfrak{v} - \mathfrak{n})(\mathfrak{v}-1)-1} dr \right)^{\frac{1}{\mathfrak{v}-1}} \int_{\mathcal{R}_\eta} |\psi(\sigma)| d\sigma \\ &\lesssim \sum_{\eta \geq \kappa+3} 2^{\eta \frac{\beta \mathfrak{v} - \mathfrak{n}}{\mathfrak{v}-1}} \int_{\mathcal{R}_\eta} |\psi(\sigma)| d\sigma. \end{aligned}$$

Applying the Hölder inequality and Lemma 2.11, $\int_{\mathcal{R}_\eta} |\psi| d\sigma \lesssim 2^{\eta \mathfrak{n}/\mathfrak{p}'_\infty} \|\psi \chi_\eta\|_{L^{\mathfrak{p}(\cdot)}(\cdot)}$, so that

$$|\mathcal{W}_{\beta, \mathfrak{v}} \psi_{\text{III}}(\epsilon)| \lesssim \sum_{\eta \geq \kappa+3} 2^{\eta \gamma} \|\psi \chi_\eta\|_{L^{\mathfrak{p}(\cdot)}(\cdot)}, \quad \gamma := \frac{\beta \mathfrak{v} - \mathfrak{n}}{\mathfrak{v} - 1} + \frac{\mathfrak{n}}{(\mathfrak{v} - 1) \mathfrak{p}'_\infty}.$$

Taking the $L^{\mathfrak{p}^*(\cdot)}(\mathcal{R}_\kappa)$ -norm and applying Lemma 2.11 to obtain $\|\chi_\kappa\|_{L^{\mathfrak{p}^*(\cdot)}(\cdot)} \approx 2^{\kappa \mathfrak{n}/\mathfrak{p}'_\infty}$,

$$\|\mathcal{W}_{\beta, \mathfrak{v}} \psi_{\text{III}} \cdot \chi_\kappa\|_{L^{\mathfrak{p}^*(\cdot)}(\cdot)} \lesssim 2^{\kappa \mathfrak{n}/\mathfrak{p}'_\infty} \sum_{\eta \geq \kappa+3} 2^{\eta \gamma} \|\psi \chi_\eta\|_{L^{\mathfrak{p}(\cdot)}(\cdot)}. \quad (3.16)$$

Multiplying (3.16) by $2^{\kappa \mathfrak{a}_\infty}$ and re-expressing the sum as a convolution with the kernel $\Psi(\eta - \kappa) := 2^{(\eta - \kappa) \delta_2}$, where

$$\delta_2 := \gamma - \mathfrak{a}_\infty = \frac{\beta \mathfrak{v} - \mathfrak{n}}{\mathfrak{v} - 1} + \frac{\mathfrak{n}}{(\mathfrak{v} - 1) \mathfrak{p}'_\infty} - \mathfrak{a}_\infty < 0$$

by the assumption $\mathfrak{a}^- > \beta - \mathfrak{n}/\mathfrak{p}^-$, the kernel $\{\Psi(k)\}_{k \geq 0}$ is ℓ^1 -summable, and Lemma 2.12 gives

$$\left\{ \sum_{\kappa \in \mathbb{Z}} 2^{\kappa \mathfrak{a}_\infty \mathfrak{p}(1+\epsilon)} \|\mathcal{W}_{\beta, \mathfrak{v}} \psi_{\text{III}} \cdot \chi_\kappa\|_{L^{\mathfrak{p}^*(\cdot)}(\cdot)}^{\mathfrak{p}(1+\epsilon)} \right\}^{\frac{1}{\mathfrak{p}(1+\epsilon)}} \lesssim \left\{ \sum_{\kappa \in \mathbb{Z}} 2^{\kappa \mathfrak{a}(\cdot) \mathfrak{p}(1+\epsilon)} \|\psi \cdot \chi_\kappa\|_{L^{\mathfrak{p}(\cdot)}(\cdot)}^{\mathfrak{p}(1+\epsilon)} \right\}^{\frac{1}{\mathfrak{p}(1+\epsilon)}}. \quad (3.17)$$

Conclusion. Taking $L^{\mathfrak{p}^*(\cdot)}(\mathcal{R}_\kappa)$ -norms in (3.10), multiplying by $2^{\kappa \mathfrak{a}_\infty}$, raising to the power $\mathfrak{p}(1+\epsilon)$, and summing over $\kappa \in \mathbb{Z}$, the quasi-triangle inequality for $\ell^{\mathfrak{p}(1+\epsilon)}$ together with (3.13), (3.15), and (3.17) yields

$$\left\{ \sum_{\kappa \in \mathbb{Z}} 2^{\kappa \mathfrak{a}_\infty \mathfrak{p}(1+\epsilon)} \|\mathcal{W}_{\beta, \mathfrak{v}} \psi \cdot \chi_\kappa\|_{L^{\mathfrak{p}^*(\cdot)}(\cdot)}^{\mathfrak{p}(1+\epsilon)} \right\}^{\frac{1}{\mathfrak{p}(1+\epsilon)}} \lesssim \mathcal{M} \left\{ \sum_{\kappa \in \mathbb{Z}} 2^{\kappa \mathfrak{a}(\cdot) \mathfrak{p}(1+\epsilon)} \|\psi \cdot \chi_\kappa\|_{L^{\mathfrak{p}(\cdot)}(\cdot)}^{\mathfrak{p}(1+\epsilon)} \right\}^{\frac{1}{\mathfrak{p}(1+\epsilon)}}.$$

Multiplying both sides by ϵ^θ and taking $\sup_{\epsilon > 0}$, the left-hand side equals $\|\mathcal{W}_{\beta, \mathfrak{v}}(\psi)\|_{\dot{\mathbf{K}}_{\mathfrak{p}^*(\cdot)}^{\mathfrak{a}(\cdot), \mathfrak{p}(\cdot), \theta}}$ and the right-hand side equals $\mathcal{M}\|\psi\|_{\dot{\mathbf{K}}_{\mathfrak{p}(\cdot)}^{\mathfrak{a}(\cdot), \mathfrak{p}(\cdot), \theta}}$ by Definition 2.5, giving (3.9). $\square$

Remark 3.3. The theorems above refine and generalise the results in [10, 19, 18, 3, 22], where analogous conclusions were derived for Riesz and sublinear-type operators within classical and variable Herz-type spaces. The transition to grand variable Herz spaces recovers all known Herz-space boundedness results as limiting cases (by sending $\epsilon \rightarrow 0$, equivalently $\theta \rightarrow 0^+$) and simultaneously reaches borderline embedding cases that were beyond reach in the standard framework.

3.2. Birkhoff's Ergodic Theorem in Grand Variable Herz Spaces. The pointwise ergodic theorem, first established by George David Birkhoff in 1931 [9], constitutes a cornerstone of ergodic theory. It forges a profound connection between the time average of a function along the orbits of a measure-preserving transformation and the spatial average of that function over the entire probability space. Let (Ω, μ) be a probability space and $\mathcal{T} : \Omega \rightarrow \Omega$ a measure-preserving transformation. For $\psi \in L^1(\Omega)$, the theorem asserts that the limit

$$\psi_{\text{av}}(\eta) = \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{i=0}^{m-1} \psi(\mathcal{T}^i(\eta)) \quad (3.18)$$

holds for almost every $\eta \in \Omega$, with the limiting function ψ_{av} being $\mathcal{T}$ -invariant and satisfying $\int_{\Omega} \psi_{\text{av}} d\mu = \int_{\Omega} \psi d\mu$. The significance of this result is difficult to overstate: it provides rigorous justification for replacing time averages by ensemble averages in dynamical systems, thereby linking ergodic theory, probability theory, statistical mechanics [21], and information theory. Extensions to non-measure-preserving transformations, amenable groups, and various function spaces have since been developed [12, 20, 8]. The present work establishes a variant of the ergodic theorem within the grand variable Herz space framework $\mathbf{K}_{q(\cdot)}^{\alpha, \mathfrak{p}, \theta}(\Omega)$ (defined precisely in Definition 2.8), in which the exponents $\mathfrak{p}(\cdot)$ and $\alpha(\cdot)$ are required to be $\mathcal{T}$ -invariant; this hypothesis allows one to capture refined local and global regularity properties of dynamical systems concurrently.

Theorem 3.4. *Let (Ω, μ) be a probability space and $\mathcal{T} : \Omega \rightarrow \Omega$ a measure-preserving transformation. Let $\{\mathcal{B}_k\}_{k \geq 0}$ be a non-decreasing sequence of measurable sets satisfying $\mu(\mathcal{B}_0) > 0$, $\mu(\mathcal{B}_k) \rightarrow 1$, and $\mathcal{R}_k = \mathcal{B}_k \setminus \mathcal{B}_{k-1}$ for $k \geq 1$ with $\mathcal{R}_0 = \mathcal{B}_0$. Assume $\mathcal{T}^{-1}(\mathcal{R}_k) = \mathcal{R}_k$ modulo null sets for each k . Moreover, suppose $\mathfrak{p}(\cdot)$ is a $\mathcal{T}$ -invariant variable exponent on Ω with $\mathfrak{p}^+ < \infty$, and that $\alpha(\cdot)$ is likewise $\mathcal{T}$ -invariant. The space $\mathbf{K}_{q(\cdot)}^{\alpha, \mathfrak{p}, \theta}(\Omega)$ is as in Definition 2.8, with the annuli $\{\mathcal{R}_k\}$ playing the role of the dyadic shells. Then:*

- (i) *For any $\psi \in \mathbf{K}_{q(\cdot)}^{\alpha, \mathfrak{p}, \theta}(\Omega)$, the time average defined in (3.18) exists for almost every $\eta \in \Omega$, and ψ_{av} belongs to $\mathbf{K}_{q(\cdot)}^{\alpha, \mathfrak{p}, \theta}(\Omega)$.*
- (ii) *Every $\psi \in \mathbf{K}_{q(\cdot)}^{\alpha, \mathfrak{p}, \theta}(\Omega)$ satisfies*

$$\lim_{m \rightarrow \infty} \left\| \psi_{\text{av}} - \frac{1}{m} \sum_{i=0}^{m-1} \psi \circ \mathcal{T}^i \right\|_{\mathbf{K}_{q(\cdot)}^{\alpha, \mathfrak{p}, \theta}(\Omega)} = 0, \quad (3.19)$$

together with $\psi_{\text{av}}(\eta) = \psi_{\text{av}}(\mathcal{T}(\eta))$ a.e. and $\int_{\Omega} \psi_{\text{av}} d\mu = \int_{\Omega} \psi d\mu$.

Proof. Existence of the Pointwise Limit. For each $k \geq 0$, the Hölder inequality in the variable-exponent setting gives

$$\|\psi\|_{L^1(\mathcal{R}_k)} = \int_{\mathcal{R}_k} |\psi(\eta)| \cdot 1 \, d\mu(\eta) \leq \|\psi\chi_{\mathcal{R}_k}\|_{L^{q(\cdot)}(\Omega, \mu)} \|\chi_{\mathcal{R}_k}\|_{L^{q'(\cdot)}(\Omega, \mu)} \leq \|\psi\chi_{\mathcal{R}_k}\|_{L^{q(\cdot)}(\Omega, \mu)} \mu(\mathcal{R}_k)^{1-1/q_{\Omega}^-},$$

where $q_{\Omega}^- := \operatorname{ess\,inf}_{\eta \in \Omega} q(\eta)$ and we used $\|\chi_{\mathcal{R}_k}\|_{L^{q'(\cdot)}(\Omega, \mu)} \leq \mu(\mathcal{R}_k)^{1-1/q_{\Omega}^-}$. Summing over $k \geq 0$, and using the fact that $\{\mathcal{R}_k\}$ partitions Ω up to a null set (since $\mu(\mathcal{B}_k) \rightarrow 1$),

$$\|\psi\|_{L^1(\Omega)} = \sum_{k \geq 0} \|\psi\|_{L^1(\mathcal{R}_k)} \leq \sum_{k \geq 0} \mu(\mathcal{R}_k)^{1-1/q_{\Omega}^-} \|\psi\chi_{\mathcal{R}_k}\|_{L^{q(\cdot)}(\Omega, \mu)} \lesssim \|\psi\|_{\mathbf{K}_{q(\cdot)}^{\alpha, p, \theta}(\Omega)} < \infty, \quad (3.20)$$

establishing the continuous embedding $\mathbf{K}_{q(\cdot)}^{\alpha, p, \theta}(\Omega) \hookrightarrow L^1(\Omega)$. With $\psi \in L^1(\Omega)$ and $\mathcal{T}$ measure-preserving, the classical Birkhoff Ergodic Theorem asserts that the limit

$$\psi_{\text{av}}(\eta) := \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{i=0}^{m-1} \psi(\mathcal{T}^i(\eta)) \quad (3.21)$$

exists for μ -a.e. $\eta \in \Omega$, $\psi_{\text{av}} \circ \mathcal{T} = \psi_{\text{av}}$ a.e., and $\int_{\Omega} \psi_{\text{av}} \, d\mu = \int_{\Omega} \psi \, d\mu$.

Membership of ψ_{av} in the Grand Variable Herz Space. Fix $\epsilon > 0$ and $k \geq 0$. Since $t \mapsto |t|^{p(\eta)(1+\epsilon)}$ is convex and $\psi_{\text{av}}(\eta) = \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{i=0}^{m-1} \psi(\mathcal{T}^i(\eta))$, Jensen's inequality gives, for a.e. η ,

$$|\psi_{\text{av}}(\eta)|^{p(\eta)(1+\epsilon)} \leq \liminf_{m \rightarrow \infty} \frac{1}{m} \sum_{i=0}^{m-1} |\psi(\mathcal{T}^i(\eta))|^{p(\eta)(1+\epsilon)}. \quad (3.22)$$

Integrating (3.22) over $\mathcal{R}_k$ and applying Fatou's lemma,

$$\int_{\mathcal{R}_k} |\psi_{\text{av}}|^{p(\cdot)(1+\epsilon)} \, d\mu \leq \liminf_{m \rightarrow \infty} \frac{1}{m} \sum_{i=0}^{m-1} \int_{\mathcal{R}_k} |\psi(\mathcal{T}^i(\eta))|^{p(\eta)(1+\epsilon)} \, d\mu(\eta). \quad (3.23)$$

For each fixed i , the substitution $\eta' = \mathcal{T}^i(\eta)$, combined with the measure-preserving property $d\mu(\eta) = d\mu(\eta')$, the $\mathcal{T}$ -invariance $p(\mathcal{T}^i(\eta)) = p(\eta)$ a.e., and the invariance $\mathcal{T}^{-i}(\mathcal{R}_k) = \mathcal{R}_k \bmod \text{null sets}$, yields

$$\int_{\mathcal{R}_k} |\psi(\mathcal{T}^i(\eta))|^{p(\eta)(1+\epsilon)} \, d\mu(\eta) = \int_{\mathcal{R}_k} |\psi(\eta')|^{p(\eta')(1+\epsilon)} \, d\mu(\eta'). \quad (3.24)$$

Since the right-hand side of (3.24) is independent of i , every term in the Cesàro sum is identical, so substituting back into (3.23),

$$\int_{\mathcal{R}_k} |\psi_{\text{av}}|^{p(\cdot)(1+\epsilon)} \, d\mu \leq \int_{\mathcal{R}_k} |\psi|^{p(\cdot)(1+\epsilon)} \, d\mu,$$

which, upon taking the appropriate power $1/(p(\cdot)(1+\epsilon))$ and applying the norm equivalence on $\mathcal{R}_k$, gives

$$\|\psi_{\text{av}}\chi_k\|_{L^{p(\cdot)(1+\epsilon)}(\Omega, \mu)} \leq C_0 \|\psi\chi_k\|_{L^{p(\cdot)(1+\epsilon)}(\Omega, \mu)} \quad (3.25)$$

with C_0 independent of k and ϵ . Multiplying (3.25) by $2^{k\alpha q_\Omega^-}$, raising to the power q_Ω^- , summing over $k \geq 0$, taking the $1/q_\Omega^-$ power, multiplying by ϵ^θ , and taking $\sup_{\epsilon > 0}$ in accordance with Definition 2.8,

$$\|\psi_{av}\|_{\mathbf{K}_{q(\cdot)}^{\alpha, p, \theta}(\Omega)} \leq C_0 \|\psi\|_{\mathbf{K}_{q(\cdot)}^{\alpha, p, \theta}(\Omega)} < \infty, \quad (3.26)$$

which gives both the norm bound and $\psi_{av} \in \mathbf{K}_{q(\cdot)}^{\alpha, p, \theta}(\Omega)$, completing part (i).

Convergence in Norm. Bounded case. Suppose first $\|\psi\|_{L^\infty(\Omega)} \leq M < \infty$. For each $k \geq 0$ and $\epsilon > 0$, the pointwise bound

$$\left| \psi_{av}(\eta) - \frac{1}{m} \sum_{i=0}^{m-1} \psi(\mathcal{T}^i(\eta)) \right|^{p(\eta)(1+\epsilon)} \leq (2M)^{p^+(1+\epsilon)} =: g_\epsilon \in L^1(\mathcal{R}_k, \mu)$$

holds uniformly in m , since $p^+ < \infty$ and $\mu(\mathcal{R}_k) \leq 1$. Birkhoff's theorem gives pointwise a.e. convergence of the Cesàro averages to ψ_{av} , so the Dominated Convergence Theorem yields

$$\lim_{m \rightarrow \infty} \int_{\mathcal{R}_k} \left| \psi_{av} - \frac{1}{m} \sum_{i=0}^{m-1} \psi \circ \mathcal{T}^i \right|^{p(\cdot)(1+\epsilon)} d\mu = 0 \quad \text{for each fixed } k. \quad (3.27)$$

Since $\|\psi\|_{\mathbf{K}} < \infty$, for any $\delta > 0$ there exists K_0 such that $\sum_{k > K_0} 2^{k\alpha q_\Omega^-} \|\psi \chi_k\|_{L^{p(\cdot)(1+\epsilon)}(\Omega, \mu)}^{q_\Omega^-} < \delta^{q_\Omega^-}$, and the bound (3.25) controls the tail of the Cesàro average by the same quantity. Combining the tail estimate with (3.27) applied to the finite sum $k \leq K_0$, there exists $m_0 = m_0(\delta, M)$ such that for all $m \geq m_0$,

$$\left\| \psi_{av} - \frac{1}{m} \sum_{i=0}^{m-1} \psi \circ \mathcal{T}^i \right\|_{\mathbf{K}_{q(\cdot)}^{\alpha, p, \theta}(\Omega)} < \delta. \quad (3.28)$$

General case. For arbitrary $\psi \in \mathbf{K}_{q(\cdot)}^{\alpha, p, \theta}(\Omega)$, the truncations $\psi_N := \psi \cdot \mathbf{1}_{\{|\psi| \leq N\}}$ satisfy $\|\psi - \psi_N\|_{\mathbf{K}} \rightarrow 0$ as $N \rightarrow \infty$ by monotone convergence. Fix $\delta > 0$ and choose $\phi \in L^\infty(\Omega)$ with $\|\psi - \phi\|_{\mathbf{K}} \leq \delta$. The decomposition

$$\begin{aligned} \psi_{av} - \frac{1}{m} \sum_{i=0}^{m-1} \psi \circ \mathcal{T}^i &= (\psi - \phi)_{av} + \left(\phi_{av} - \frac{1}{m} \sum_{i=0}^{m-1} \phi \circ \mathcal{T}^i \right) \\ &\quad + \frac{1}{m} \sum_{i=0}^{m-1} (\phi - \psi) \circ \mathcal{T}^i \\ &=: E_1 + E_2 + E_3 \end{aligned} \quad (3.29)$$

holds in $\mathbf{K}_{q(\cdot)}^{\alpha, p, \theta}(\Omega)$. For E_1 , the bound (3.26) gives directly

$$\|E_1\|_{\mathbf{K}} = \|(\psi - \phi)_{av}\|_{\mathbf{K}} \leq C_0 \|\psi - \phi\|_{\mathbf{K}} \leq C_0 \delta. \quad (3.30)$$

For E_3 , since $\mathcal{T}$ is measure-preserving and both $p(\cdot)$, $\alpha(\cdot)$ are $\mathcal{T}$ -invariant with $\mathcal{T}^{-i}(\mathcal{R}_k) = \mathcal{R}_k \bmod \text{null sets}$, composition with $\mathcal{T}^i$ is an isometry of $\mathbf{K}_{q(\cdot)}^{\alpha, p, \theta}(\Omega)$:

$$\|(\phi - \psi) \circ \mathcal{T}^i\|_{\mathbf{K}} = \|\phi - \psi\|_{\mathbf{K}} \leq \delta \quad \text{for every } i \geq 0,$$

and the triangle inequality in $\mathbf{K}$ applied to the Cesàro average yields

$$\|E_3\|_{\mathbf{K}} \leq \frac{1}{m} \sum_{i=0}^{m-1} \|(\phi - \psi) \circ \mathcal{T}^i\|_{\mathbf{K}} = \|\phi - \psi\|_{\mathbf{K}} \leq \delta. \quad (3.31)$$

For E_2 , since $\phi \in L^\infty(\Omega)$, the bounded case (3.28) provides $m_0 = m_0(\delta, \phi)$ such that

$$\|E_2\|_{\mathbf{K}} < \delta \quad \text{for all } m \geq m_0. \quad (3.32)$$

Combining (3.30), (3.32), and (3.31) via the triangle inequality,

$$\left\| \psi_{\text{av}} - \frac{1}{m} \sum_{i=0}^{m-1} \psi \circ \mathcal{T}^i \right\|_{\mathbf{K}^{\alpha, p), \theta}_{q(\cdot)}(\Omega)} \leq (C_0 + 2) \delta \quad \text{for all } m \geq m_0, \quad (3.33)$$

and since $\delta > 0$ was arbitrary, (3.19) follows. $\square$

Remark 3.5. The foregoing theorem improves upon recent results obtained in the variable Lebesgue space setting [16], which are carried over here to the grand variable Herz framework that simultaneously captures local and global behaviour of dynamical systems. In particular, the grand integrability structure furnishes control at borderline integrability thresholds inaccessible in standard variable Lebesgue or Herz space settings. Setting $\theta \rightarrow 0^+$ (equivalently, working within the standard Herz space) recovers the ergodic result of [16] as a special case.

4. CONCLUSION

We have proved the boundedness of the Wolff potential operator in grand variable Herz spaces, extending earlier results confined to Lebesgue and classical Herz spaces. The central novelty lies in the simultaneous use of the grand integrability parameter, governing borderline Sobolev-type embeddings, and the variable exponent structure, adapting to local integrability of the medium. We have further established Birkhoff's ergodic theorem in this setting, proving convergence of time averages both pointwise almost everywhere and in the grand variable Herz norm. These results provide effective tools for regularity and well-posedness of quasilinear elliptic and parabolic PDEs in irregular domains. Open directions include multilinear extensions of the Wolff potential, commutator estimates with BMO functions, and applications to nonlinear elliptic systems with non-standard growth.

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¹ DEPARTMENT OF MATHEMATICS, COLLEGE OF ARTS AND SCIENCES, MINDANAO STATE UNIVERSITY – TAWI-TAWI COLLEGE OF TECHNOLOGY AND OCEANOGRAPHY, 7500 BONGAO, TAWI-TAWI, PHILIPPINES.

Email address: rosalioartes@msutawi-tawi.edu.ph

³ ABDUS SALAM SCHOOL OF MATHEMATICAL SCIENCES, GOVERNMENT COLLEGE UNIVERSITY, LAHORE 54600, PAKISTAN.

⁴ CENTER FOR THEORETICAL PHYSICS, KHAZAR UNIVERSITY, 41 MEHSETI STR., BAKU, AZ1096, AZERBAIJAN.

Email address: waqar2989@gmail.com

⁵ DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF ALABAMA AT BIRMINGHAM, BIRMINGHAM, AL, USA.

Email address: sasjad@uab.edu

⁶ MATHEMATICS RESEARCH CENTER, NEAR EAST UNIVERSITY, NEAR EAST BOULEVARD, NICOSIA, MERSIN, 99138, TURKEY.

Email address: captaintariq2187@gmail.com

⁷ IRFAN SUAT GUNSEL OPERATIONAL RESEARCH INSTITUTE, NEAR EAST UNIVERSITY, NICOSIA/TRNC, MERSIN 10, 99138, TURKEY.

⁸ DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE, KOREA UNIVERSITY, 145 ANAM-RO, SEONGBUK-GU, SEOUL 02841, SOUTH KOREA.

⁹ SUSTAINABILITY COMPETENCE CENTRE, SZÉCHENYI ISTVÁN UNIVERSITY, EGYETEM TÉR 1, H-9026 GYŐR, HUNGARY.

¹⁰ VIZJA UNIVERSITY, OKOPOWA 59, 01-043 WARSAW, POLAND.

Email address: hijazahmad@korea.ac.kr