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SHARP THRESHOLD DYNAMICS FOR A STOCHASTIC SIRS MODEL WITH GENERAL NONLINEAR INCIDENCE

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Abstract. This paper studies the dynamics of a stochastic SIRS epidemic model with a general nonlinear incidence rate. The model incorporates environmental noise through white noise perturbations. We first establish basic properties of the total population, including boundedness, recurrence, and the existence of a stationary distribution. Then, under suitable hypotheses on the incidence function, we derive conditions for disease extinction and persistence. The theoretical results are illustrated by a corollary providing explicit lower bounds for the time averages of the infected, recovered, and susceptible compartments. Numerical simulations are used to confirm the theoretical threshold dynamics and demonstrate the practical applicability of the model.

Keywords. Stochastic SIRS model, general incidence rate, extinction, persistence in mean, threshold dynamics.
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1. Introduction

Mathematical epidemic models have played a fundamental role in understanding infectious disease transmission since the seminal work of Kermack and McKendrick [11]. Classical compartmental models, such as SIR and its variants, have provided valuable insights into disease dynamics, control strategies, and threshold phenomena [3, 4]. However, deterministic models often neglect environmental variability and random fluctuations that significantly influence disease spread in real populations. To address this limitation, stochastic epidemic models have received considerable attention [2, 18]. Environmental factors such as climate variability, demographic fluctuations, and human mobility can alter transmission dynamics and disease outcomes. Stochastic differential equations (SDEs), particularly those driven by Brownian motion, offer a natural framework for incorporating such uncertainties into epidemiological models [10, 19, 20]. A key component of epidemic modeling is the incidence function, which describes the generation of new infections. While bilinear and standard incidence rates are

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widely used, more general nonlinear incidence functions have been proposed to capture saturation effects, behavioral responses, and other complex transmission mechanisms [1, 5, 17]. Such generalized incidence functions provide greater modeling flexibility and encompass many commonly used transmission laws as special cases. The SIRS framework is particularly suitable for diseases in which immunity is temporary and recovered individuals may return to the susceptible class [8]. Numerous stochastic SIRS models have been investigated, focusing on extinction, persistence, stability, and the effects of environmental noise [6, 13, 20, 22]. Despite substantial progress, the analysis of stochastic SIRS systems with general incidence functions remains challenging, especially regarding threshold dynamics that determine whether a disease dies out or persists. In stochastic settings, the classical basic reproduction number often requires modification to account for noise effects [7, 15, 21]. Motivated by these challenges, we consider a stochastic SIRS epidemic model with a general incidence function $f(S, I)$ and environmental perturbations affecting all compartments. The main contributions of this work are as follows:

(1) We establish fundamental properties of the model, including positivity, boundedness, recurrence, and the existence of a stationary distribution for the total population process.
(2) We derive a stochastic threshold criterion for disease extinction. When the stochastic reproduction number $\mathcal{R}_0^s$ is below unity, the infected and recovered populations converge exponentially to zero almost surely, while the susceptible population converges to the disease-free stationary distribution.
(3) We establish persistence in the mean results when $\mathcal{R}_0^s > 1$, obtaining explicit lower bounds for the long-term averages of the epidemiological compartments.

Our analysis combines Itô's formula, martingale techniques, and ergodic arguments to characterize the impact of environmental noise on disease dynamics. The obtained threshold parameter explicitly incorporates noise intensity and extends classical extinction and persistence results for SIRS models with general incidence functions. The remainder of the paper is organized as follows. Section 2 introduces the model and assumptions. Section 3 establishes fundamental properties of the total population process. Sections 4 and 5 present the extinction and persistence analyses, respectively. Numerical simulations are provided in Section 6, followed by concluding remarks in Section 7.

2. The SIRS model with general incidence rate

Consider the stochastic SIRS model:

$$\begin{cases} dS = (\Lambda - \mu S - f(S,I) + \delta R)\,dt + \sigma S\,dB, \\ dI = (-(\mu+\lambda)I + f(S,I))\,dt + \sigma I\,dB, \\ dR = (-(\mu+\delta)R + \lambda I)\,dt + \sigma R\,dB, \end{cases} \tag{2.1}$$

where $S(t)$, $I(t)$, and $R(t)$ represent the numbers of susceptible, infected, and recovered individuals at time t, respectively. The parameters are: Λ (recruitment

rate), μ (natural death rate), λ (recovery rate), δ (rate of immunity loss), and σ (noise intensity). The function $f(S,I)$ is a general incidence rate, and B_t is a standard Brownian motion. Under the regularity assumptions on $f(S,I)$ (in particular $f \in \mathcal{C}^2(\mathbb{R}^2_+)$), classical stochastic differential equation theory guarantees that system 2.1 admits a unique global positive solution for any initial condition $(S(0),I(0),R(0)) \in \mathbb{R}^3_+$.

3. Basic properties of the total population

Theorem 3.1. *Let $N(t)$ be the total population in system (2.1) with initial value $(S(0),I(0),R(0)) \in \mathbb{R}^3_+$. Then:*

(i) For any $q \geq 1$ such that

$$\mu - \frac{1}{2}(q-1)\sigma^2 > 0,$$

we have

$$\limsup_{t\to\infty} \mathbb{E}\, N^q(t) \leq \left(\frac{\Lambda}{\mu - \frac{1}{2}(q-1)\sigma^2}\right)^q.$$

(ii) $N(t)$ is positive recurrent and admits a unique stationary density

$$\nu^*(x) = \frac{b^a}{\Gamma(a)} x^{-(a+1)} e^{\frac{-b}{x}}, \qquad x > 0,$$

where

$$a = \frac{2\mu + \sigma^2}{\sigma^2}, \qquad b = \frac{2\Lambda}{\sigma^2},$$

and $\Gamma(\cdot)$ is the Gamma function.

(iii)

$$\mathbb{P}\Big(\liminf_{t\to\infty} N(t) > 0\Big) = \mathbb{P}\Big(\limsup_{t\to\infty} N(t) < \infty\Big) = 1.$$

(iv)

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t S(u)\, dB(u) = \lim_{t\to\infty} \frac{1}{t}\int_0^t I(u)\, dB(u) = \lim_{t\to\infty} \frac{1}{t}\int_0^t R(u)\, dB(u) = 0.$$

Proof. For (i)–(ii), see Lahrouz *et al.* [14].

(iii) Firstly, we show that

$$\liminf_{t\to\infty} N(t) > 0 \quad \text{a.s.}$$

The equation of $N(t)$ is given by

$$dN = (\Lambda - \mu N)\, dt + \sigma N\, dB.$$

By Itô's formula applied to $e^t/N(t)$, we obtain

$$d\left(\frac{e^t}{N(t)}\right) = \frac{e^t}{N(t)}\, dt + \left(-\frac{\Lambda}{N(t)^2} + \frac{\mu}{N(t)} + \frac{\sigma^2}{N(t)}\right) e^t\, dt - \frac{\sigma e^t}{N(t)}\, dB(t),$$

$$= \left(-\frac{\Lambda}{N(t)^2} + \frac{\mu + \sigma^2 + 1}{N(t)}\right) e^t dt - \frac{\sigma e^t}{N(t)}\, dB(t)$$

$$\leq \frac{(\mu+\sigma^2+1)^2}{4\Lambda}e^t dt - \frac{\sigma e^t}{N(t)}\,dB(t). \tag{1}$$

Where the elementary inequality $-\frac{\Lambda}{x^2}+\frac{\mu+\sigma^2+1}{x} \leq \frac{(\mu+\sigma^2+1)^2}{4\Lambda}$ is used in the last estimation (1), which gives by integration

$$\frac{e^t}{N(t)} - \frac{1}{N(0)} \leq \frac{(\mu+\sigma^2+1)^2}{4\Lambda}\int_0^t e^u du - \int_0^t \frac{\sigma e^u}{N(u)} dB(u). \tag{3.1}$$

Simplifying and taking expectation on both sides of (3.1) yields

$$\mathbb{E}\left(\frac{1}{N(t)}\right) \leq \frac{1}{N(0)} + \frac{(\mu+\sigma^2+1)^2}{4\Lambda} = C.$$

Now, for $m > \frac{1}{N(0)}$, define the stopping time

$$\tau_m^1 = \inf\left\{t \geq 0 : N(t) \leq \frac{1}{m}\right\}$$

and the measurable set

$$\Omega_1 = \left\{\omega \in \Omega : \liminf_{t\to\infty} N(t) = 0\right\}.$$

So,

$$m\,\mathbb{P}(\Omega_1) = m\,\mathbb{E}(\mathbf{1}_{\Omega_1}) \leq \mathbb{E}\left(\frac{1}{N(\tau_m^1)}\right) \leq C.$$

Hence

$$\mathbb{P}(\Omega_1) \leq \frac{C}{m}.$$

As m tends to ∞ gives

$$\mathbb{P}(\Omega_1) = 0.$$

Which completes the proof of the first part of (iii). Similarly, if we consider

$$\tau_m^2 = \inf\{t \geq 0 : N(t) \geq m\}$$

and

$$\Omega_2 = \left\{\omega \in \Omega : \limsup_{t\to\infty} N(t) = \infty\right\}.$$

We obtain

$$\begin{aligned} m\,\mathbb{P}(\Omega_2) &= m\,\mathbb{E}(\mathbf{1}_{\Omega_2}) \\ &\leq \mathbb{E}(N(\tau_m^2)) \\ &\leq \sup_{t>0}\mathbb{E}(N(t)) < \infty \qquad \text{(By using (i))}. \end{aligned}$$

Then $\mathbb{P}(\Omega_2) \leq \left(\sup_{t>0}\mathbb{E}N(t)\right) m^{-1}$, which implies by letting $m \to \infty$, that $\mathbb{P}(\Omega_2) = 0$.

(iv). Let $\left\langle \int_0^t S(u)\,dB(u)\right\rangle$ be the quadratic variation of the local martingale of $\int_0^t S(u)\,dB(u)$, we have

$$\begin{aligned}\left\langle \int_0^t S(u)\,dB(u) \right\rangle &= \int_0^t S^2(u)\,du \\ &\leq \int_0^t N^2(u)\,du \\ &\leq \left(\sup_{t>0} N^2(u)\right) t,\end{aligned}$$

where $\sup_{t>0} N^2(u) < \infty$. a.s according to (iii). Hence, the law of large numbers for local martingales implies that

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t S(u)\,dB(u) = 0.$$

Using the same reasoning, we obtain the remainder results. □

4. Disease Extinction

Theorem 4.1. *Suppose that the incidence rate $f(S,I)$ is $\mathcal{C}^2(\mathbb{R}^2_+)$ and verifies the hypothesis*

(H1) $f(S,0) = f(0,I) = 0$ *for all* $S, I > 0$.
(H2) $I \to \frac{f(S,I)}{I}$ *is decreasing for all* $S > 0$.
(H3) $\frac{\partial f}{\partial y}(.,0)$ *is increasing on* $(0,\infty)$.
(H4) $\frac{\partial f}{\partial y}(.,0)$ *is integrable with respect to* ν^*.

Let us define the epidemic threshold:

$$\mathcal{R}_0^s := \frac{\int_0^\infty \frac{\partial f}{\partial y}(x,0)\nu^*(dx)}{\mu + \lambda + \frac{1}{2}\sigma^2}.$$

If $\mathcal{R}_0^s < 1$, *then:*

(i) $I(t)$ *tends to 0 exponentially with probability 1.*
(ii) $\lim_{t\to\infty} R(t) = 0$ *a.s.*
(iii) $S(t)$ *converges weakly to the stationary distribution* ν^*.

Proof. (i) By Itô's formula we derive from the I-equation that

$$d\log I = -(\mu + \lambda + \frac{1}{2}\sigma^2) + \frac{f(S,I)}{I} + \sigma dB.$$

From (H1), one can deduce that

$$\frac{f(S,I)}{I} \leq \frac{\partial f}{\partial y}(S,0).$$

Hence

$$\begin{aligned} d\log I &\leq -(\mu + \lambda + \frac{1}{2}\sigma^2) + \frac{\partial f}{\partial y}(S,0) + \sigma dB \\ &\leq -(\mu + \lambda + \frac{1}{2}\sigma^2) + \frac{\partial f}{\partial y}(N,0) + \sigma dB, \end{aligned} \tag{4.1}$$

where the last inequality follows from the fact that $S \leq N$ and the monotonicity of $\frac{\partial f}{\partial y}(\cdot, 0)$. Integrating (4.1) and dividing both sides by t gives

$$\frac{\log I(t)}{t} \leq -(\mu + \lambda + \frac{1}{2}\sigma^2) + \frac{1}{t}\int_0^t \frac{\partial f}{\partial y}(N(u), 0)du + \frac{\log I(0) + B(t)}{t}. \tag{4.2}$$

Using the large number theorem for the Brownian motion $B(t)$ and the ergodicity of $N(t)$, yields that

$$\lim_{t\to\infty} \frac{B(t)}{t} = 0 \tag{4.3}$$

and

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \frac{\partial f}{\partial y}(N(u), 0)du = \int_0^\infty \frac{\partial f}{\partial y}(x, 0)\nu^*(dx). \tag{4.4}$$

Combining (4.2), (4.3) and (4.4) gives

$$\limsup_{t\to\infty} \frac{\log I(t)}{t} \leq (\mu + \lambda + \frac{1}{2}\sigma^2)(\mathcal{R}_0^s - 1) < 0.$$

This completes the proof of (i).

(ii) Let $\epsilon > 0$. An application of Itô's formula gives

$$de^{\epsilon t}R(t) = [\lambda e^{\epsilon t}I(t) - (\mu + \delta - \epsilon)e^{\epsilon t}R(t)]dt + \sigma e^{\epsilon t}R(t)dB(t).$$

Then

$$e^{\epsilon t}R(t) = R(0) + \lambda\int_0^t e^{\epsilon u}I(u)du - (\mu+\delta-\epsilon)\int_0^t e^{\epsilon u}R(u)du + \sigma\int_0^t e^{\epsilon u}R(u)dB(u).$$

From (i), we have

$$\int_0^\infty e^{\epsilon u}I(u)du \leq const\int_0^\infty e^{\epsilon u}e^{-\left(\mu+\lambda+\frac{1}{2}\sigma^2-\int_0^\infty \frac{\partial f}{\partial y}(x,0)\nu^*(dx)\right)u}du < \infty,$$

where ϵ is chosen such that

$$0 < \epsilon < \min\left(\mu + \delta, \mu + \lambda + \frac{1}{2}\sigma^2 - \int_0^\infty \frac{\partial f}{\partial y}(x, 0)\nu^*(dx)\right).$$

.

Therefore, using the nonnegative semi-martingale convergence theorem by Liptser and Shiryayev [16], one can derive that

$$\lim_{t\to\infty} e^{\epsilon t}R(t) = X(\omega) < \infty \quad \text{a.s.}$$

Thereby, for almost all $\omega \in \Omega$, and $\eta > 0$ there exists $T(\omega) > 0$ such that

$$\forall t > T(\omega) \quad e^{\epsilon t}R(t) \leq X(\omega) + \eta.$$

Hence

$$\frac{\log R(t)}{t} \leq -\epsilon + \frac{\log(X(\omega) + \eta)}{t}$$

and

$$\limsup_{t\to\infty} \frac{\log R(t)}{t} \leq -\epsilon$$

which implies, using the arbitrariness of ϵ, that

$$\limsup_{t\to\infty}\frac{\log R(t)}{t}\leq -\min\left(\mu+\delta,\mu+\lambda+\frac{1}{2}\sigma^2-\int_0^\infty\frac{\partial f}{\partial y}(x,0)\nu^*(dx)\right).$$

(iii). From (i) and (ii), we have

$$N(t)-S(t)=I(t)+R(t)\to 0.$$

Moreover, since $N(t)$ is a positive recurrent Feller diffusion with a unique invariant measure ν^*, its law converges weakly to ν^* (see e.g. Khasminskii [12]). Therefore, using Slutsky theorem, we obtain the convergence

$$S(t)\xrightarrow{\mathcal{W}}\nu^*\quad\text{as } t\to\infty.$$

□

5. Disease Persistence

Theorem 5.1. *Suppose that the incidence rate $f(S,I)\in\mathcal{C}^2(\mathbb{R}^2_+)$ and verifies the conditions $(H_1)-(H_4)$. If the condition*

$$\mathcal{R}_0^s>1$$

holds, then we have the following assertions :

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t I(u)du>0\qquad a.s.$$

That is, the disease is persistent in mean.

Proof. We know that

$$d\log I(t)=-(\mu+\lambda+\frac{1}{2}\sigma^2)+\frac{f(S(t),I(t))}{I(t)}+\sigma dB(t).\tag{5.1}$$

Applying the mean value theorem on the function

$$g_S(I)=\begin{cases}\frac{f(S,I)}{I} & \text{if } I>0,\\ \frac{\partial f}{\partial y}(S,0) & \text{if } I=0.\end{cases}$$

So

$$\frac{f(S,I)}{I}=\frac{\partial f}{\partial y}(S,0)+I\frac{\partial}{\partial y}\left[\frac{f(S,I)}{I}\right]_{I=C_{S,I}}.$$

where $0<C_{S,I}<I$.

Let $\omega\in\{\omega\in\Omega,\ 0<\liminf N(t)<\limsup N(t)<\infty\}=\Omega'$. According to Theorem 3.1 we have $\mathbb{P}(\Omega')=1$. Hence,

$$\left|\frac{\partial}{\partial y}\left[\frac{f(S(t,\omega),I(t,\omega)}{I(t,\omega)}\right]\right|\leq X_1(\omega)\tag{5.2}$$

where $X_1(\omega)=\sup_{t>0}\left|\frac{\partial}{\partial y}\left[\frac{f(S(t,\omega),I(t,\omega)}{I(t,\omega)}\right]\right|<\infty$ a.s. This follows from the assumption that $f\in\mathcal{C}^2(\mathbb{R}^2_+)$ and the fact that $S(t)$ and $I(t)$ are almost surely bounded by $N(t)$, which satisfies $\limsup_{t\to\infty}N(t)<\infty$ a.s. (as established in

Theorem 3.1).
Injecting (5.2) into (5.1) yields

$$\begin{aligned} d\log I(t,\omega) &\geq -(\mu+\lambda+\frac{1}{2}\sigma^2)+\frac{\partial f}{\partial y}(S(t,\omega),0)-X_1(\omega)I(t,\omega)+\sigma dB(t,\omega). \\ &= \int_0^\infty \frac{\partial f}{\partial y}(x,0)\nu^*(dx)-(\mu+\lambda+\frac{1}{2}\sigma^2)+\sigma dB(t,\omega) \\ &\quad -\int_0^\infty \frac{\partial f}{\partial y}(x,0)\nu^*(dx)+\frac{\partial f}{\partial y}(N(t,\omega),0) \\ &\quad -\left[\frac{\partial f}{\partial y}(N(t,\omega),0)-\frac{\partial f}{\partial y}(S(t,\omega),0)\right]-X_1(\omega)I(t,\omega). \end{aligned} \tag{5.3}$$

Another application of the mean value theorem gives

$$\frac{\partial f}{\partial y}(N,0)-\frac{\partial f}{\partial y}(S,0)=(N-S)\frac{\partial^2 f}{\partial x\partial y}(C_{S,N},0).$$

Similarly as for $X_1(\omega)$, the fact that $f\in\mathcal{C}^2(\mathbb{R}_+^2)$ and $S(t)$ is almost surely bounded leads to

$$\sup_{t>0}\frac{\partial^2 f}{\partial x\partial y}(C_{S(t,\omega),N(t,\omega)},0)=X_2(\omega)<\infty \quad \text{a.s.}$$

So,

$$\frac{\partial f}{\partial y}(N(t,\omega),0)-\frac{\partial f}{\partial y}(S(t,\omega),0)\leq (N(t,\omega)-S(t,\omega))X_2(\omega)$$

which implies with (5.3) that

$$\begin{aligned} d\log I(t,\omega) &\geq \int_0^\infty \frac{\partial f}{\partial y}(x,0)\nu^*(dx)-(\mu+\lambda+\frac{1}{2}\sigma^2)+\sigma dB(t) \\ &\quad -\int_0^\infty \frac{\partial f}{\partial y}(x,0)\nu^*(dx)+\frac{\partial f}{\partial y}(N(t,\omega),0) \\ &\quad -X_2(\omega)(N(t,\omega)-S(t,\omega))-X_1(\omega)I(t,\omega) \\ &= \int_0^\infty \frac{\partial f}{\partial y}(x,0)\nu^*(dx)-(\mu+\lambda+\frac{1}{2}\sigma^2)+\sigma dB(t) \\ &\quad -\int_0^\infty \frac{\partial f}{\partial y}(x,0)\nu^*(dx)+\frac{\partial f}{\partial y}(N(t,\omega),0) \\ &\quad -X_2(\omega)R(t,\omega)-(X_1(\omega)+X_2(\omega))I(t,\omega), \end{aligned}$$

where $N(t,\omega)-S(t,\omega)=R(t,\omega)+I(t,\omega)$ is used.
Using the equation of R, one can write

$$
\begin{aligned}
d\log I(t,\omega) - \frac{X_2(\omega)}{\mu+\delta}\,dR(t,\omega) \;\geq\; & \int_0^\infty \frac{\partial f}{\partial y}(x,0)\nu^*(dx) - (\mu+\lambda+\tfrac{1}{2}\sigma^2) \\
& - \int_0^\infty \frac{\partial f}{\partial y}(x,0)\nu^*(dx) + \frac{\partial f}{\partial y}(N(t,\omega),0) \\
& - \Big(X_1(\omega) + X_2(\omega) + \frac{\lambda X_2(\omega)}{\mu+\delta}\Big) I(t,\omega) \\
& + \sigma\Big(1 - \frac{X_2(\omega)}{\mu+\delta}R(t,\omega)\Big)\,dB(t,\omega).
\end{aligned}
$$

It follows by integrating and dividing by t that

$$
\begin{aligned}
\frac{\log I(t,\omega)}{t} - \frac{\log I(0)}{t} - \frac{X_2(\omega)}{\mu+\delta}\left(\frac{R(t,\omega)-R(0)}{t}\right) & \\
\geq \int_0^\infty \frac{\partial f}{\partial y}(x,0)\nu^*(dx) - (\mu+\lambda+\tfrac{1}{2}\sigma^2) & \\
- \int_0^\infty \frac{\partial f}{\partial y}(x,0)\nu^*(dx) & \\
+ \frac{1}{t}\int_0^t \frac{\partial f}{\partial y}(N(u,\omega),0)\,du & \\
- \Big(X_1(\omega) + X_2(\omega) + \frac{\lambda X_2(\omega)}{\mu+\delta}\Big)\frac{1}{t}\int_0^t I(u,\omega)\,du & \\
- \frac{\sigma}{t}\int_0^t \Big(1 - \frac{X_2(\omega)}{\mu+\delta}R(u,\omega)\Big)\,dB(u,\omega). &
\end{aligned}
\tag{5.4}
$$

Using the ergodicity of $N(t)$ and Theorem 3.1(iv), we obtain

$$
\lim_{t\to\infty}\Big(-\int_0^\infty \frac{\partial f}{\partial y}(x,0)\nu^*(dx) + \frac{1}{t}\int_0^t \frac{\partial f}{\partial y}(N(u,\omega),0)\,du\Big) = 0,
$$

and

$$
\lim_{t\to\infty}\frac{1}{t}\int_0^t \Big(1 - \frac{X_2(\omega)}{\mu+\delta}R(u,\omega)\Big)\,dB(u,\omega) = 0.
$$

Moreover, since $\limsup_{t\to\infty} N(t,\omega) < \infty$, we also have

$$
\limsup_{t\to\infty}\frac{\log I(t,\omega)}{t} \leq \limsup_{t\to\infty}\frac{I(t,\omega)}{t} \leq \limsup_{t\to\infty}\frac{N(t,\omega)}{t} \leq \sup_{t>0}\big(N(t,\omega)\big)\times 0 = 0,
$$

and

$$
\limsup_{t\to\infty}\frac{R(t,\omega)}{t} \leq \limsup_{t\to\infty}\frac{N(t,\omega)}{t} = 0.
$$

Taking lim sup on both sides of (5.4) and using the above limits yields

$$
\begin{aligned}
0 \;\geq\; & \int_0^\infty \frac{\partial f}{\partial y}(x,0)\nu^*(dx) - (\mu+\lambda+\tfrac{1}{2}\sigma^2) \\
& - \Big(X_1(\omega) + X_2(\omega) + \frac{\lambda X_2(\omega)}{\mu+\delta}\Big)\liminf_{t\to\infty}\frac{1}{t}\int_0^t I(u,\omega)\,du.
\end{aligned}
$$

Thus

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t I(u,\omega)\,du \geq \frac{(\mu+\lambda+\frac{1}{2}\sigma^2)(\mathcal{R}_0^S-1)}{X_1(\omega)+X_2(\omega)+\dfrac{\lambda X_2(\omega)}{\mu+\delta}} > 0 \quad \text{a.s.} \tag{5.5}$$

This completes the proof of Theorem 5.1. □

Corollary 5.2. *Let $f(S,I)\in C^2(\mathbb{R}^2_+)$ and verifies the hypotheses $(H_1)-(H_4)$. Moreover, suppose that:*

(H5) $(S,I)\to \dfrac{\partial}{\partial I}\left[\dfrac{f(S,I)}{I}\right]$ *is bounded by* $M_1>0$.

(H6) $S\to \frac{\partial^2 f}{\partial x\partial y}(S,0)$ *is bounded above by* $M_2>0$.

If the condition

$$\mathcal{R}_0^s > 1$$

holds, then we have the following estimates:

(i) $\liminf_{t\to\infty}\frac{1}{t}\int_0^t I(u)\,du > \epsilon_I$,

(ii) $\liminf_{t\to\infty}\frac{1}{t}\int_0^t R(u)\,du > \epsilon_R$,

(iii) $\liminf_{t\to\infty}\frac{1}{t}\int_0^t S(u)\,du > \epsilon_S$,

where

$$\epsilon_I = \frac{\displaystyle\int_0^\infty \frac{\partial f}{\partial y}(x,0)\,\nu^*(dx) - (\mu+\lambda+\frac{1}{2}\sigma^2)}{M_1+M_2+\lambda\dfrac{M_2}{\mu+\delta}} > 0, \qquad \epsilon_R = \frac{\lambda}{\mu+\delta}\,\epsilon_I > 0,$$

$$\epsilon_S = \frac{\mu+\lambda+\frac{1}{2}\sigma^2}{M_2}.$$

Proof. (i) By the definition of $X_1(\omega)$ and $X_2(\omega)$ in the proof of the previous theorem, and from hypotheses (H5) and (H6), we have

$$X_1(\omega)\leq M_1 \quad \text{and} \quad X_2(\omega)\leq M_2.$$

Then, part (i) of the corollary follows from equation (5.5).

(ii) From the R-equation, one can write

$$\frac{1}{t}\int_0^t R(u)\,du = \frac{R(0)-R(t)}{(\mu+\delta)t} + \frac{\lambda}{(\mu+\delta)t}\int_0^t I(u)\,du + \frac{\sigma}{t}\int_0^t R(u)\,dB(u).$$

From Theorem (3.1), we have

$$\lim_{t\to\infty}\frac{R(t)}{t} = \lim_{t\to\infty}\frac{1}{t}\int_0^t R(u)\,dB(u) = 0.$$

Thus,

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t R(u)\,du \geq \frac{\lambda}{\mu+\delta}\liminf_{t\to\infty}\frac{1}{t}\int_0^t I(u)\,du = \frac{\lambda}{\mu+\delta}\,\epsilon_I = \epsilon_R.$$

(iii) From the equation of I,

$$\begin{aligned} d\log I(t) &= -\Big(\mu+\lambda+\frac{1}{2}\sigma^2\Big)+\frac{f(S,I)}{I}+\sigma\, dB(t) \\ &\le -\Big(\mu+\lambda+\frac{1}{2}\sigma^2\Big)+\frac{\partial f}{\partial y}(S,0)+\sigma\, dB(t), \end{aligned}$$

where we used the fact that $I \mapsto \frac{f(S,I)}{I}$ is decreasing.
By the mean value theorem applied to the function $S \mapsto \frac{\partial f}{\partial y}(S,0)$ (which is C^1 by (H6)) and using $\frac{\partial f}{\partial y}(0,0)=0$ (which follows from (H1)),

$$\frac{\partial f}{\partial y}(S,0)=\frac{\partial f}{\partial y}(0,0)+S\frac{\partial^2 f}{\partial x\partial y}(C,0)\le M_2 S\le M_2(N-I),$$

where $C\in(0,S)$ and the last inequality uses $S\le N-I$ (since $R\ge 0$). Therefore,

$$d\log I(t)\le M_2 N-\Big(\mu+\lambda+\frac{1}{2}\sigma^2\Big)-M_2 I+\sigma\, dB(t).$$

By integrating, using the ergodic behaviour of N (the term $\int_0^t N(s)ds/t\to\Lambda/\mu$) and the strong law for Brownian motion, we obtain after standard calculations

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t I(u)\,du\le\frac{\Lambda}{\mu}-\frac{\mu+\lambda+\frac{1}{2}\sigma^2}{M_2}.$$

Finally,

$$\begin{aligned} \liminf_{t\to\infty}\frac{1}{t}\int_0^t S(u)\,du &\ge \liminf_{t\to\infty}\frac{1}{t}\int_0^t N(u)\,du-\limsup_{t\to\infty}\frac{1}{t}\int_0^t I(u)\,du \\ &\ge \frac{\Lambda}{\mu}-\Big(\frac{\Lambda}{\mu}-\frac{\mu+\lambda+\frac{1}{2}\sigma^2}{M_2}\Big) \\ &= \epsilon_S>0. \end{aligned}$$

This completes the proof of the corollary.

□

6. Numerical simulations

We present numerical experiments that confirm the theoretical threshold dynamics derived in Sections 4 and 5. The simulations illustrate the sharp dichotomy between disease extinction and persistence described by the stochastic basic reproduction number $\mathcal{R}_0^S$. All computations are performed for the bilinear incidence rate

$$f(S,I)=\beta SI,$$

which satisfies hypotheses (H1)–(H4) (and consequently (H5)–(H6) with $M_1=0$ and $M_2=\beta$). System (2.1) is approximated by Milstein's Higher Order method [9]. With initial condition $(S_0,I_0,R_0)=(20,2,0)$. Throughout the simulations we use the step size $\Delta t=0.01$ and a final time $T=50$. The demographic and disease-related parameters are fixed as follows:

$$\Lambda=5,\qquad \mu=0.2,\qquad \lambda=0.5,\qquad \delta=0.1,\qquad \sigma=0.1.$$

The transmission rate β will be chosen differently in the two scenarios to place $\mathcal{R}_0^S$ on either side of unity. For the extinction scenario we set $\beta = 0.01$. Using the stationary distribution ν^* of the total population (Theorem 3.1) and the fact that $\mathbb{E}_{\nu^*}[N] = \Lambda/\mu = 25$, we obtain

$$\int_0^\infty \frac{\partial f}{\partial y}(x,0)\,\nu^*(\mathrm{d}x) = \beta \int_0^\infty x\,\nu^*(\mathrm{d}x) = 0.01 \times 25 = 0.25.$$

The denominator is $\mu + \lambda + \frac{1}{2}\sigma^2 = 0.2 + 0.5 + 0.005 = 0.705$. Hence

$$\mathcal{R}_0^S = \frac{0.25}{0.705} \approx 0.3546 < 1.$$

According to Theorem 4.1 the infected and recovered compartments should vanish almost surely, while the susceptibles converge weakly to ν^*. Figure 1 displays the time evolution of $I(t)$ and $R(t)$. Both compartments vanish, as predicted by Theorem 4.1. The exponential decay is evident; the empirical decay rate, estimated from the slope after the initial transient, is approximately -0.25, consistent with the theoretical upper bound

$$\limsup_{t\to\infty} \frac{\log I(t)}{t} \le (\mu + \lambda + \tfrac{1}{2}\sigma^2)(\mathcal{R}_0^S - 1) \approx -0.455.$$

The numerical values of $I(t)$ at selected times are recorded in Table 1. After a small stochastic bump around $t = 3$, the infected population becomes negligible (below 10^{-2}) around $t = 30$, in agreement with the exponential decay proved in Theorem 4.1.

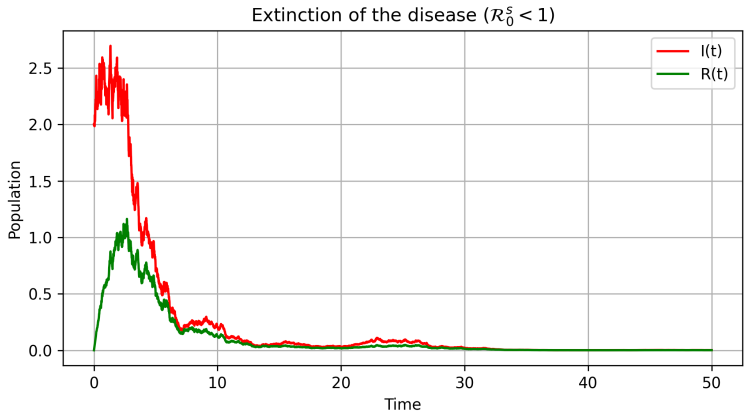

FIGURE 1. Extinction of the disease for $\beta = 0.01$ ($\mathcal{R}_0^S \approx 0.35$). Both the infected population $I(t)$ and the recovered population $R(t)$ decay exponentially to zero.

To illustrate persistence we take $\beta = 0.3$. Then

$$\int_0^\infty \frac{\partial f}{\partial y}(x,0)\,\nu^*(\mathrm{d}x) = 0.3 \times 25 = 7.5, \qquad \mathcal{R}_0^S = \frac{7.5}{0.705} \approx 10.64 > 1.$$

TABLE 1. Infected population $I(t)$ in the extinction scenario $\beta = 0.01$ ($\mathcal{R}_0^S \approx 0.35$). The recovered population $R(t)$ exhibits a similar exponential decay; both I and R become negligible around $t = 30$. Only a selection of times is shown.

t	$I(t)$	t	$I(t)$	t	$I(t)$	t	$I(t)$
0	2.00	10	0.25	20	0.01	30	0.00
2	2.30	12	0.10	22	0.015	40	0.00
4	1.50	14	0.11	24	0.016	50	0.00
6	0.90	16	0.12	26	0.005		
8	0.30	18	0.02	28	0.003		

Figure 2 shows the trajectories of $S(t)$, $I(t)$ and $R(t)$. The infected and recovered compartments do not die out; they fluctuate around positive endemic values. Long-time averages computed from the simulation yield

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t I(u)\,\mathrm{d}u \approx 2.8, \qquad \liminf_{t\to\infty} \frac{1}{t}\int_0^t R(u)\,\mathrm{d}u \approx 14.0,$$

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t S(u)\,\mathrm{d}u \approx 23.2.$$

These positive values confirm the persistence in mean established in Theorem 5.1. Corollary 5.2 additionally provides deterministic lower bounds for these averages, guaranteeing that they cannot approach zero. The empirical time averages are fully consistent with the theoretical prediction of persistence. The simulations

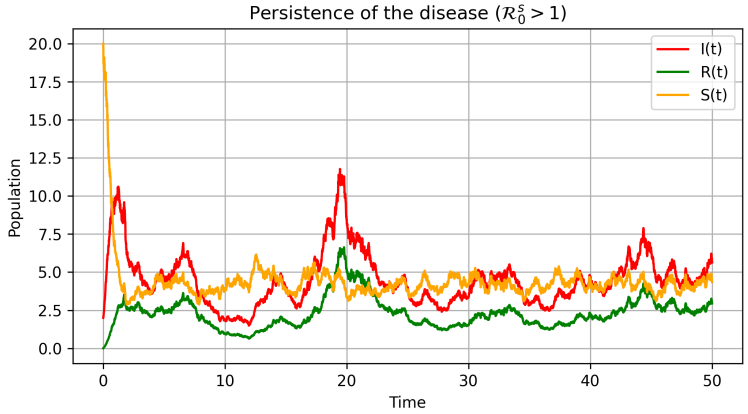

FIGURE 2. Persistence of the disease for $\beta = 0.3$ ($\mathcal{R}_0^S \approx 10.64$). All compartments persist; the infected and recovered populations fluctuate around stable positive levels.

fully support the theoretical threshold theorem:

- when $\mathcal{R}_0^S < 1$ the disease disappears exponentially fast and the recovered population vanishes.

- when $\mathcal{R}_0^S > 1$ the infection becomes endemic, with all compartments persisting in mean and long-time averages strictly bounded away from zero.

In both regimes the total population $N(t)$ fluctuates around the deterministic equilibrium $\Lambda/\mu = 25$. The numerical results thus provide a solid confirmation of the sharp analytic conditions derived in the paper and demonstrate the practical relevance of the stochastic threshold $\mathcal{R}_0^S$.

7. Discussion and Conclusion

We have analyzed a stochastic SIRS epidemic model with a general nonlinear incidence rate, where all compartments are subject to white noise. The main result is a sharp threshold theorem based on the quantity

$$\mathcal{R}_0^S = \frac{\int_0^\infty \frac{\partial f}{\partial y}(x,0)\,\nu^*(\mathrm{d}x)}{\mu + \lambda + \frac{1}{2}\sigma^2},$$

where ν^* is the stationary distribution of the total population $N(t)$.

- If $\mathcal{R}_0^S < 1$, the infected and recovered compartments vanish almost surely, and the susceptible population converges weakly to the disease-free stationary distribution ν^* (Theorem 4.1).
- If $\mathcal{R}_0^S > 1$, the disease persists in mean, and we gave explicit positive lower bounds for the time averages of S, I, and R (Theorem 5.1 and Corollary 5.2).

These results extend the classical deterministic dichotomy to the stochastic setting. The noise term $\frac{1}{2}\sigma^2$ in the denominator makes extinction more likely for larger noise intensities, reflecting that random fluctuations can break transmission even when the deterministic reproduction number would predict endemicity. Numerical simulations for the bilinear incidence $f(S,I) = \beta SI$ fully confirm the theoretical predictions. Future work could relax the hypotheses on f (e.g., non-monotone incidence), consider more realistic noise (Ornstein-Uhlenbeck, telegraph noise), include additional compartments (exposed, vaccinated, spatial structure), or calibrate the model to real epidemic data. In conclusion, this work provides a rigorous mathematical framework for understanding the interplay between nonlinear transmission, temporary immunity, and environmental stochasticity in infectious disease dynamics.

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