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ON SOME COHOMOLOGICAL INVARIANTS ASSOCIATED TO A REDUCED NORMAL CROSSING DIVISOR IN THE AFFINE PLANE

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Abstract. This work focuses on the study of cohomological invariants associated with a reduced normal crossing divisor in the affine plane. After determining the classical Poisson cohomology groups and the logarithmic Poisson cohomology groups along the associated ideal, we establish that, for a Poisson bivector attached to the log-symplectic structure under consideration, there exists an isomorphism of vector bundles between the module of logarithmic derivations and the module of logarithmic differential 1-forms relative to the ideal. This allows us to construct a cochain morphism related to the de Rham complex and subsequently compute the de Rham cohomology groups in order to carry out a comparison between the resulting cohomology groups.

Keywords. Poisson algebra, normal crossing divisor, Saito basis, Koszul bracket, cohomology.
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1. Introduction

Let $\mathbb{F}$ be a field of characteristic zero, and let $\mathcal{A}$ be an associative, commutative, unital $\mathbb{F}$-algebra. A Poisson structure on $\mathcal{A}$ is an $\mathbb{F}$-biderivation $\{.,.\}$ on $\mathcal{A}$ that is a Lie bracket. Any Poisson structure $\{.,.\}$ on $\mathcal{A}$ induces an $\mathcal{A}$-module homomorphism $H : \Omega_{\mathbb{F}}(\mathcal{A}) \to Der_{\mathbb{F}}(\mathcal{A})$ such that $d_{\mathbb{F}/\mathcal{A}}(a) \mapsto X_a := \{a, .\}$, where $\Omega_{\mathbb{F}}(\mathcal{A})$ denotes the $\mathcal{A}$-module of Kähler differentials of $\mathcal{A}$, $Der_{\mathbb{F}}(\mathcal{A})$ is the $\mathcal{A}$-module of derivations of $\mathcal{A}$, and $d_{\mathbb{F}/\mathcal{A}}$ is the canonical derivation of $\mathcal{A}$. From this map we obtain the following 2-form on $\Omega_{\mathbb{F}}(\mathcal{A})$, called the Poisson form of $\{.,.\}$ and denoted $\pi_{\{.,.\}}$, defined by $d_{\mathbb{F}}(a) \otimes d_{\mathbb{F}}(b) \mapsto \pi_{\{.,.\}} d_{\mathbb{F}}(a) \otimes d_{\mathbb{F}}(b) := \{a, b\}$. In general, an arbitrary element π of $Der_{\mathbb{F}}(\mathcal{A}) \wedge Der_{\mathbb{F}}(\mathcal{A})$ is a Poisson 2-form on $\mathcal{A}$ if and only if $[\pi, \pi] = 0$, where $[.,.]$ is the Schouten–Nijenhuis bracket of multiderivations of $\mathcal{A}$. One of the oldest Poisson structures on the algebra $C^\infty(\mathbb{R}^{2n})$ of smooth maps from $\mathbb{R}^{2n}$ to $\mathbb{R}$ was discovered in 1809 by the French mathematician Siméon Denis

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Poisson and is defined by

$$\{f,g\}=\sum_{i=1}^{n}\left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}-\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i}\right). \tag{1.1}$$

This Poisson structure plays a fundamental role in analytical mechanics. Lichnerowicz, A. (see [8]) and Weinstein, A. (see [17]) were the first researchers to extend it into a broad theory now known as Poisson geometry. However, this theory can be traced back to Lie, S. (see [9]). The Poisson bracket (1.1) is derived from a symplectic structure on $\mathbb{R}^{2n}$ and appears as one of the main ingredients of symplectic geometry. The literature shows that the notion of Poisson algebra goes back to Braconnier, J.'s work in [1] and to the work of Vinogradov, M. A. and Krasil'shchik, I. S. (see [16]).

One of the most important notions related to Poisson structures is Poisson cohomology, which was introduced by Lichnerowicz, A. (see [8]) and Vaisman, I. (see [15]) in the geometric framework and, in an algebraic setting, by Krasil'shchik, I. S. (see [7]) and Huebschmann, J. (see [4]). It was proved by Huebschmann, J. (see [4]) that Poisson cohomology has poor functorial properties. Poisson cohomology is difficult to compute due to the lack of general methods. Nevertheless, it is very interesting because it allows us to describe various properties of Poisson structures. One of the main applications of Poisson cohomology is geometric quantization. It was shown by Huebschmann, J. (see [4]) and Vaisman, I. (see [15]) that the obstruction to the prequantization of a Poisson structure is measured by the Poisson cohomology class of the associated Poisson 2-form. This provides a strong motivation for the explicit computation of the Poisson cohomology of a given Poisson structure.

As recalled above, the explicit calculation of Poisson cohomology remains a difficult problem in homological algebra. In certain cases, the Poisson structure contains elements that allow the construction of a new cochain complex, which can also be used in its prequantization. This is the case for log-symplectic and, more generally, logarithmic Poisson structures (see [3]).

In this paper, we prove that the second Poisson cohomology group and its logarithmic counterpart of the Poisson structure $\{x,y\}=xy$ on the Poisson algebra $(\mathbb{F}[x,y],\{.,.\})$ are different. However, its logarithmic Poisson cohomology and logarithmic de Rham cohomology are isomorphic. Consequently, we summarise the results of this paper in the following theorems.

Theorem 1.1. *The Poisson cohomology groups of $\{x,y\}=xy$ on $\mathcal{A}=\mathbb{F}[x,y]$, denoted H^*_{Pois}, are given by: $H^0_{Pois}\backsimeq\mathbb{F}$, $H^1_{Pois}\backsimeq x\mathbb{F}[y]\times y\mathbb{F}$ and $H^2_{Pois}\backsimeq\mathbb{F}\oplus xy\mathbb{F}$.*

Theorem 1.2. *The logarithmic Poisson cohomology groups of $\{x,y\}=xy$ along the ideal $xy\mathcal{A}$ on $\mathcal{A}=\mathbb{F}[x,y]$ satisfy $H^0_{LP}\backsimeq\mathbb{F}$, $H^1_{LP}\backsimeq\mathbb{F}\times\mathbb{F}$ and $H^2_{LP}\backsimeq\mathbb{F}$.*

Theorem 1.3. *The logarithmic de Rham cohomology groups of $\{x,y\}=xy$ on $\mathcal{A}=\mathbb{F}[x,y]$ along $xy\mathcal{A}$ are $H^0_{DR-log}\backsimeq\mathbb{F}$, $H^1_{DR-log}\backsimeq\mathbb{F}\times\mathbb{F}$ and $H^2_{DR-log}\backsimeq\mathbb{F}$.*

From these theorems we deduce the following corollary.

Corollary 1.4. $H^*_{LP}\backsimeq H^*_{DR-log}$.

The paper is structured as follows. After the first introductory part, Section 2 presents preliminary notions on logarithmic derivation fields, the Saito basis, the Hamiltonian map and the Poisson complex. Section 3 is devoted to the computation of the Poisson cohomology of $\{x, y\} = xy$ on $\mathcal{A} = \mathbb{F}[x, y]$. In Section 4, we compute the logarithmic Poisson cohomology of $\{x, y\} = xy$ on $\mathcal{A} = \mathbb{F}[x, y]$ along the ideal $\mathcal{I} = xy\mathcal{A}$. In Section 5, we compute the logarithmic de Rham cohomology of $\mathcal{A} = \mathbb{F}[x, y]$ along $\mathcal{I}$ and prove that it agrees with the corresponding logarithmic Poisson cohomology.

2. Preliminaries

Definition 2.1. [5] A Poisson algebra is an $\mathbb{F}$-vector space $\mathcal{A}$ equipped with two operations $(f, g) \mapsto f.g$ and $(f, g) \mapsto \{f, g\}$ such that

(i) $(\mathcal{A}, .)$ is a commutative associative algebra over $\mathbb{F}$ with unit 1;
(ii) $(\mathcal{A}, \{., .\})$ is a Lie algebra over $\mathbb{F}$;
(iii) The two operations are compatible in the sense that for all $a, b, c \in \mathcal{A}$,

$$\{a.b, c\} = a.\{b, c\} + b.\{a, c\}. \tag{2.1}$$

From the Leibniz rule (2.1), we deduce that for each $a \in \mathcal{A}$, the map $ad_a : \mathcal{A} \to \mathcal{A}$, $b \longmapsto \{a, b\}$ is a derivation on $\mathcal{A}$. Moreover, for all $a, b \in \mathcal{A}$ we have

$$ad_{a.b}(x) = a.\{b, x\} + b.\{a, x\} = a \cdot ad_b(x) + b \cdot ad_a(x).$$

The map $ad : \mathcal{A} \to Der_{\mathcal{A}}$, $a \longmapsto \{a, .\}$ is a derivation of $\mathcal{A}$ with values in the $\mathcal{A}$-module $Der_{\mathcal{A}}$. By the universal property of $(\Omega_{\mathcal{A}}, d)$, the derivation $ad : \mathcal{A} \to Der_{\mathcal{A}}$ induces an $\mathcal{A}$-module homomorphism called the Hamiltonian map, defined as

$$H : \Omega_{\mathcal{A}} \to Der_{\mathcal{A}}, \quad dx \longmapsto \mathcal{H}_x = \{x, .\}, \tag{2.2}$$

such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{d} & \Omega_{\mathcal{A}} \\ & \searrow^{ad} & \downarrow H \\ & & Der_{\mathcal{A}} \end{array} \tag{2.3}$$

Example 2.2. For $\{x, y\} = xy$ and $\mathcal{A} = \mathbb{F}[x, y]$, the Hamiltonian operator is $H : \Omega_{\mathcal{A}} \longrightarrow Der_{\mathcal{A}}$ with $df \mapsto xy\mathcal{H}_f$, where $\mathcal{H}_f = \partial_x f\, \partial_y - \partial_y f\, \partial_x \in Der_{\mathcal{A}}$.

Example 2.3. Let $\mathcal{A} = \mathbb{F}[x, y]$ and $\varphi \in \mathcal{A}$. Define $\{., .\}_\varphi$ by $\{f, g\}_\varphi = \varphi\mathcal{H}_f(g)$ for all $f, g \in \mathcal{A}$. Then $(\mathcal{A}, \{., .\}_\varphi)$ is a Poisson algebra.

Consider the equation $\varphi = 0$ defining a divisor (or hypersurface) on $\mathbb{F}^q$, $q > 0$.

Definition 2.4. [3] A divisor $D = \{\varphi = 0\}$ is said to have normal crossings at a point p if, locally at p, there exists a complex coordinate system $(x_1, \ldots, x_q)$ such that D is defined by the equation $x_1 \ldots x_n = 0$ for some $q \geqslant n \geqslant 0$.

Example 2.5. Let $(x, y) \in \mathbb{F}^2$. The divisor $D = \{xy = 0\}$ is a reduced normal crossing divisor on the affine plane $\mathbb{F}^2$.

Definition 2.6. [11] Let $(\mathcal{A},.)$ be an algebra. $Der_{\mathbb{F}}(\mathcal{A})$ is the $\mathcal{A}$-module of derivations of $\mathcal{A}$; that is, $\delta \in Der_{\mathbb{F}}(\mathcal{A})$ if δ is a linear map $\delta : \mathcal{A} \longrightarrow \mathcal{A}$ satisfying $\delta(a.b) = a.\delta(b) + \delta(a).b$ for all $a, b \in \mathcal{A}$.

Denote by $Der_{\mathcal{A}}$ the set of all derivations on $\mathcal{A} = \mathbb{F}[x_1, \ldots, x_q]$ and by $\Omega_{\mathcal{A}}$ its dual.

Definition 2.7. [3] A derivation δ of $\mathcal{A}$ is said to be logarithmic along $\mathcal{I}$ if $\delta(\mathcal{I}) \subset \mathcal{I}$. Denote by $Der_{\mathcal{A}}(\log \mathcal{I})$ the set of all derivations logarithmic along $\mathcal{I}$, and by $\Omega^1_{\mathcal{A}}(\log \mathcal{I})$ its dual, which is the $\mathcal{A}$-module of Kähler differential 1-forms logarithmic along $\mathcal{I}$.

Theorem 2.8. [14]

(i) $\Omega^1_{\mathcal{A}}(\log \mathcal{I})$ *is free if and only if* $\Omega^1_{\mathcal{A}}(\log \mathcal{I}) = \wedge^q \Omega^1_{\mathcal{A}}(\log \mathcal{I}) = \Omega^q_{\mathcal{A}}(\log \mathcal{I})$*; i.e., if there exist* q *elements* $\omega_1, \ldots, \omega_q$ *of* $\Omega^1_{\mathcal{A}}(\log \mathcal{I})$ *such that* $\omega_1 \wedge \cdots \wedge \omega_q = \frac{unit}{h} dx_1 \wedge \cdots \wedge dx_q$*. Then the set* $\{\omega_1, \ldots, \omega_q\}$ *forms a free basis for* $\Omega^1_{\mathcal{A}}(\log \mathcal{I})$*.*

(ii) $Der_{\mathcal{A}}(\log \mathcal{I})$ *is free if and only if there exist* $\delta^1, \ldots, \delta^q \in Der_{\mathcal{A}}$ *with* $\delta^i = \sum_{j=1}^q a_i^j(z)\partial_{z_j}$*,* $j = 1, \ldots, q$*, such that the determinant* $\det(a_i^j(z))_{i,j=1,\ldots,q}$ *is a unit multiple of* h*. Then the set of vector fields* $\{\delta^1, \ldots, \delta^q\}$ *is a free basis of* $Der_{\mathcal{A}}(\log \mathcal{I})$*.*

Definition 2.9. [3] The Poisson structure $\{x, y\} = \varphi$ is said to be logarithmic along the ideal $\mathcal{I} = \varphi\mathcal{A}$, if the divisor $D = \{\varphi = 0\}$ is logarithmic along $\mathcal{I}$.

Definition 2.10. [6] Let $\mathcal{A}$ be an $\mathbb{F}$-Poisson algebra and k an integer.

(i) A k-multiderivation of $\mathcal{A}$ is an alternating k-linear map f with values in $\mathcal{A}$ such that for each i, the map $x \mapsto f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_k)$ is a derivation. The $\mathbb{F}$-vector space of alternating k-derivations is denoted $\mathfrak{X}^k(\mathcal{A})$.

(ii) The Poisson coboundary operator $\delta^*_\pi : \mathfrak{X}^* \longrightarrow \mathfrak{X}^{*+1}$ is given by

$$\begin{aligned}\delta^q_\pi(Q)(x_0, \ldots, x_q) \;:=\; & \sum_{i=0}^{q} (-1)^i \{x_i, Q(x_0, \ldots, \hat{x}_i, \ldots, x_q)\} + \\ & \sum_{0 \le i \le j \le q} (-1)^{i+j} Q(\{x_i, x_j\}, x_0, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_q);\end{aligned}$$

for all $x_0, \ldots, x_q \in \mathcal{A}$ and $Q \in \mathfrak{X}^q(\mathcal{A})$, where $\delta^{q+1}_\pi \circ \delta^q_\pi = 0$ for $q \in \mathbb{N}$.

(iii) $(\mathfrak{X}^*, \delta^*_\pi)$ is the Poisson complex of the Poisson algebra $(\mathcal{A}, \{.,.\})$.

(iv) The Poisson cohomology of $(\mathcal{A}, \{.,.\})$ is defined by $H^*_{Pois}(\mathcal{A}, \{.,.\}) = H^*(\mathfrak{X}^*, \delta^*_\pi)$.

The k-th Poisson cohomology of $(\mathcal{A}, \{.,.\})$ is $H^k_{Pois}(\mathcal{A}, \{.,.\}) = H^k(\mathfrak{X}^*, \delta^*_\pi) = \dfrac{\ker \delta^{k+1}_\pi}{\operatorname{im} \delta^k_\pi}$.

3. Computation of the Poisson cohomology over $\mathcal{A}$ of $\{x, y\} = xy$

Let $\mathcal{A} = \mathbb{F}[x, y]$. The module of derivations $Der_{\mathcal{A}}$ is a free $\mathcal{A}$-module of rank 2, with a basis given by the partial derivatives ∂_x and ∂_y. We can therefore identify

the module $\wedge^2 Der_{\mathcal{A}}$ with $Der_{\mathcal{A}} \wedge Der_{\mathcal{A}}$. This allows us to establish a link between the complex from Definition 2.10 and that of Roger, C. and Vanhaecke, P. (2002) in the affine setting with which we are concerned.

Remark 3.1. Let $\mathcal{A} = \mathbb{F}[x, y]$. We have the following natural isomorphisms θ^i:
$\mathfrak{X}^0 = \wedge^0 Der_{\mathcal{A}} \backsimeq \mathcal{A}$, $\theta^0 := \mathrm{id}_{\mathcal{A}}$,
$\mathfrak{X}^1 = \wedge^1 Der_{\mathcal{A}} = Der_{\mathcal{A}} \backsimeq \mathcal{A} \times \mathcal{A}$ and $\theta^1 : Der_{\mathcal{A}} \longrightarrow \mathcal{A} \times \mathcal{A}$, $a_1\partial_x + a_2\partial_y \mapsto (a_1, a_2)$,
$\mathfrak{X}^2 = \wedge^2 Der_{\mathcal{A}} \backsimeq \mathcal{A}$ and $\theta^2 : \wedge^2 Der_{\mathcal{A}} \longrightarrow \mathcal{A}$, $a\partial_x \wedge \partial_y \mapsto a$,
$\mathfrak{X}^k = \wedge^k Der_{\mathcal{A}} = 0$ for all $k > 2$ (and for $k < 0$). Here $\theta^k = 0$.

Consider the bivector $\pi_0 = \{x, y\}\partial_x \wedge \partial_y$ with $\{x, y\} = xy$ over $\mathcal{A}$. According to Definition 2.10, we have $H^*_{Pois}(\mathcal{A}, \{., .\}) = H^*(\mathfrak{X}^*, \delta^*_\pi) = H^*_{Pois}(\mathfrak{X}^*_0, d^*_{\pi_0})$. This relation together with Remark 3.1 give rise to the following Poisson complex.

Lemma 3.2. *The Poisson complex of* $\{x, y\} = xy$ *over* $\mathcal{A} = \mathbb{F}[x, y]$ *is*

$$(\mathfrak{X}^*, d^*_{\pi_0}) : \ 0 \xrightarrow{d^0_{\pi_0}} \mathcal{A} \xrightarrow{d^1_{\pi_0}} \mathcal{A} \times \mathcal{A} \xrightarrow{d^2_{\pi_0}} \mathcal{A} \xrightarrow{d^3_{\pi_0}} 0$$

where $d^1_{\pi_0}(f) = xy(\partial_y f, -\partial_x f)$ *and* $d^2_{\pi_0}(f_1, f_2) = xy(\partial_x f_1 + \partial_y f_2) - yf_1 - xf_2$, *with* $d^0_{\pi_0} = d^3_{\pi_0} = 0$.

Proof. One verifies that $d^{i+1}_{\pi_0} \circ d^i_{\pi_0}(a) = 0$ for all $a \in \mathcal{A}$. This completes the proof. □

In the following subsections (3.1, 3.2 and 3.3), we will simply denote $d^*_{\pi_0} := d^*$.

3.1. Module of 0-Poisson cohomology over $\mathcal{A}$ of $\{x, y\} = xy$.

Proposition 3.3. $H^0_{Pois} \backsimeq \mathbb{F}$.

Proof. Let $f \in \mathcal{A}$. Then $f \in \ker d^1$ if and only if $\partial_y f = \partial_x f = 0$, which implies $f \in \mathbb{F}$. Hence $\ker d^1 \backsimeq \mathbb{F}$. Since $\operatorname{im} d^0 \backsimeq 0$, we obtain $H^0_{Pois} \backsimeq \mathbb{F}$. □

3.2. Module of 1-Poisson cohomology over $\mathcal{A}$ of $\{x, y\} = xy$.

From Proposition 3.2 we have $d^1(\mathcal{A}) \subset (xy\mathcal{A} \times xy\mathcal{A}) \cap \ker d^2$. Let's consider $(xyu, xyv) \in (xy\mathcal{A} \times xy\mathcal{A}) \cap \ker d^2$. Then $(xyu, xyv) \in \ker d^2$ is equivalent to

$$\begin{aligned} 0 &= xy(\partial_x(xyu) + \partial_y(xyv)) - xy^2u - x^2yv \\ &= xy(yu + xy\partial_x u + xv + xy\partial_y v) - xy^2u - x^2yv \\ &= xy^2u + x^2yv + x^2y^2\partial_x u + x^2y^2\partial_y v - xy^2u - x^2yv \\ &= x^2y^2(\partial_x u + \partial_y v). \end{aligned}$$

Hence $\partial_x u + \partial_y v = 0$. For any $f \in \mathcal{A}$, we consider the antiderivative $\int f\,dx$ as the inversion operator of ∂_x in $\mathbb{F}[x]$, with the integration constant in $\mathbb{F}[y]$ (and symmetrically). In other words, $\int f\,dx$ is the primitive of f with constant equal to the zero polynomial. For example, if $f = \sum_{i,j\geq 0} f_{ij}x^i y^j$, then we have $\int f\,dx = \sum_{i,j\geq 0} \frac{1}{i+1} f_{ij}x^{i+1}y^j$ and $\int f\,dy = \sum_{i,j\geq 0} f_{ij}x^i y^{j+1}$. Therefore, $\partial_x u + \partial_y v = 0$ implies $v = -\int \partial_x u\,dy + g(x)$ and then

$$(xyu, xyv) = (xyu, -xy\int \partial_x u\,dy) + (0, xyg(x)).$$

Consequently,

$$d^1(\mathcal{A}) \subseteq \ker d^2 \cap \left(\{(xyu, -xy \int \frac{\partial u}{\partial x} dy) \mid u \in \mathcal{A}\} \oplus (0 \times xy\mathbb{F}[x]) \right). \tag{3.1}$$

Lemma 3.4. *Let K, B, C be submodules of $\mathcal{A}$. If $B \subseteq K$ and $C \subseteq K$, then*

$$K \cap (B \oplus C) = (K \cap B) \oplus (K \cap C).$$

Using Lemma 3.4, we prove the following lemmas to express $d^1(\mathcal{A})$ explicitly.

Lemma 3.5.

$$d^1(\mathcal{A}) \subseteq \ker d^2 \cap \{(xyu, -xy \int \frac{\partial u}{\partial x} dy) \mid u \in \mathcal{A}\} \oplus \ker d^2 \cap (0 \times xy\mathbb{F}[x]).$$

Proof. Let $(a, b) = (xyu, -xy \int \partial_x u \, dy)$. Then

$$\frac{\partial(xyu)}{\partial x} - \frac{\partial(xy \int \partial_x u \, dy)}{\partial y} = yu + xy\frac{\partial u}{\partial x} - x \int \partial_x u \, dy - xy\frac{\partial u}{\partial x} = yu - x \int \partial_x u \, dy.$$

It follows that $d^2(a, b) = 0$ and hence

$$\ker d^2 \cap \{(xyu, -xy \int \frac{\partial u}{\partial x} dy)\}_{u \in \mathcal{A}} = \{(xyu, -xy \int \frac{\partial u}{\partial x} dy)\}_{u \in \mathcal{A}}.$$

Furthermore, for all $f \in \mathbb{F}[x]$, one verifies that $d^2(0, xyf) = 0$, therefore we get $\ker d^2 \cap (0 \times xy\mathbb{F}[x]) = 0 \times xy\mathbb{F}[x]$. Thus $\{(xyu, -xy \int \frac{\partial u}{\partial x} dy)\}_{u \in \mathcal{A}}$ and $0 \times xy\mathbb{F}[x]$ are submodules of $\ker d^2$. From (3.1) and Lemma 3.4 we obtain the required inclusion. □

Consequently,

$$d^1(\mathcal{A}) \subseteq \{(xyu, -xy \int \frac{\partial u}{\partial x} dy)\}_{u \in \mathcal{A}} \oplus (0 \times xy\mathbb{F}[x]),$$

and we obtain the explicit expression of the 1-coboundary $d^1(\mathcal{A})$ as follows.

Lemma 3.6.

$$d^1(\mathcal{A}) \backsimeq \{xy(u, -\int \frac{\partial u}{\partial x} dy) \mid u \in \mathcal{A}\} \oplus 0 \times xy\mathbb{F}[x].$$

Proof. By Lemma 3.4, it suffices to show both inclusions $0 \times xy\mathbb{F}[x] \subseteq d^1(\mathcal{A})$ and $\{xy(u, -\int \frac{\partial u}{\partial x} dy)\}_{u \in \mathcal{A}} \subseteq d^1(\mathcal{A})$. For $u \in \mathbb{F}[x]$, we have $(0, xyu(x)) \in d^1(\mathcal{A})$ if and only if there exists $f \in \mathcal{A}$ such that $\partial_y f = 0$ and $-\partial_x f = u$. The condition $\partial_y f = 0$ implies $f \in \mathbb{F}[x]$, and then $-\partial_x f = u$ gives $f = -\int u(x)dx$. Hence $d^1(\mathcal{A}) \cap (0 \times xy\mathbb{F}[x]) = (0 \times xy\mathbb{F}[x])$. On the other hand, $xy(u, -\int \frac{\partial u}{\partial x} dy) \in d^1(\mathcal{A})$ if and only if there exists $f \in \mathcal{A}$ such that $xy(u, -\int \frac{\partial u}{\partial x} dy) = (xy\partial_y f, -xy\partial_x f)$. This implies $u = \partial_y f$ and $\int \frac{\partial u}{\partial x} dy = \partial_y f$, i.e., $f = \int u \, dy$ and $\partial_x f = \int \frac{\partial u}{\partial x} dy$. Therefore

$$d^1(\mathcal{A}) \subseteq \{xy(u, -\int \frac{\partial u}{\partial x} dy)\}_{u \in \mathcal{A}} \oplus (0 \times xy\mathbb{F}[x]) \subseteq d^1(\mathcal{A}),$$

which completes the proof. □

Lemma 3.7.

$$\ker d^2 \subseteq \{(xy\int f dx, -yx\int f dy) \mid f \in \mathcal{A}\} \oplus x\mathbb{F}[y] \times y\mathbb{F}[x].$$

Proof. Let $(f_1, f_2) \in \mathcal{A} \times \mathcal{A}$. Then $(f_1, f_2) \in \ker d^2$ if and only if

$$x\big[y(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}) - f_2\big] = yf_1 = yxu.$$

In other words, $y(x\frac{\partial u}{\partial x} + \frac{\partial f_2}{\partial y}) = yv$. This relation yields $x\frac{\partial u}{\partial x} + y\frac{\partial v}{\partial y} = 0$. Thus

$$\ker d^2 = \{(xu, yv) \mid x\frac{\partial u}{\partial x} + y\frac{\partial v}{\partial y} = 0\}.$$

From $x\frac{\partial u}{\partial x} + y\frac{\partial v}{\partial y} = 0$ there exists $f \in \mathcal{A}$ such that $\frac{\partial u}{\partial x} = yf$, whence $xf = -\frac{\partial v}{\partial y}$. Consequently, there exists $g \in \mathbb{F}[x]$ such that $v = -x\int f\, dy + g$. Since $\frac{\partial u}{\partial x} = yf$, there exists $h \in \mathbb{F}[y]$ such that $u = y\int f\, dx + h$. Therefore

$$\ker d^2 \subseteq \{(xy\int f dx, -yx\int f dy) \mid f \in \mathcal{A}\} \oplus x\mathbb{F}[y] \times y\mathbb{F}[x].$$

□

Lemma 3.8.

$$\ker d^2 \backsimeq \{(xy\int f dx, -yx\int f dy) \mid f \in \mathcal{A}\} \oplus 0 \times xy\mathbb{F}[x] \oplus x\mathbb{F}[y] \times 0 \oplus 0 \times y\mathbb{F}.$$

Proof. We have the canonical isomorphisms

$$x\mathbb{F}[y] \times y\mathbb{F}[x] = (x\mathbb{F}[y] \times y\mathbb{F}) \oplus (0 \times xy\mathbb{F}[x]) = (0 \times xy\mathbb{F}[x]) \oplus (x\mathbb{F}[y] \times 0) \oplus (0 \times y\mathbb{F}).$$

By Lemma 3.7, $\ker d^2$ is contained in the direct sum of the three terms. Conversely, take any element of the form $(xy\int f dx + xa(y), -yx\int f dy + yb(x))$. A direct computation shows d^2 vanishes on it, giving the reverse inclusion. Hence

$$\ker d^2 = \{(xy\int f dx, -yx\int f dy)\}_{f\in\mathcal{A}} \oplus x\mathbb{F}[y] \times y\mathbb{F}[x].$$

□

Proposition 3.9. $H^1_{Pois} \backsimeq x\mathbb{F}[y] \times 0 \oplus 0 \times y\mathbb{F} \backsimeq x\mathbb{F}[y] \times y\mathbb{F}$.

Proof. Recall the classical isomorphisms

$$x\mathbb{F}[y] \times y\mathbb{F}[x] \backsimeq x\mathbb{F}[y] \times 0 \oplus 0 \times y\mathbb{F}[x] \backsimeq x\mathbb{F}[y] \times 0 \oplus 0 \times y\mathbb{F} \oplus 0 \times xy\mathbb{F}[x].$$

Using these to compute the 2-cocycle, we obtain

$$\ker d^2 \backsimeq d^1(\mathcal{A}) \oplus x\mathbb{F}[y] \times 0 \oplus 0 \times y\mathbb{F}.$$

Thus $H^1_{Pois} \backsimeq x\mathbb{F}[y] \times 0 \oplus 0 \times y\mathbb{F}$. □

3.3. The module of 2-Poisson cohomology of $\{x, y\} = xy$ over $\mathcal{A}$.

Lemma 3.10. *The 2-coboundary associated to the Poisson structure $\{x, y\} = xy$ is given by $\operatorname{im} d^2 = y\mathbb{F}[y] \oplus x\mathbb{F}[x] \oplus (x^2y\mathcal{A} + xy^2\mathcal{A})$.*

Proof. From the cochain complex (3.2), for all $f^1, f^2 \in \mathcal{A}$,

$$d^2(f^1, f^2) = y(x\partial_x f^1 - f^1) + x(y\partial_y f^2 - f^2) \in y\mathcal{A} + x\mathcal{A}.$$

Thus $\operatorname{im} d^2 = \operatorname{im} d^2 \cap (y\mathcal{A} + x\mathcal{A})$. Let $\psi \in \operatorname{im} d^2$ be written as $\psi = y\psi^1 + x\psi^2$ with $\psi^i \in \mathcal{A}$. Then $d^2(f^1, f^2) = \psi$ implies

$$y(x\partial_x f^1 - f^1 - \psi^1) + x(y\partial_y f^2 - f^2 - \psi^2) = 0,$$

which leads to the following

$$\psi^1 = \sum_{i,j\geq 0} (i-1) f^1_{ij} x^i y^j, \qquad \psi^2 = \sum_{i,j\geq 0} (j-1) f^2_{ij} x^i y^j.$$

Hence $\psi_{11} = 0$. Therefore

$$\operatorname{im} d^2 \cap (y\mathcal{A} + x\mathcal{A}) = y \bigoplus_{\substack{i,j\geq 0 \\ i\neq 1}} (i-1) f^1_{ij} x^i y^j + x \bigoplus_{\substack{i,j\geq 0 \\ j\neq 1}} (j-1) f^2_{ij} x^i y^j,$$

which is isomorphic to $y\mathbb{F}[y] \oplus x\mathbb{F}[x] \oplus (x^2y\mathcal{A} + xy^2\mathcal{A})$. □

Corollary 3.11. *The Poisson structure $\{x, y\} = xy$ is cohomological, i.e., $\{x, y\} \notin \operatorname{im} d^2$.*

Proof. Every $\psi \in \operatorname{im} d^2$ has $\psi_{11} = 0$, whereas $\{x, y\} = xy$ has a non-zero xy term. Hence $xy \notin \operatorname{im} d^2$. □

From the above corollary we deduce $y\mathcal{A} + x\mathcal{A} = \operatorname{im} d^2 \oplus xy\mathbb{F}$, and we can compute the 2-Poisson cohomology.

Proposition 3.12. *The 2-Poisson cohomology module of $\{x, y\} = xy$ over $\mathcal{A}$ is $H^2_{Pois} \backsimeq \mathbb{F} \oplus xy\mathbb{F}$.*

Proof. We have $x\mathcal{A}+y\mathcal{A} = \operatorname{im} d^2 \oplus xy\mathbb{F}$ and $\mathcal{A} \backsimeq (x\mathcal{A}+y\mathcal{A}) \oplus \mathbb{F}$. Since $\ker d^3 = \mathcal{A}$, we obtain $\ker d^3 \backsimeq \operatorname{im} d^2 \oplus (\mathbb{F} \oplus xy\mathbb{F})$, and therefore $H^2_{Pois} \backsimeq \mathbb{F} \oplus xy\mathbb{F}$. □

The three propositions above conclude the proof of the following theorem.

Theorem 3.13. *The Poisson cohomology groups of $\{x, y\} = xy$ on $\mathcal{A} = \mathbb{F}[x, y]$, denoted H^*_{Pois}, are given by: $H^0_{Pois} \backsimeq \mathbb{F}$, $H^1_{Pois} \backsimeq x\mathbb{F}[y] \times y\mathbb{F}$, $H^2_{Pois} \backsimeq \mathbb{F} \oplus xy\mathbb{F}$, and $H^k_{Pois} = 0$ for all $k > 2$.*

4. On logarithmic Poisson cohomology of $(\mathcal{A}, \{x, y\} = xy)$ along $\mathcal{I} = xy\mathcal{A}$

Lemma 4.1. *Let $\mathcal{B} = (\delta^1 = x\partial_x, \delta^2 = y\partial_y)$ and $\mathcal{B}^* = (\omega_1 = \dfrac{dx}{x}, \omega_2 = \dfrac{dy}{y})$. Then $\mathcal{B}$ and $\mathcal{B}^*$ are respectively the dual Saito basis of $Der_{\mathcal{A}}(\log \mathcal{I})$ and $\Omega^1_{\mathcal{A}}(\log \mathcal{I})$.*

Proof. According to Theorem 2.8, we have $\delta^1(h) = \delta^2(h) = h \in (h)$ and $\det(\delta^1, \delta^2) = h$, which is a unit multiple of h. Hence $\mathcal{B}$ is the free Saito basis of $Der_{\mathcal{A}}(\log \mathcal{I})$. Moreover, $\omega_1 \wedge \omega_2 = \frac{1}{h} dx \wedge dy$, so $\mathcal{B}^*$ is a Saito basis of $\Omega^1_{\mathcal{A}}(\log \mathcal{I})$. Finally, $\delta^i \lrcorner \omega_j = \delta_{ij}$ (Kronecker symbol), so $\mathcal{B}^*$ is the dual of $\mathcal{B}$. □

According to Definition 2.10, take $x_0 = x$, $x_1 = y$, $q = 1$ and $Q = \tilde{H}$, where $\tilde{H} : \Omega^1_{\mathcal{A}}(\log \mathcal{I}) \longrightarrow Der_{\mathcal{A}}(\log \mathcal{I})$ is the Hamiltonian operator defined for all $\omega = a\omega_1 + b\omega_2 \in \Omega^1_{\mathcal{A}}(\log \mathcal{I})$ by $\tilde{H}(\omega) = a\delta^2 - b\delta^1$. Recall that the Koszul bracket of two 1-forms satisfies $[dx, dy] = d\{x, y\}$ and $[\frac{dx}{x}, \frac{dy}{y}] = d(\frac{1}{xy}\{x, y\})$. From these relations and Definition 2.10, the logarithmic Poisson complex $(\wedge^* Der_{\mathcal{A}}(\log \mathcal{I}), d^*_{\tilde{H}})$ becomes

$$0 \xrightarrow{d^0_{\tilde{H}}} \mathcal{A} \xrightarrow{d^1_{\tilde{H}}} Der_{\mathcal{A}}(\log \mathcal{I}) \xrightarrow{d^2_{\tilde{H}}} \wedge^2 Der_{\mathcal{A}}(\log \mathcal{I}) \xrightarrow{d^3_{\tilde{H}}} 0. \tag{4.1}$$

Remark 4.2. We have the following natural isomorphisms:
$\wedge^0 Der_{\mathcal{A}}(\log \mathcal{I}) \backsimeq \mathcal{A}$, $Der_{\mathcal{A}}(\log \mathcal{I}) \backsimeq \mathcal{A} \times \mathcal{A}$, $\wedge^2 Der_{\mathcal{A}}(\log \mathcal{I}) \backsimeq \mathcal{A}$, and for $k > 2$ (or $k < 0$), $\wedge^k Der_{\mathcal{A}}(\log \mathcal{I}) = 0$.

Hence the complex (4.1) is equivalent to the following.

Lemma 4.3. *Let $\mathcal{A} = \mathbb{F}[x, y]$, $\mathcal{I} = xy\mathcal{A}$ and $\tilde{H}$ the logarithmic Hamiltonian map. The logarithmic Poisson complex is*

$$0 \xrightarrow{d^0_{\tilde{H}}} \mathcal{A} \xrightarrow{d^1_{\tilde{H}}} \mathcal{A}^2 \xrightarrow{d^2_{\tilde{H}}} \mathcal{A} \xrightarrow{d^3_{\tilde{H}}} 0$$

where $d^0_{\tilde{H}} = d^3_{\tilde{H}} = 0$, $d^1_{\tilde{H}}(f) = (\delta^2 f, -\delta^1 f)$ and $d^2_{\tilde{H}}(f_1, f_2) = \delta^1 f_1 + \delta^2 f_2$.

One verifies that $d^{k+1}_{\tilde{H}} \circ d^k_{\tilde{H}} = 0$, so $\operatorname{im} d^k_{\tilde{H}} \subset \ker d^{k+1}_{\tilde{H}}$. The corresponding cohomology is the Lichnerowicz-Poisson cohomology, called logarithmic Poisson cohomology and denoted H^*_{LP}, with $H^k_{LP} = \ker d^{k+1}_{\tilde{H}} / \operatorname{im} d^k_{\tilde{H}}$.

Proposition 4.4. $H^0_{LP} \backsimeq \mathbb{F}$.

Proof. For $f \in \mathcal{A}$, $f \in \ker d^1_{\tilde{H}}$ iff $\delta^2 f = \delta^1 f = 0$, i.e., $y\partial_y f = x\partial_x f = 0$, which implies $f \in \mathbb{F}$. Hence $\ker d^1_{\tilde{H}} \backsimeq \mathbb{F}$ and $\operatorname{im} d^0_{\tilde{H}} \backsimeq 0$, so $H^0_{LP} \backsimeq \mathbb{F}$. □

Proposition 4.5. $H^1_{LP} \backsimeq \mathbb{F} \times \mathbb{F}$.

Proof. Let $(\varphi_1, \varphi_2) \in (\mathcal{A} \times \mathcal{A}) \cap \ker d^2_{\tilde{H}}$, i.e., $\delta^1\varphi_1 + \delta^2\varphi_2 = 0$. Then there exist $u \in \mathcal{A}$, $a \in \mathbb{F}[x]$, $b \in \mathbb{F}[y]$ such that $\varphi_2 = x \int u\, dy + a(x)$ and $\varphi_1 = -y \int u\, dx + b(y)$. Conversely, for any $\vec{\varphi} = (p + b(y), q + a(x))$ with (p, q) in the image of the map $\theta : \mathcal{A} \to \mathcal{A} \times \mathcal{A}$, $u \mapsto (-y \int u\, dx,\ x \int u\, dy)$, we have $\delta^1(p + b(y)) + \delta^2(q + a(x)) = 0$. Hence

$$\ker d^2_{\tilde{H}} \backsimeq \big(\operatorname{im} \theta \oplus y\mathbb{F}[y] \times x\mathbb{F}[x]\big) \oplus \mathbb{F} \times \mathbb{F}.$$

Since $\operatorname{im} \theta \oplus y\mathbb{F}[y] \times x\mathbb{F}[x] \backsimeq \operatorname{im} d^1_{\tilde{H}}$, we obtain $H^1_{LP} \backsimeq \mathbb{F} \times \mathbb{F}$. □

Proposition 4.6. $H^2_{LP} \backsimeq \mathbb{F}$.

Proof. Let $\psi \in \operatorname{im} d^2_{\tilde{H}}$. Then there exists $(\varphi_1, \varphi_2) \in \mathcal{A} \times \mathcal{A}$ such that $\psi = d^2_{\tilde{H}}(\varphi_1, \varphi_2) = \delta^1\varphi_1 + \delta^2\varphi_2$, which implies $\psi = xa + yb$ for some $a, b \in \mathcal{A}$. Hence $x(\partial_x\varphi_1 - a) + y(\partial_y\varphi_2 - b) = 0$, so $\partial_x\varphi_1 = a$, $\partial_y\varphi_2 = b$. Thus $\varphi_1 = \int a\, dx + a_1(y)$,

$\varphi_2 = \int b\,dy + b_1(x)$. Consequently, $\operatorname{im} d^2_{\tilde{H}} \backsimeq x\mathcal{A} + y\mathcal{A}$. Since $\mathcal{A} \backsimeq \mathbb{F} \oplus (x\mathcal{A} + y\mathcal{A})$ and $\ker d^3_{\tilde{H}} \backsimeq \mathcal{A}$, we get $H^2_{LP} \backsimeq \mathbb{F}$. □

Hence the three propositions above complete the proof of the following theorem.

Theorem 4.7. *The logarithmic Poisson cohomology groups of $\{x, y\} = xy$ along the ideal $xy\mathcal{A}$ on $\mathcal{A} = \mathbb{F}[x, y]$ satisfy $H^0_{LP} \backsimeq \mathbb{F}$, $H^1_{LP} \backsimeq \mathbb{F} \times \mathbb{F}$, $H^2_{LP} \backsimeq \mathbb{F}$, and $H^k_{LP} = 0$ for $k > 2$.*

5. On logarithmic de Rham cohomology of $\{x, y\} = xy$ along $\mathcal{I}$

$\mathcal{I} = xy\mathcal{A}$ is a principal ideal of $\mathcal{A}$. According to Saito, K. (see [14]), in dimension two the sheaves $\Omega^1_{\mathcal{A}}(\log \mathcal{I})$ and $Der_{\mathcal{A}}(\log \mathcal{I})$ are reflexive and locally free $\mathcal{O}_{\mathcal{A}}$-modules for any ideal $\mathcal{I}$ of $\mathcal{A}$. Therefore, by the natural isomorphisms in Remark 4.2, we obtain the following diagram:

$$\begin{array}{ccccccccc} 0 & \xrightarrow{d^0} & \mathcal{A} & \xrightarrow{d^1} & \mathcal{A}\times\mathcal{A} & \xrightarrow{d^2} & \mathcal{A} & \xrightarrow{d^3} & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{A} & \longrightarrow & \Omega^1_{\mathcal{A}}(\log \mathcal{I}) & \longrightarrow & \Omega^2_{\mathcal{A}}(\log \mathcal{I}) & \longrightarrow & 0 \end{array}$$

According to Vaisman, I. (see [15]), for each symplectic Poisson structure π, the sharp map $\sharp : \Omega^k_{\mathcal{A}} \to \wedge^k Der_{\mathcal{A}}$ defined by $\alpha^\sharp(\beta) = \pi(\alpha \otimes \beta)$ and, for a k-form $\omega \in \Omega^k_{\mathcal{A}}$, $\omega^\sharp(\alpha_1, \ldots, \alpha_k) = (-1)^k \omega(\alpha_1^\sharp, \ldots, \alpha_k^\sharp)$, is a cochain complex homomorphism. In the present case, $\pi = xy\partial_x \wedge \partial_y$ arises from the log-symplectic structure. Therefore we need the following lemma.

Lemma 5.1.
Let $\mathcal{A} = \mathbb{F}[x, y]$ and $\mathcal{I} = xy\mathcal{A}$. The map $\pi^\sharp : \Omega^1_{\mathcal{A}}(\log \mathcal{I}) \longrightarrow Der_{\mathcal{A}}(\log \mathcal{I})$ defined by $\alpha \mapsto i_\alpha \pi = \pi(\alpha, \cdot)$ is an isomorphism of logarithmic vector bundles.

Proof. Extend $\sharp$ to $\Omega^1_{\mathcal{A}}(\log \mathcal{I})$ by $(\frac{dx}{x})^\sharp = i_{\frac{dx}{x}}(xy\partial_x \wedge \partial_y) = y\partial_y$ and $(\frac{dy}{y})^\sharp = i_{\frac{dy}{y}}(xy\partial_x \wedge \partial_y) = -x\partial_x$. The matrix of $\pi^\sharp$ is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, with determinant 1, hence invertible over $\mathcal{A}$. Explicitly, for $\alpha = \alpha_1 \dfrac{dx}{x} + \alpha_2 \dfrac{dy}{y}$ $(\alpha_i \in \mathcal{A})$, $\pi^\sharp(\alpha) = \alpha_1 y\partial_y - \alpha_2 x\partial_x$. Let $\mathcal{X} = ax\partial_x + by\partial_y$ be an arbitrary logarithmic vector field. Solving $\pi^\sharp(\alpha) = \mathcal{X}$ gives $\alpha_2 = -a$, $\alpha_1 = b$, so $\alpha = b\dfrac{dx}{x} - a\dfrac{dy}{y}$. The inverse map $\omega^\flat : Der_{\mathcal{A}}(\log \mathcal{I}) \longrightarrow \Omega^1_{\mathcal{A}}(\log \mathcal{I})$ is therefore $\omega^\flat(ax\partial_x + by\partial_y) = b\dfrac{dx}{x} - a\dfrac{dy}{y}$. Thus $\pi^\sharp$ is an isomorphism of free $\mathcal{A}$-modules, i.e., of logarithmic vector bundles. □

It follows that the logarithmic Poisson structure induces a cohomology homomorphism $[\alpha]^\sharp = [\alpha^\sharp]$ from the logarithmic de Rham complex to the Poisson

counterpart. In summary, we have the following complex morphism $(\sharp^k)_{k=0,1,2}$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{A} & \longrightarrow & \Omega^1_{\mathcal{A}}(\log \mathcal{I}) & \longrightarrow & \Omega^2_{\mathcal{A}}(\log \mathcal{I}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \sharp^0} & & \downarrow{\scriptstyle \sharp^1} & & \downarrow{\scriptstyle \sharp^2} & & \\ 0 & \longrightarrow & \mathcal{A} & \longrightarrow & Der_{\mathcal{A}}(\log \mathcal{I}) & \longrightarrow & \wedge^2 Der_{\mathcal{A}}(\log \mathcal{I}) & \longrightarrow & 0 \end{array}$$

where $\sharp^0(a) = a$, $\sharp^1(a\frac{dx}{x} + b\frac{dy}{y}) = ay\partial_y - bx\partial_x$, and $\sharp^2(\psi\frac{dx\wedge dy}{xy}) = -\psi xy\partial_x \wedge \partial_y$. From the definition of $\sharp$ and the facts above, we have the following lemma.

Lemma 5.2. *Let $\mathcal{A} = \mathbb{F}[x,y]$ and $\mathcal{I} = xy\mathcal{A}$. The logarithmic de Rham complex $(\Omega^*_{\mathcal{A}}(\log \mathcal{I}), \partial^*)$ is given by*

$$0 \xrightarrow{\partial^0} \mathcal{A} \xrightarrow{\partial^1} (\mathcal{A}\frac{dx}{x} \oplus \mathcal{A}\frac{dy}{y}) \xrightarrow{\partial^2} \mathcal{A}\frac{dx\wedge dy}{xy} \xrightarrow{\partial^3} 0$$

where the logarithmic de Rham differentials $(\partial^k)_{k=0}^3$ are respectively

$$\partial^0 = \partial^3 = 0, \quad \partial^1(f) = x\partial_x f\frac{dx}{x} + y\partial_y f\frac{dy}{y}, \quad \partial^2(a\frac{dx}{x} + b\frac{dy}{y}) = (y\partial_y a - x\partial_x b)\frac{dx\wedge dy}{xy}.$$

The associated cohomological invariants $H^k_{DR-log} = \ker \partial^{k+1} / \operatorname{im} \partial^k$ are given in the following propositions.

Proposition 5.3. $H^0_{DR-log} \backsimeq \mathbb{F}$.

Proof. $f \in \ker \partial^1$ iff $x\partial_x f = y\partial_y f = 0$, so $f \in \mathbb{F}$. Hence $\ker \partial^1 \backsimeq \mathbb{F}$ and $\operatorname{im} \partial^0 \backsimeq 0$, giving $H^0_{DR-log} \backsimeq \mathbb{F}$. □

Proposition 5.4. $H^1_{DR-log} \backsimeq \mathbb{F} \times \mathbb{F}$.

Proof. Write $\partial^1(a) = a_1\frac{dx}{x} + a_2\frac{dy}{y}$. Then $a_1\frac{dx}{x} + a_2\frac{dy}{y} \in \operatorname{im} \partial^1$ iff there exist $a_0, b_0 \in \mathcal{A}$ such that $\frac{\partial a_0}{\partial y} = \frac{\partial b_0}{\partial x}$ and $a_1 = xa_0$, $a_2 = yb_0$. Now let $(f,g) \in \ker \partial^2$; then $f = xa$, $g = yb$ with $\frac{\partial a}{\partial y} = \frac{\partial b}{\partial x}$. Thus $\ker \partial^2 \backsimeq \operatorname{im} \partial^1 \oplus \mathbb{F} \times \mathbb{F}$, and the result follows. □

Proposition 5.5. $H^2_{DR-log} \backsimeq \mathbb{F}$.

Proof. For $f, g \in \mathcal{A}$, $\partial^2(f,g) = y\partial_y f - x\partial_x g \in y\mathcal{A} + x\mathcal{A}$. Conversely, given $\varphi = yb + xa \in \operatorname{im} \partial^1 \cap (y\mathcal{A} + x\mathcal{A})$, we can choose $g = -\int a\,dx + u(y)$ and $f = \int b\,dy + v(x)$ to obtain $\partial^2(f,g) = \varphi$. Hence $\operatorname{im} \partial^2 \backsimeq x\mathcal{A} + y\mathcal{A}$. Since $\ker \partial^3 = \mathcal{A}$ and $\mathcal{A} \backsimeq \mathbb{F} \oplus (x\mathcal{A} + y\mathcal{A})$, we get $H^2_{DR-log} \backsimeq \mathbb{F}$. □

The three propositions of this section suffice to complete the proof of the following theorem.

Theorem 5.6. *The logarithmic de Rham cohomology groups of $\{x,y\} = xy$ on $\mathcal{A} = \mathbb{F}[x,y]$ along $xy\mathcal{A}$ are $H^0_{DR-log} \backsimeq \mathbb{F}$, $H^1_{DR-log} \backsimeq \mathbb{F} \times \mathbb{F}$, $H^2_{DR-log} \backsimeq \mathbb{F}$, and $H^k_{DR-log} = 0$ for $k > 2$.*

More explicitly, we deduce the following corollary.

Corollary 5.7. $H^*_{LP} \backsimeq H^*_{DR-log}$.

6. Conclusion

Let $(x, y) \in \mathbb{F}^2$ and $D = \{xy = 0\}$ a reduced normal crossing divisor. The classical Poisson cohomology group of degree 2 and its logarithmic counterpart of the Poisson structure $\{x, y\} = xy$ on the Poisson algebra $(\mathcal{A} = \mathbb{F}[x, y], xy\mathcal{A})$ are different. However, its logarithmic Poisson cohomology and logarithmic de Rham cohomology groups along the ideal $\mathcal{I} = xy\mathcal{A}$ are isomorphic. This allows us to conclude that the obstruction to its prequantization can be measured by the associated logarithmic de Rham cohomology.

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