

ANTI-INVARIANT SUBMANIFOLDS OF A NORMAL PARACONTACT METRIC MANIFOLD

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ABSTRACT. In this paper, anti-invariant submanifolds of a normal paracontact metric manifold are studied and characterizing the submanifold with respect to covariant derivative of the second fundamental form of anti-invariant submanifold. Furthermore, some special cases are also discussed and we give a non-trivial example which satisfies the statements of theorems.

1. INTRODUCTION

In modern the geometry of submanifolds has become a subject of growing interest for its significant application in applied mathematics and physics. For instance, the notion of invariant submanifold is used to discuss properties of nonlinear autonomous system. On the other hand, the notion of geodesics plays an important role in the theory of relativity. But, an invariant submanifold inherit almost all properties of the ambient manifold and so the study of invariant submanifolds is not so interesting from the point of view of the geometry of submanifolds. On the other hand, the theory of anti-invariant submanifolds proved to be a very nice topic is modern differential geometry and it has been studied by many geometers. For totally geodesic submanifolds, the geodesics of the ambient manifolds remain geodesics in the submanifolds. Therefore, totally geodesic submanifolds are also very much important in physical sciences. So it is very important to work totally geodesic submanifolds. The study of geometry of invariant submanifolds was initiated by Bejancu A. and Papaghuic N. [7, 8]. After then semiparallel anti-invariant submanifolds continued to work by many geometers.

Arslan K. and et. al. [5] defined and studied 2-semiparallel surfaces in space forms. Ishihara I. [11], Yano K. and Kon M. [20] researched anti-invariant submanifolds of a Sasakian space form. In [6, 7, 8], authors studied semi-invariant submanifolds in Sasakian and cosymplectic manifolds.

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Motivated by the above studies, the present paper deal with the study of anti-invariant submanifolds of a normal paracontact metric manifold.

Let M be a $(2n + 1)$ -dimensional manifold and ϕ , ξ and η be a tensor field of type $(1, 1)$, a vector field, 1-form on M , respectively. If ϕ , ξ and η satisfy the conditions

$$\phi^2 X = X - \eta(X)\xi, \quad \eta(\xi) = 1, \quad (1.1)$$

for any vector field X on M , then M is said to be have an almost contact manifold. In addition, it called almost contact metric manifold if M has a Riemannian metric tensor such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi), \quad (1.2)$$

for any $X, Y \in \Gamma(TM)$, where $\Gamma(TM)$ denotes set of the differentiable vector fields on M .

Furthermore, M is called a normal paracontact metric manifold if we have

$$(\bar{\nabla}_X \phi)Y = -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi \quad (1.3)$$

and

$$\bar{\nabla}_X \xi = \phi X, \quad (1.4)$$

for any $X, Y \in \chi(M)$, where $\bar{\nabla}$ denotes the Levi-Civita connection determined by g .

The concircular curvature tensor, projective curvature tensor, conformal curvature tensor, quasi-conformal curvature tensor and pseudo-projective curvature tensor of a normal paracontact metric manifold M^{2n+1} are, respectively, defined by

$$\tilde{Z}(X, Y)Z = R(X, Y)Z - \frac{\tau}{2n(2n+1)}\{g(Y, Z)X - g(X, Z)Y\}, \quad (1.5)$$

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{2n}\{S(Y, Z)X - S(X, Z)Y\}, \quad (1.6)$$

$$\begin{aligned} C(X, Y)Z &= R(X, Y)Z - \frac{1}{2n-1}\{S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX \\ &\quad - g(X, Z)QY\} + \frac{\tau}{2n(2n-1)}\{g(Y, Z)X - g(X, Z)Y\}, \end{aligned} \quad (1.7)$$

$$\begin{aligned} \tilde{C}(X, Y)Z &= \lambda R(X, Y)Z + \mu\{S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX \\ &\quad - g(X, Z)QY\} - \frac{\tau}{2n+1}\left\{\frac{\lambda}{2n} + 2\mu\right\}\{g(Y, Z)X - g(X, Z)Y\} \end{aligned} \quad (1.8)$$

and

$$\begin{aligned} \tilde{P}(X, Y)Z &= \lambda R(X, Y)Z + \mu\{S(Y, Z)X - S(X, Z)Y\} \\ &\quad - \frac{\tau}{2n+1}\left\{\frac{\lambda}{2n} + \mu\right\}\{g(Y, Z)X - g(X, Z)Y\}, \end{aligned} \quad (1.9)$$

for any $X, Y, Z \in \chi(M)$, where λ, μ are arbitrary constants such that $\lambda, \mu \neq 0$, τ is the scalar curvature, S is the Ricci tensor and R denotes the Riemannian curvature tensor of M .

Also, in a normal paracontact metric manifold M^{2n+1} , the following relations are satisfied

$$R(\xi, X)Y = -g(X, Y)\xi + \eta(Y)X, \quad (1.10)$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \quad (1.11)$$

$$R(\xi, X)\xi = X - \eta(X)\xi, \quad (1.12)$$

$$S(X, \xi) = 2n\eta(X), \quad \text{and} \quad Q\xi = 2n\xi. \quad (1.13)$$

for any $X, Y, Z \in \Gamma(TM)$.

Now let $\bar{M}$ be a submanifold of a normal paracontact metric manifold M with induced metric tensor g . We also denote the induced connections on the tangent bundle $\Gamma(T\bar{M})$ and the normal bundle $T^\perp\bar{M}$ by $\bar{\nabla}$ and $\bar{\nabla}^\perp$, respectively. Then the Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \quad (1.14)$$

and

$$\bar{\nabla}_X V = -A_V X + \bar{\nabla}_X^\perp V, \quad (1.15)$$

for any $X, Y \in \Gamma(T\bar{M})$ and $V \in \Gamma(T^\perp\bar{M})$, where h and A_V are second fundamental form and shape operator, respectively, for the immersion of $\bar{M}$ into M . $\bar{M}$ (or immersion) is called totally geodesic submanifold if $h = 0$. Furthermore, h and A_V are related by

$$g(A_V X, Y) = g(h(X, Y), V), \quad (1.16)$$

The covariant derivation of h is defined by

$$(\bar{\nabla}_X h)(Y, Z) = \bar{\nabla}_X^\perp h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z), \quad (1.17)$$

for any $X, Y, Z \in \Gamma(T\bar{M})$. h is said to be parallel if $(\bar{\nabla}_X h)(Y, Z) = 0$.

For a $\bar{M}$ is a submanifold of a normal paracontact metric manifold M and $X \in \Gamma(T\bar{M})$, we can write

$$\phi X = fX + \omega X, \quad (1.18)$$

where fX and ωX are the tangent and normal components of ϕX , respectively, with respect to submanifold $\bar{M}$. $\bar{M}$ is said to be an anti-invariant submanifold if $f = 0$. In the rest of this paper, we suppose that $\bar{M}$ is an anti-invariant submanifold of a normal paracontact metric manifold M . In this case, we have $\phi\Gamma(T\bar{M}) \subseteq \Gamma(T^\perp\bar{M})$ and $\phi\Gamma(T^\perp\bar{M}) \subseteq \Gamma(T\bar{M})$.

2. PRELIMINARIES

Let (M, g) be a semi Riemannian manifold and $\bar{M}$ be a submanifold of M . We denote the Levi-Civita connection of g and the second fundamental form of $\bar{M}$ by ∇ and h , respectively. The submanifold $\bar{M}$ said to be semiparallel if

$$R(X, Y)h = 0, \quad (2.1)$$

for any $X, Y \in \chi(\bar{M})$, where R denote the Riemannian curvature tensor of M and $R(X, Y)h = 0$ is defined by

$$(R(X, Y)h)(Z, U) = R^\perp(X, Y)h(Z, U) - h(R(X, Y)Z, U) - h(Z, R(X, Y)U),$$

for any $X, Y, Z, U \in \chi(\bar{M})$.

In [5], Arslan K. et.al. defined and studied 2-semiparallel submanifolds. Such submanifolds are defined by

$$R(X, Y)\nabla h = 0, \quad (2.2)$$

for any $X, Y \in \Gamma(T\bar{M})$, where

$$\begin{aligned} (R(X, Y)\nabla h)(Z, U, W) &= R^\perp(X, Y)(\nabla_Z h)(U, W) - (\nabla_{R(X, Y)Z} h)(U, W) \\ &\quad - (\nabla_Z h)(R(X, Y)U, W) - (\nabla_Z h)(U, R(X, Y)W) \end{aligned} \quad (2.3)$$

for any $X, Y, Z, U, W \in \Gamma(T\bar{M})$.

A Riemannian manifold of n -dimension ($n > 3$) is said to be conformally flat if its conformal curvature tensor vanishes identically.

Now, let us assume that normal Paracontact metric manifold M^{2n+1} is conformal flat. Then from (1.7) we have

$$\begin{aligned} R(X, Y)\xi &= \frac{1}{2n-1} \{S(Y, \xi)X - S(X, \xi)Y + \eta(Y)QX - \eta(X)QY\} \\ &\quad - \frac{\tau}{2n(2n-1)} \{\eta(Y)X - \eta(X)Y\}, \end{aligned} \quad (2.4)$$

which implies that

$$\begin{aligned} \eta(Y)X - \eta(X)Y &= \frac{2n}{2n-1} \{\eta(Y)X - \eta(X)Y\} + \frac{1}{2n-1} \{\eta(Y)QX - \eta(X)QY\} \\ &\quad - \frac{\tau}{2n(2n-1)} \{\eta(Y)X - \eta(X)Y\} \end{aligned} \quad (2.5)$$

from which for $Y = \xi$,

$$QX = \left(\frac{\tau}{2n} - 1\right)X + \left(2n - \frac{\tau}{2n} + 1\right)\eta(X)\xi. \quad (2.6)$$

Thus we have the following statement.

Theorem 2.1. *A Conformal flat normal paracontact metric manifold is an η -Einstein manifold.*

Similarly a Riemannian manifold of n -dimension ($n > 3$) is said to be quasi-conformally flat if its quasi-conformal curvature tensor vanishes identically.

Now, let us suppose that normal paracontact metric manifold is quasi-conformally flat. Then from (1.8), we have

$$\begin{aligned} R(X, Y)Z &= -\frac{\mu}{\lambda}\{S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY\} \\ &+ \frac{\tau}{\lambda(2n+1)}\left(\frac{\lambda}{2n} + 2\mu\right)\{g(Y, Z)X - g(X, Z)Y\}. \end{aligned}$$

By direct calculations, we conclude that

$$\begin{aligned} S(X, Y) &= \left(\frac{(\lambda + 2n\mu)(\tau - 2n(2n+1)) + 2n\mu\tau}{2n\mu(2n+1)}\right)g(X, Y) \\ &+ \left(\frac{(\lambda + 4n\mu)(2n(2n+1) - \tau)}{2n\mu(2n+1)}\right)\eta(X)\eta(Y). \end{aligned}$$

for any $X, Y, Z \in \Gamma(TM)$, that is, a quasi-conformal flat normal paracontact metric manifold is also an η -Einstein manifold. This leads to the following results.

$$\lambda = (1 - 2n)\mu. \quad (2.7)$$

Thus we have

$$\begin{aligned} R(X, Y)Z &= \frac{1}{2n-1}\{S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY\} \\ &- \frac{\tau}{2n(2n-1)}\{g(Y, Z)X - g(X, Z)Y\}, \end{aligned} \quad (2.8)$$

that is, quasi-conformal flat is equivalent to conformal flat.

3. ANTI-INVARIANT SUBMANIFOLDS OF A NORMAL PARACONTACT METRIC MANIFOLD

In this section, we study anti-invariant submanifolds of a normal paracontact metric manifold satisfying $\tilde{Z}(X, Y)h = 0$ and $\tilde{Z}(X, Y)\nabla h = 0$. Finally we see that these conditions are satisfied if and only if the second fundamental form h of an anti-invariant submanifold is parallel.

Proposition 3.1. *Let $\bar{M}$ be an anti-invariant submanifold of a normal paracontact metric manifold M . Then the following relations hold:*

- 1.) $\nabla_X \xi = 0$, $h(X, \xi) = -\phi X$,
 - 2.) $\nabla_X^\perp \phi Y = \phi \nabla_X Y$,
 - 3.) $A_{\phi Y} \xi = \phi h(\xi, Y) = \phi^2 Y = Y - \eta(Y)\xi$,
- for any $X, Y \in \Gamma(TM)$.

Proof. By using (1.4) and taking into account of $\bar{M}$ being anti-invariant submanifold, 1.) statement is obvious. On the other hand, making use of (1.3) and (1.14), we have

$$\begin{aligned} (\nabla_X \phi)Y &= \nabla_X \phi Y - \phi \nabla_X Y = -A_{\phi Y} X + \nabla_X^\perp \phi Y - \phi h(X, Y) - \phi \nabla_X Y \\ &= -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi, \end{aligned}$$

from which

$$A_{\phi Y}X + \phi h(X, Y) = g(X, Y)\xi + \eta(Y)X - 2\eta(X)\eta(Y)\xi \quad (3.1)$$

and

$$\nabla_X^\perp \phi Y = \phi \nabla_X Y \quad (3.2)$$

for any $X, Y \in \chi(\bar{M})$, which proves 2.) and 3.) statements. On the other hand, we have

$$\begin{aligned} g(h(X, \xi), \phi Y) &= -g(\phi X, \phi Y) \\ g(A_{\phi Y}\xi, X) &= g(\phi^2 Y, X). \end{aligned}$$

This proves our assertion. $\square$

Theorem 3.2. *Let $\bar{M}$ be an anti-invariant submanifold of a normal paracontact metric manifold M . Then $\bar{M}$ is semiparallel if and only if $\bar{M}$ is totally geodesic submanifold.*

Proof. If $\bar{M}$ is semiparallel, then $R.h = 0$. This implies that

$$\begin{aligned} (R(X, Y)h)(Z, U) &= R^\perp(X, Y)h(Z, U) - h(R(X, Y)Z, U) \\ &\quad - h(Z, R(X, Y)U), \end{aligned} \quad (3.3)$$

for any $X, Y, Z, U \in \chi(\bar{M})$. Putting $Z = U = \xi$ in (3.3), we obtain

$$\begin{aligned} h(R(X, Y)\xi, \xi) &= h(\eta(Y)X - \eta(X)Y, \xi) = \eta(Y)h(X, \xi) - \eta(X)h(Y, \xi) \\ &= \eta(X)\phi Y - \eta(Y)\phi X = 0. \end{aligned} \quad (3.4)$$

Since $\dim(\bar{M}) > 1$, X and Y are linearly independent vector fields. Thus $\eta(X) = 0$. Now let $Y = U = \xi$ be in (3.3), then we have

$$R^\perp(X, \xi)h(Z, \xi) - h(R(X, \xi)Z, \xi) - h(R(X, \xi)\xi, Z) = 0,$$

that is,

$$\begin{aligned} R^\perp(X, \xi)\phi Z &+ h(\eta(Z)X - g(X, Z)\xi, \xi) + h(X - \eta(X)\xi, Z) = \phi R(X, \xi)Z \\ &+ \eta(Z)h(X, \xi) + h(X, Z) - \eta(X)h(Z, \xi) = \phi\{\eta(Z)X - g(X, Z)\xi\} \\ &- \eta(Z)\phi X + h(X, Z) + \eta(X)\phi Z = h(X, Z) = 0. \end{aligned}$$

Thus anti-invariant semiparallel submanifold is totally geodesic. Conversely, It is clear that $\bar{M}$ is semiparallel when it is totally geodesic. $\square$

Theorem 3.3. *Let $\bar{M}$ be an anti-invariant submanifold of a normal paracontact metric manifold M . Then $\bar{M}$ is 2-semiparallel if and only if the second fundamental form h of $\bar{M}$ is parallel.*

Proof. Let us suppose $\bar{M}$ be 2-semiparallel. This implies that

$$\begin{aligned} (R(X, Y)\nabla h)(Z, U, W) &= R^\perp(X, Y)(\nabla_Z h)(U, W) - (\nabla_{R(X, Y)Z}h)(U, W) \\ &\quad - (\nabla_Z h)(R(X, Y)U, W) - (\nabla_Z h)(U, R(X, Y)W) \end{aligned} \quad (3.5)$$

for all $X, Y, Z, U, W \in \chi(\bar{M})$. Here taking $X = U = \xi$ in (3.5) and we calculate each of expression;

$$\begin{aligned} R^\perp(\xi, Y)(\nabla_Z h)(\xi, W) &= R^\perp(\xi, Y)\{\nabla_Z^\perp h(\xi, W) - h(\bar{\nabla}_Z \xi, W) - h(\xi, \bar{\nabla}_Z W)\} \\ &= R^\perp(\xi, Y)\{-\nabla_Z^\perp \phi W + \phi \nabla_Z W\} = 0, \end{aligned} \quad (3.6)$$

$$\begin{aligned} (\nabla_{R(\xi, Y)Z} h)(\xi, W) &= \nabla_{R(\xi, Y)Z}^\perp h(\xi, W) - h(\bar{\nabla}_{R(\xi, Y)Z} \xi, W) - h(\bar{\nabla}_{R(\xi, Y)Z} W, \xi) \\ &= -\nabla_{R(\xi, Y)Z}^\perp \phi W + \phi \nabla_{R(\xi, Y)Z} W = 0. \end{aligned} \quad (3.7)$$

$$\begin{aligned} (\nabla_Z h)(R(\xi, Y)\xi, W) &= \nabla_Z^\perp h(R(\xi, Y)\xi, W) - h(\bar{\nabla}_Z R(\xi, Y)\xi, W) \\ &\quad - h(R(\xi, Y)\xi, \bar{\nabla}_Z W) \\ &= \nabla_Z^\perp h(R(\xi, Y)\xi, W) - h(\nabla_Z R(\xi, Y)\xi, W) \\ &\quad - h(R(\xi, Y)\xi, \nabla_Z W) \\ &= -\nabla_Z^\perp h(Y - \eta(Y)\xi, W) + h(\nabla_Z Y - \eta(Y)\xi, W) \\ &\quad + h(\nabla_Z W, Y - \eta(Y)\xi) \\ &= -\nabla_Z^\perp h(Y, W) + \nabla_Z^\perp h(\eta(Y)\xi, W) + h(\nabla_Z Y, W) \\ &\quad - h(\nabla_Z \eta(Y)\xi, W) + h(\nabla_Z W, Y) - h(\nabla_Z W, \eta(Y)\xi) \\ &= -\nabla_Z^\perp h(Y, W) + h(\nabla_Z Y, W) + h(Y, \nabla_Z W) \\ &\quad + Z[\eta(Y)]h(\xi, W) + \eta(Y)\nabla_Z^\perp h(\xi, W) \\ &\quad - h(Z[\eta(Y)]\xi + \eta(Y)\nabla_Z \xi, W) + \eta(Y)h(\nabla_Z W, \xi) \\ &= -(\nabla_Z h)(Y, W)\eta(\nabla_Z Y)\phi W + \eta(Y)\nabla_Z^\perp \phi W \\ &\quad + \eta(\nabla_Z Y)h(\xi, W) + \eta(Y)h(\nabla_Z \xi, W) - \eta(Y)\phi \nabla_Z W \\ &= -(\nabla_Z h)(Y, W) + \eta(\nabla_Z Y)\phi W + \eta(Y)\nabla_Z^\perp \phi W \\ &\quad - \eta(\nabla_Z Y)\phi W - \eta(Y)\phi \nabla_Z W \\ &= -(\nabla_Z h)(Y, W), \end{aligned} \quad (3.8)$$

and

$$\begin{aligned} (\nabla_Z h)(\xi, R(\xi, Y)W) &= \nabla_Z^\perp h(\xi, R(\xi, Y)W) - h(\nabla_Z \xi, R(\xi, Y)W) \\ &\quad - h(\nabla_Z R(\xi, Y)W, \xi) \\ &= \nabla_Z^\perp h(\xi, \eta(W)Y - g(Y, W)\xi) - h(\nabla_Z \xi, \eta(W)Y - g(Y, W)\xi) \\ &\quad - h(\nabla_Z \eta(W)Y - g(Y, W)\xi, \xi) \\ &= \nabla_Z^\perp h(\xi, \eta(W)Y) - \nabla_Z^\perp h(\xi, g(Y, W)\xi) - h(\nabla_Z \eta(W)Y, \xi) \\ &\quad + h(\nabla_Z g(Y, W)\xi, \xi) \\ &= Z[\eta(W)]h(Y, \xi) + \eta(W)\nabla_Z^\perp h(Y, \xi) - Z[g(Y, W)]\nabla_Z^\perp h(\xi, \xi) \\ &\quad - g(Y, W)\nabla_Z^\perp h(\xi, \xi) - Z[\eta(W)]h(Y, \xi) - \eta(W)h(\nabla_Z Y, \xi) \\ &\quad + Z[g(Y, W)]h(\xi, \xi) + g(Y, W)h(\nabla_Z \xi, \xi) \\ &= -\eta(\nabla_Z W)\phi Y - \eta(W)\nabla_Z^\perp \phi Y + \eta(\nabla_Z W)\phi Y \\ &\quad + \eta(W)\phi \nabla_Z Y = 0. \end{aligned} \quad (3.9)$$

Thus we summarize all of the statements

$$(R(\xi, Y)\nabla h)(Z, \xi, W) = (\nabla_Z h)(Y, W). \quad (3.10)$$

This proves our assertion. The converse is obvious. $\square$

Theorem 3.4. *Let $\bar{M}$ be an invariant submanifold of a normal paracontact metric manifold M . $\tilde{Z}(X, Y)h = 0$ if and only if $\bar{M}$ is totally geodesic submanifold.*

Proof. $\tilde{Z}(X, Y)h = 0$ implies that

$$\begin{aligned} (\tilde{Z}(X, Y)h)(Z, U) &= R^\perp(X, Y)h(Z, U) - h(\tilde{Z}(X, Y)Z, U) \\ &\quad - h(Z, \tilde{Z}(X, Y)U), \end{aligned} \quad (3.11)$$

for any $X, Y, Z, U \in \chi(\bar{M})$. By using (1.5), we have

$$\tilde{Z}(\xi, Y)Z = \left(1 - \frac{\tau}{2n(2n+1)}\right) (g(Y, Z)\xi - \eta(Z)Y), \quad (3.12)$$

$$\tilde{Z}(X, Y)\xi = \left(1 - \frac{\tau}{2n(2n+1)}\right) (\eta(Y)X - \eta(X)Y), \quad (3.13)$$

Thus

$$\begin{aligned} 0 &= R^\perp(\xi, Y)h(Z, \xi) - h(\tilde{Z}(\xi, Y)Z, \xi) - h(Z, \tilde{Z}(\xi, Y)\xi) \\ &= -R^\perp(X, \xi)\phi Z + \left(1 - \frac{\tau}{2n(2n+1)}\right) h(g(X, Z)\xi - \eta(Z)X, \xi) \\ &\quad - \left(1 - \frac{\tau}{2n(2n+1)}\right) h(X - \eta(X)\xi, Z) = 0, \end{aligned}$$

or,

$$\begin{aligned} 0 &= -\phi R(X, \xi)Z + \left(1 - \frac{\tau}{2n(2n+1)}\right) (\eta(X)h(Z, \xi) - \eta(Z)h(X, \xi) - h(Z, X)) \\ &= \phi(g(g(X, Z)\xi - \eta(Z)X) + \left(1 - \frac{\tau}{2n(2n+1)}\right) (\eta(Z)\phi X - \eta(X)\phi Z - h(X, Z)) \\ &= -\eta(Z)\phi X + \left(1 - \frac{\tau}{2n(2n+1)}\right) (\eta(Z)\phi X - \eta(X)\phi Z - h(X, Z)). \end{aligned}$$

Taking X, Y orthogonal to ξ , we get $h(X, Z) = 0$. This proves our assertion. $\square$

Theorem 3.5. *Let $\bar{M}$ be an anti-invariant submanifold of a normal paracontact metric manifold M . $\tilde{Z}(X, Y)\nabla h = 0$ if and only if the second fundamental form h of $\bar{M}$ is either parallel or the scalar curvature τ of M $\tau = 2n(2n+1)$.*

Proof. $\tilde{Z}(X, Y)\nabla h = 0$ means that

$$\begin{aligned} R^\perp(X, Y)(\nabla_Z h)(U, W) &= (\nabla_{\tilde{Z}(X, Y)Z} h)(U, W) - (\nabla_Z h)(\tilde{Z}(X, Y)U, W) \\ &\quad - (\nabla_Z h)(U, \tilde{Z}(X, Y)W) = 0, \end{aligned} \quad (3.14)$$

for any $X, Y, Z, U, W \in \chi(\bar{M})$. Here,

$$\begin{aligned} R^\perp(\xi, Y)(\nabla_Z h)(\xi, W) &= R^\perp(\xi, Y)\{\nabla_Z^\perp h(\xi, W) - h(\nabla_Z \xi, W) - h(\nabla_Z W, \xi)\} \\ &= R^\perp(\xi, Y)\{-\nabla_Z^\perp \phi W + \phi \nabla_Z W\} = 0, \end{aligned} \quad (3.15)$$

$$\begin{aligned}
(\nabla_{\tilde{Z}(\xi, Y)Z}h)(\xi, W) &= \nabla_{\tilde{Z}(\xi, Y)Z}^\perp h(\xi, W) - h(\nabla_{\tilde{Z}(\xi, Y)Z}\xi, W) - h(\nabla_{\tilde{Z}(\xi, Y)Z}W, \xi) \\
&= -\nabla_{\tilde{Z}(\xi, Y)Z}^\perp \phi W + \phi \nabla_{\tilde{Z}(\xi, Y)Z}W = 0,
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
(\nabla_Z h)(\tilde{Z}(\xi, Y)\xi, W) &= \nabla_Z^\perp h(\tilde{Z}(\xi, Y)\xi, W) - h(\nabla_Z \tilde{Z}(\xi, Y)\xi, W) \\
&\quad - h(\nabla_Z W, \tilde{Z}(\xi, Y)\xi) \\
&= \left(1 - \frac{\tau}{2n(2n+1)}\right) \{ \nabla_Z^\perp h(\eta(Y)\xi - Y, W) \\
&\quad - h(\nabla_Z \eta(Y)\xi - Y, W) - h(\nabla_Z W, \eta(Y)\xi - Y) \} \\
&= \left(1 - \frac{\tau}{2n(2n+1)}\right) \{ \nabla_Z^\perp h(\eta(Y)\xi, W) - \nabla_Z^\perp h(Y, W) \\
&\quad - h(\nabla_Z \eta(Y)\xi, W) + h(\nabla_Z Y, W) - \eta(Y)h(\nabla_Z W, \xi) \\
&\quad + h(\nabla_Z W, Y) \} \\
&= \left(1 - \frac{\tau}{2n(2n+1)}\right) \{ \eta(Y)\nabla_Z^\perp h(\xi, W) + Z\eta(Y)h(\xi, W) - \\
&\quad - h(Z\eta(Y)\xi + \eta(Y)\nabla_Z \xi, W) + \eta(Y)\phi \nabla_Z W - (\nabla_Z h)(Y, W) \} \\
&= \left(1 - \frac{\tau}{2n(2n+1)}\right) \{ -\eta(Y)\nabla_Z^\perp \phi W + Z\eta(Y)h(\xi, W) \\
&\quad - Z\eta(Y)h(\xi, W) - \eta(Y)h(\nabla_Z \xi, W) + \eta(Y)\phi \nabla_Z W \\
&\quad - (\nabla_Z h)(Y, W) \} \\
&= -\left(1 - \frac{\tau}{2n(2n+1)}\right) (\nabla_Z h)(Y, W),
\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}
(\nabla_Z h)(\xi, \tilde{Z}(\xi, Y)W) &= \left(1 - \frac{\tau}{2n(2n+1)}\right) \{ \nabla_Z^\perp h(\xi, g(Y, W)\xi - \eta(W)Y) \\
&\quad - h(\nabla_Z g(Y, W)\xi - \eta(W)Y, \xi) \} \\
&= \left(1 - \frac{\tau}{2n(2n+1)}\right) \{ \phi \nabla_Z g(Y, W)\xi - \eta(W)Y \\
&\quad - \nabla_Z^\perp \phi(g(Y, W)\xi - \eta(W)Y) \} \\
&= 0
\end{aligned} \tag{3.18}$$

Thus we obtain

$$(\tilde{Z}(\xi, Y)\nabla h)(Z, \xi) = -\left(1 - \frac{\tau}{2n(2n+1)}\right) (\nabla_Z h)(Y, W) \tag{3.19}$$

The converse is obvious. This proves our assertion. $\square$

Example 3.6. Let us consider submanifold M of $\mathbb{R}^7$ with the cartesian coordinates $(x_1, y_1, x_2, y_2, x_3, y_3, t)$ and the almost paracontact metric structure

$$\phi\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}, \quad \phi\left(\frac{\partial}{\partial y_i}\right) = -\frac{\partial}{\partial x_i}, \quad \phi\left(\frac{\partial}{\partial t}\right) = 0, \quad 1 \leq i \leq 3.$$

It is easy to check that (ϕ, ξ, η, g) is an almost paracontact metric structure with usual inner product $\mathbb{R}^7$. Now, we consider the immersion φ of M into $\mathbb{R}^7$ as

$$\varphi(u, v, w, t) = (v, u, u \cos w, v \cos w, u \sin w, v \sin w, t).$$

Then the tangent bundle TM of M is spanned by the vector fields

$$\begin{aligned} e_1 &= \frac{\partial}{\partial x_1} + \cos w \frac{\partial}{\partial y_2} + \sin w \frac{\partial}{\partial y_3}, & e_2 &= \frac{\partial}{\partial y_1} + \cos w \frac{\partial}{\partial x_2} + \sin w \frac{\partial}{\partial x_3} \\ e_3 &= -u \sin w \frac{\partial}{\partial x_2} - v \sin w \frac{\partial}{\partial y_2} + u \cos w \frac{\partial}{\partial x_3} + v \cos w \frac{\partial}{\partial y_3}, & e_4 &= \frac{\partial}{\partial t}. \end{aligned}$$

Furthermore, $\phi(TM)$ is spanned by the vector fields

$$\begin{aligned} \phi e_1 &= \frac{\partial}{\partial y_1} - \cos w \frac{\partial}{\partial x_2} - \sin w \frac{\partial}{\partial x_3}, & \phi e_2 &= -\frac{\partial}{\partial x_1} + \cos w \frac{\partial}{\partial y_2} + \sin w \frac{\partial}{\partial y_3} \\ \phi e_3 &= -u \sin w \frac{\partial}{\partial y_2} + v \sin w \frac{\partial}{\partial x_2} + u \cos w \frac{\partial}{\partial y_3} - v \cos w \frac{\partial}{\partial x_3}, & \phi e_4 &= 0. \end{aligned}$$

Since $g(\phi e_i, e_j) = 0$, $1 \leq i, j \leq 4$, ϕe_1 , ϕe_2 and ϕe_3 are orthogonal to TM and so M is a 4-dimensional anti-invariant submanifold of $\mathbb{R}^7$. Furthermore, by direct calculations, we have

$$\begin{aligned} [e_1, e_2] &= \sin x_2 \cos x_2 \frac{\partial}{\partial y_2} - \cos^2 x_2 \frac{\partial}{\partial y_3}, \\ [e_1, e_3] &= -\sin x_2 \frac{\partial}{\partial y_2} + \cos x_2 \frac{\partial}{\partial y_3} - y_1 \sin^2 x_2 \frac{\partial}{\partial y_2} + y_1 \sin x_2 \cos x_2 \frac{\partial}{\partial y_3}, \\ [e_2, e_3] &= -(\sin x_2 + y_1) \frac{\partial}{\partial x_2} - x_1 \cos^2 x_2 \frac{\partial}{\partial y_2} + \cos x_2 \frac{\partial}{\partial x_3} - x_1 \sin x_2 \cos x_2 \frac{\partial}{\partial y_3}. \end{aligned}$$

By using Kozsul formulae, we obtain

$$\begin{aligned} \bar{\nabla}_{e_2} e_1 &= \left(\frac{x_1 \cos x_2}{x_1^2 + x_2^2} \right) e_3, & \bar{\nabla}_{e_1} e_3 &= \left(\frac{x_1}{x_1^2 + y_1^2} \right) e_3, \\ \bar{\nabla}_{e_2} e_2 &= \left(\frac{y_1 \cos x_2 \cos 2x_2 + (\cos x_2 - 1)(\sin x_2 - y_1 \cos x_2)}{x_1^2 + y_1^2} \right) e_3, \\ \bar{\nabla}_{e_2} e_3 &= -\left(\frac{x_1 \cos x_2}{2} \right) e_1 - \left(\frac{y_1 \cos x_2}{2} \right) e_2 + \left(\frac{y_1}{x_1^2 + y_1^2} \right) e_3 \\ \bar{\nabla}_{e_3} e_1 &= -\left(\frac{x_1 y_1 \sin x_2}{x_1^2 + y_1^2} \right) e_3, & \bar{\nabla}_{e_3} e_2 &= y_1 \left(\frac{\sin^2 x_2}{x_1^2 + y_1^2} \right) e_3, \\ \bar{\nabla}_{e_3} e_3 &= \left(\frac{y_1^2 (\sin x_2 \cos x_2 - \sin x_2 \cos^2 x_2 + \sin^3 x_2) + y_1 (\cos^2 x_2 - 1)}{2} \right) e_2 \\ &\quad + \left(\frac{x_1 y_1 \sin x_2}{2} \right) e_1 \end{aligned}$$

$$\bar{\nabla}_{e_1} e_1 = \bar{\nabla}_{e_1} e_2 = \bar{\nabla}_{e_4} e_1 = \bar{\nabla}_{e_4} e_2 = \bar{\nabla}_{e_4} e_3 = \bar{\nabla}_{e_4} e_4 = 0,$$

Since M is totally geodesic submanifold, this example provide all of the expressions of the theorems given in the work.

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