Gulf Journal of Mathematics
Vol 10, Issue 2 (2021) 31-37

STATISTICAL CONVERGENCE OF SEQUENCES OF SUBSPACES

FATİH NURAY[1*]

Abstract. In this study, we will give the definition of statistical convergence of a sequences of subspaces of a k-dimensional inner product space and examine some inclusion relations.

1. Introduction and preliminaries

Let F be the field reals or complex numbers and X be a vector space over F. An inner product on X is a function $<,>: X \times X \to F$ such that

(1) $< ax + by, z >= a < x, z > +b < y, z >$, for $a, b \in F$ and $x, y, z \in X$
(2) $< x, y >= \overline{< y, x >}$ for $x, y \in X$
(3) $< x, x > \geq 0$ for $x \in X$
(4) $< x, x >= 0$ if and only if $x = 0$.

Recall that every real number $x \in \mathbb{R}$ equals its complex conjugate. Hence for real vector spaces the condition (2) becomes $< x, y >=< y, x >$ for $x, y \in X$.
A norm can be defined from an inner product via $\|x\| = \sqrt{< x, x >}$ for all $x \in X$. Let X be a inner product space over F. Let V be a subspace of X and $x \in X$. An element $v_0 \in V$ is said to be a *best approximation* to x or *nearest* to x in X, if $\|x - v_0\| \leq \|x - v\|$ for all $v \in V$. Suppose V is finite dimensional and $\{v_1, v_2, ..., v_k\}$ is an orthonormal basis of V. Then

$$v_0 = \sum_{i=1}^{k} < v, v_i > v_i$$

is the best approximation to v in V. Let $(X <,>)$ be an inner product space. Let V be a subspace of inner product space X and $\{v_1, v_2, ..., v_k\}$ be an orthonormal basis for V. The ortogonal projection P_V from X onto the subspace V is defined by

$$P_V u = \sum_{i=1}^{k} < u, v_i > v_i$$

Date: Received: Oct 20, 2020; Accepted: Feb 27, 2021.
* Corresponding author.
2010 *Mathematics Subject Classification.* Primary 40A05; Secondary 40D25.
Key words and phrases. Inner product space, subspace, convergence, statistical convergence, ortogonal projection.

for every $u \in X$.
For example, let $X = \mathbb{R}^3$ and $V = \{(x_1, x_2, 0) : x_1, x_2 \in \mathbb{R}\}$ be the x_1x_2-plane in $\mathbb{R}^3$. Choose an arbitrary point $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Ortogonal projection of x onto V is $P_V x = (x_1, x_2, 0)$. In order to show this, choose an arbitrary point $v = (v_1, v_2, v_3) \in V$. Then $x - v = (x_1 - v_1, x_2 - v_2, x_3)$ and $x - P_V x = (0, 0, x_3)$. So

$$\|x - v\|^2 = |x_1 - v_1|^2 + |x_2 - v_2|^2 + |x_3|^2 \geq |x_3|^2 = \|x - P_V x\|^2.$$

Thus $P_V x$ is closer x than v, so $P_V x$ is the ortogonal projection. If we take $\{e_1, e_2, e_3\}$ as the standart basis of $\mathbb{R}^3$, then above example tells us the ortogonal projection of $x = (x_1, x_2, x_3) = x_1e_1 + x_2e_2 + x_3e_3$ onto $V = span\{e_1, e_2\}$ is $P_V x = x_1e_1 + x_2e_2 = (x_1, x_2, 0)$. As a second example, let $X = \mathbb{R}^3$ and V be subspace of $\mathbb{R}^3$ spanned by the vectors $(1, 1, 1)$ and $(1, -2, 1)$. Then

$$\begin{aligned} P_V x &= \frac{x_1 + x_2 + x_3}{3}(1, 1, 1) + \frac{x_1 - 2x_2 + x_3}{6}(1, -2, 1) \\ &= (\frac{x_1 + x_3}{2}, x_2, \frac{x_1 + x_3}{2}). \end{aligned}$$

(For more information on this topic, see [4],[9]).
Distance or gap between two subspaces U and V of X is defined by

$$\begin{aligned} d(U, V) &= \sup\{\inf\{\|u - v\| : v \in V\} : u \in U, \|u\| = 1\} \\ &= \sup\{\|u - P_V u\| : u \in U, \|u\| = 1\} \\ &= \sup_{u \in U, \|u\|=1} \|u - P_V u\|. \end{aligned}$$

Given an inner product space $(X, <, >)$ of dimension k or higher (may be infinite). Let U_n and V be k-dimensional subspaces of X for every $n \in \mathbb{N}$. Throughout the paper, it is supposed that $U_n = span\{u_{1_n}, u_{2_n}, ..., u_{k_n}\}$ and $V = span\{v_1, v_2, ..., v_k\}$ be k-dimensional subspaces of X, where $\{u_{1_n}, u_{2_n}, ..., u_{k_n}\}$ and $\{v_1, v_2, ..., v_k\}$ are orthonormal for every $n \in \mathbb{N}$.
Convergence definition for sequence of subspaces was given in [10] as follows:
Let U_n and V be k-dimensional subspaces of X for $n = 1, 2, 3, ...$ A sequence (U_n) of subspaces of X is said to be convergent to the subspace V if

$$\lim_{n\to\infty} \sup_{u_n \in U_n, \|u_n\|=1} \|u_n - P_V u_n\| = 0,$$

Also the authors of [10] proved the following theorem that gives a condition equivalent to that in the definition of convergence.

Theorem 1.1. *A sequence (U_n) of subspaces is convergent to the subspace V if and only if*

$$\lim_{n\to\infty} \|u_{i_n} - P_V u_{i_n}\| = 0$$

for every $i = 1, 2, ..., k$ where

$$P_V u_{i_n} = \sum_{j=1}^{k} < u_{j_n}, v_j > v_j.$$

For example, consider a three-dimensional space X with orthonormal basis vectors e_1, e_2, e_3. U_n be one-dimensional subspace spanned by the vector

$$e_1 + \frac{1}{n} sinne_2 + \frac{1}{n} cosne_3.$$

Then the sequence (U_n) converges to the subspace V, spanned by the vector e_1. If W_n is an one-dimensional subspace spanned by the vector

$$sinne_2 + cosne_3$$

then the sequence (W_n) is not convergent.
In most convergence theories it is advantages to have a criterion that verify convergence without using the value of the limit. For this goal we will introduce a Cauchy convergence criterion.

Definition 1.2. Let U_n be k-dimensional subspaces of X for $n = 1, 2, 3, ...$ A sequence (U_n) of subspaces of X is said to be Cauchy sequence if

$$\sup_{u_n \in U_n, \|u_n\|=1} \|u_n - P_{U_m} u_n\| < \epsilon$$

for every $m, n > n_0 \in \mathbb{N}$.

We can give another definition for Cauchy sequence equivalent to the Definition 1.1.

Definition 1.3. A sequence (U_n) of subspaces is convergent to the subspace V if and only if

$$\|u_{i_n} - P_{U_m} u_{i_n}\| < \epsilon$$

for every $i = 1, 2, ..., k$ and $m, n > n_0 \in \mathbb{N}$.

2. Statistical Convergence

In this section, we will give the definition of statistical convergence of a sequences of subspaces of a k-dimensional inner product space and examine some inclusion relations.
Let $A \subseteq \mathbb{N}$. Define

$$\underline{\delta}(A) = \liminf_{m\to\infty} \frac{|A \cap \{1, 2, 3, ..., m\}|}{m} \quad \text{and } \bar{\delta}(A) = \limsup_{m\to\infty} \frac{|A \cap \{1, 2, 3, ..., m\}|}{m}.$$

We call $\underline{\delta}$ and $\overline{\delta}$ the lower and upper asymptotic densities, respectively. If $\underline{\delta}(A) = \overline{\delta}(A)$, the common value is referred as asymptotic density and denoted by $\delta(A)$.
If (x_n) is a sequence such that x_n satisfies property P for all n except a set of asymptotic density zero, then we say that x_n satisfies P for "almost all n ", and we abbreviate this by "$a.a.$ n." A sequence (x_n) of real or complex numbers is said to be statistically convergent to the number a if for every $\epsilon > 0$,

$$\delta(\{k : \quad |x_k - a| \geq \epsilon\}) = 0.$$

A statistically convergent sequence may be bounded or unbounded. For example, define $x_k = k$ if k is a square and $x_k = 0$ otherwise. Then

$$|\{k \leq n : \quad |x_k - 0| > \epsilon\}| \leq \sqrt{n},$$

so $st-\lim x_k=0$.
The idea of statistical convergence was introduced by Steinhaus in [13] and Fast in [6] and since then has been studied by other authors including [2],[3],[7],[8], [11] and [13] and . Over the years and under different names, statistical convergence has been discussed in the mathematical analysis.

Definition 2.1. A sequence (U_n) of subspaces is said to be statistically convergent to the subspace V if for every $\epsilon>0$,

$$\lim_{m\to\infty}\frac{1}{m}\left|\left\{n\leq m:\quad \sup_{u_n\in U_n,\|u_n\|=1}\|u_n-P_Vu_n\|>\epsilon\right\}\right|=0$$

i.e.,

$$\sup_{u_n\in U_n,\|u_n\|=1}\|u_n-P_Vu_n\|<\epsilon$$

a.a. n.

In this case we write

$$st-\lim U_n=V.$$

Another definition equivalent to this definition is given below:

Definition 2.2. A sequence (U_n) of subspaces is said to be statistically convergent to the subspace V if for every $\epsilon>0$,

$$\lim_{m\to\infty}\frac{1}{m}\left|\{n\leq m:\quad \|u_{i_n}-P_Vu_{i_n}\|>\epsilon\}\right|=0$$

for every $i=1,2,3,....$

Theorem 2.3. *Definition 2.1 and Definition 2.2 are equivalent.*

Proof. Let

$$\sup_{u_n\in U_n,\|u_n\|=1}\|u_n-P_Vu_n\|<\epsilon$$

a.a. n. We can write

$$\|u_{i_n}-P_Vu_{i_n}\|\leq\sup_{u_n\in U_n,\|u_n\|=1}\|u_n-P_Vu_n\|<\epsilon$$

a.a. n for every $i=1,2,...,k$, $u_{i_n}\in U_n$ with $\|u_{i_n}\|=1$.
Now, say $\sum_{i=1}^k|\gamma_i|=K$ and suppose that

$$\|u_{i_n}-P_Vu_{i_n}\|<\frac{\epsilon}{K}$$

a.a. n for every $i = 1, 2, ..., k$.

$$\begin{aligned}
&\sup_{u_n \in U_n, \|u_n\|=1} \|u_n - P_V u_n\| \\
= &\sup_{u_n \in U_n, \|u_n\|=1} \|(\gamma_1 u_{1_n} + \gamma_2 u_{2_n} + ... + \gamma_k u_{k_n}) - P_V(\gamma_1 u_{1_n} + \gamma_2 u_{2_n} + ... + \gamma_k u_{k_n})\| \\
\leq &\sup_{u_n \in U_n, \|u_n\|=1} \sum_{i=1}^{k} \|\gamma_i u_{i_n} - P_V(\gamma_i u_{i_n})\| \\
= &\sum_{i=1}^{k} |\gamma_i| \|u_{i_n} - P_V u_{i_n}\| < \epsilon
\end{aligned}$$

a.a. n for every $i = 1, 2, ..., k$. □

If (U_n) is convergent to V then (U_n) is statistically convergent to V, but the opposite may not be true.

Example 2.4. Let X be three-dimensional space with orthonormal basis vectors e_1, e_2, e_3 and

$$U_n = \begin{cases} \operatorname{span}\{\sin ne_2 + \cos ne_3\}, & \text{if } n \text{ is square} \\ \operatorname{span}\{e_1 + \frac{1}{n}\sin ne_2 + \frac{1}{n}\cos ne_3\}, & \text{otherwise.} \end{cases} \tag{2.1}$$

Then the sequence (U_n) is not convergent but statistically convergent to the subspace V, spanned by the vector e_1.

Theorem 2.5. *If $st - \lim U_n = V$ and $st - \lim U_n = S$ then $V = S$.*

Proof. If possible let $U \neq S$. Choose $\epsilon = \frac{\|P_V u_n - P_S u_n\|}{2} > 0$. Then

$$\begin{aligned}
&\frac{1}{m}\left|\left\{n \leq m : \sup_{u_n \in U_n, \|u_n\|=1} \|P_V u_n - P_S u_n\| > \epsilon\right\}\right| \\
\leq &\frac{1}{m}\left|\left\{n \leq m : \sup_{u_n \in U_n, \|u_n\|=1} \|u_n - P_V u_n\| > \frac{\epsilon}{2}\right\}\right| \\
+ &\frac{1}{m}\left|\left\{n \leq m : \sup_{u_n \in U_n, \|u_n\|=1} \|u_n - P_S u_n\| > \frac{\epsilon}{2}\right\}\right|
\end{aligned}$$

which is impossible because the right side limit is equal to zero but not left side limit as $m \to \infty$. □

Definition 2.6. A sequence (U_n) of subspaces is said to be statistically Cauchy if for every $\epsilon > 0$ there exists a number $N(= N(\epsilon))$ such that

$$\lim_{m \to \infty} \frac{1}{m}\left|\left\{n \leq m : \sup_{u_n \in U_n, \|u_n\|=1} \|u_n - P_{U_N} u_n\| > \epsilon\right\}\right| = 0$$

i.e.,

$$\sup_{u_n \in U_n, \|u_n\|=1} \|u_n - P_{U_N} u_n\| < \epsilon$$

a.a. n.

Another definition equivalent to this definition can be given as:

Definition 2.7. A sequence (U_n) of subspaces is said to be statistically Cauchy sequence if for every $\epsilon > 0$ there exists a number $N(= N(\epsilon))$ such that

$$\lim_{m\to\infty} \frac{1}{m} \left|\{n \leq m : \quad \|u_{i_n} - P_{U_N} u_{i_n}\| > \epsilon\}\right| = 0$$

for every $i = 1, 2, 3, ...$

Theorem 2.8. *If a sequence (U_n) of subspaces is statistically convergent, then (U_n) is a statistically Cauchy sequence.*

Proof. Suppose that $st - \lim(U_n) = V$ and $\epsilon > 0$. Then

$$\sup_{u_n\in U_n, \|u_n\|=1} \|u_n - P_V u_n\| < \frac{\epsilon}{2}$$

a.a. n, and if N is chosen such that

$$\sup_{u_n\in U_n, \|u_n\|=1} \|P_{U_N} u_n - P_V u_n\| < \frac{\epsilon}{2}$$

a.a. n. Then

$$\begin{aligned} & \sup_{u_n\in U_n, \|u_n\|=1} \|u_n - P_{U_N} u_n\| \\ < & \sup_{u_n\in U_n, \|u_n\|=1} \|u_n - P_V u_n\| + \sup_{u_n\in U_n, \|u_n\|=1} \|P_{U_N} u_n - P_V u_n\| \\ < & \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

a.a. n. Hence (U_n) is a statistically Cauchy sequence. $\square$

Theorem 2.9. *Let (U_n) and (L_n)be two sequences of subspaces. If (L_n) is a convergent sequence such that $U_n = L_n$ a.a. n, then (U_n) is statistically convergent.*

Proof. Suppose that $U_n = L_n$ a.a. n and $L_n \to V$. From the inclusion

$$\begin{aligned} & \{n \leq m : \sup_{u_n\in U_n, \|u_n\|=1} \|u_n - P_V u_n\| \geq \epsilon\} \\ \subseteq & \{n \leq m : \sup_{u_n\in U_n\neq L_n, \|u_n\|=1} \|u_n - P_V u_n\| \geq \epsilon\} \\ \cup & \{n \leq m : \sup_{\ell_n\in L_n, \|\ell_n\|=1} \|\ell_n - P_V \ell_n\| \geq \epsilon\}, \end{aligned}$$

for $\epsilon > 0$, we get

$$\begin{aligned} & \lim_{m\to\infty} |\{n \leq m : \sup_{u_n\in U_n, \|u_n\|=1} \|u_n - P_V u_n\| \geq \epsilon\}| \\ \leq & \lim_{m\to\infty} |\{n \leq m : \sup_{u_n\in U_n\neq L_n, \|u_n\|=1} \|u_n - P_V u_n\| \geq \epsilon\} \\ + & \lim_{m\to\infty} |\{n \leq m : \sup_{\ell_n\in L_n, \|\ell_n\|=1} \|\ell_n - P_V \ell_n\| \geq \epsilon\}| = 0. \end{aligned}$$

Consequently we have $st - \lim U_n = V$. $\square$

Theorem 2.10. $st - \lim U_n = V$ *if and only if* $st - \lim \sum_{j=1}^{k} < u_{i_n}, v_j >^2 = 1$ *for every* $i = 1, 2, 3, ..., k$.

Proof. Since $\{v_1, v_2, ..., v_k\}$ is orthonormal basis, with an easy calculation we get

$$\|u_{i_n} - P_V u_{i_n}\|^2 = 1 - \sum_{j=1}^{k} < u_{i_n}, v_j >^2$$

for every $i = 1, 2, 3, ..., k$ and hence

$$\begin{aligned} & \frac{1}{m} \left|\{n \leq m : \quad \|u_{i_n} - P_V u_{i_n}\| > \epsilon\}\right| \\ = \quad & \frac{1}{m} \left| \left\{ n \leq m : \quad |1 - \sum_{j=1}^{k} < u_{i_n}, v_j >^2 | > \epsilon^2 \right\} \right|. \end{aligned}$$

So,we have if $st - \lim U_n = V$ if and only if $st - \lim \sum_{j=1}^{k} < u_{i_n}, v_j >^2 = 1$ for every $i = 1, 2, 3, ..., k$.. □

References

1. M. Candan and E. E. Kara, *A study on topolojical and geometrical characteristics of new Banach sequence spaces*, Gulf J. Math., **3** (2015), 67-84.
2. J. S. Connor, *The statistical and strong p-Cesàro of sequences*, Analysis, **8** (1988), 47-63.
3. J. S. Connor, *On strong matrix summability with respect to a modulus and statistical convergence*, Canad. Math. Bull., **32** (1989), 194-198.
4. Ć. B. Dolićanin and A. B. Antonevich, *Dynamical Systems Generated by Linear Maps*, Springer International Publishing Switzerland 2014.
5. E. Dündar and C. Çakan, *Rough I-convergence*, Gulf J. Math. **2(1)** (2014), 45-51.
6. H. Fast, *Sur la convergence statistique*, Cooloq. Math., **2** (1951), 241-244.
7. J. A. Fridy, *On statistical convergence*, Analysis, **5** (1985), 301-313.
8. J. A. Fridy and C. Orhan, *Lacunary statistical convergence*, Pacific. J. Math., **160** (1993), 43-51.
9. C, Heil, *Introduction to Real Analysis*, Graduate Texts in Mathematics, Springer, 2019.
10. Manuharawati et all, *The new convergence definition for sequence of k-dimensional subspaces of an inner product spaces*, J. Phys.:Conf. Ser. 1417 012020.
11. T. Salat, *On statistically convergent sequences of real numbers*, Math. Slovaca, **30** (1980), 139-150.
12. I. J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly, **66** (1959), 361-375.
13. H. Steinhaus, *Sur la convergence ordinaire et la convergence asymptotique*, Colloq. Math., **2** (1951), 73-74.

[1]Deparment of Mathematics, Afyon Kocatepe University, 03200, Afyonkarahisar, TURKEY

Email address: fnuray@aku.edu.tr